ADVANCED PRACTICAL PHYSICS FOR STUDENTS

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WITH 8 PLATES
AND 493 DIAGRAMS AND ILLUSTRATIONS

NINTH EDITION, REVISED AND ENLARGED

METHUEN & CO. LTD. LONDON
36 Essex Street, Strand, W C.2

This book was first published in April 1923
It has been reprinted seven times
Ninth edition, revised and enlarged, 1951

CATALOGUE NO. 7389/U

PRINTED IN GREAT BRITAIN

PREFACE

The course of Practical Physics described in this book is based upon that followed in King's College, London, by students who have completed their Intermediate Course, and who are proceeding to a Pass or Honours Degree. This has been extended, and it is hoped that the book will be useful to a wider circle of students of physics than those immediately concerned with University Examinations.

A number of well-known physicists have contributed to the development of the King's College course, amongst whom we may mention Professors H. A. Wilson, C. G. Barkla, H. S. Allen, and W. Wilson, who formerly worked here in the Wheatstone Laboratory, and Professor O. W. Richardson, the present occupant of the chair.

The general aim has been to provide with each experiment a short theoretical treatment which will enable the student to perform the experiment without immediate reference to theoretical treatises. To aid this scheme an introductory chapter on the calculus has been included. This chapter is an innovation in a book of this type, but it is hoped that the student will find here a bridge over that period during which his physics demands more advanced mathematics than his systematic study of that subject has yet given him.

We take this opportunity of expressing our gratitude to Professor O. W. Richardson, who has allowed us to make use of laboratory manuscripts and results of experiments. We are also greatly indebted to our colleagues and to Mr. G. Williamson, who have given us many suggestions, and to the Honours students of the past session who have supplied us with numerical and graphical results. We have been greatly helped by the ready assistance on the part of the Cambridge and Paul Scientific Instrument Co., Messrs. Elliot Bros., Gambrell, Ltd., Adam Hilger, Ltd., W. G. Pye & Co., and the Weston Electric Co., who supplied us with the blocks for many of the illustrations.

B. L. W.
H. T. F.

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March 1923

PREFACE TO NINTH EDITION

In this edition we have undertaken a complete revision of the book. The introductory chapter on the calculus which appeared in previous editions seems to us now to be out of place in a work of this kind, since students in these days acquire the necessary training in mathematics before embarking upon a final degree course in experimental physics. It has been replaced by a chapter on accuracy of observations. We have also added chapters dealing with modern developments in physics, especially in the branch of electronics, which have resulted in considerable changes in experimental teaching of the subject.

The object has been to teach the important physical principles underlying this branch of physics, and not to provide specialist instruction in this field.

It is with great pleasure that we acknowledge the help of Dr. D. Owen for his advice on electrical bridge methods.

B. L. W.
H. T. F.

May 1950

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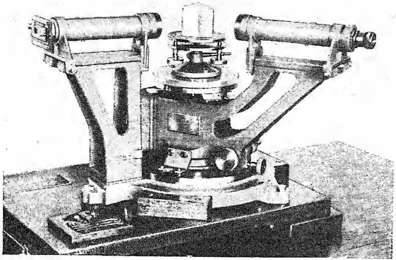

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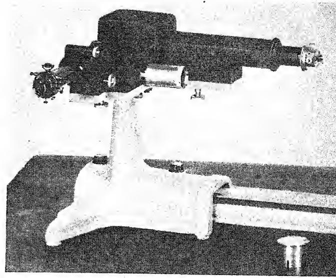

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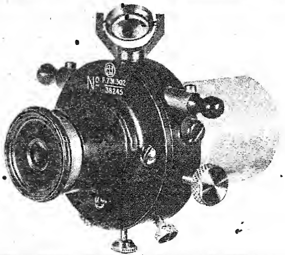

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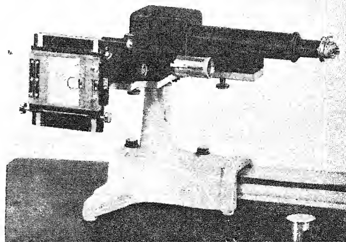

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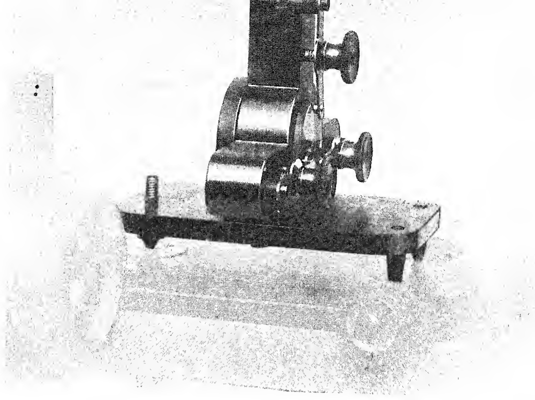

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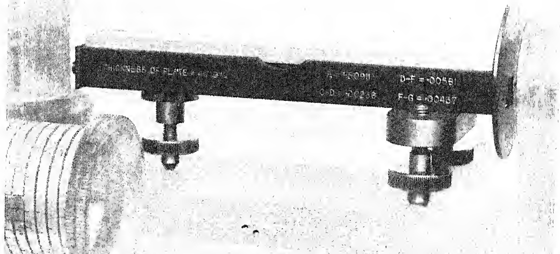

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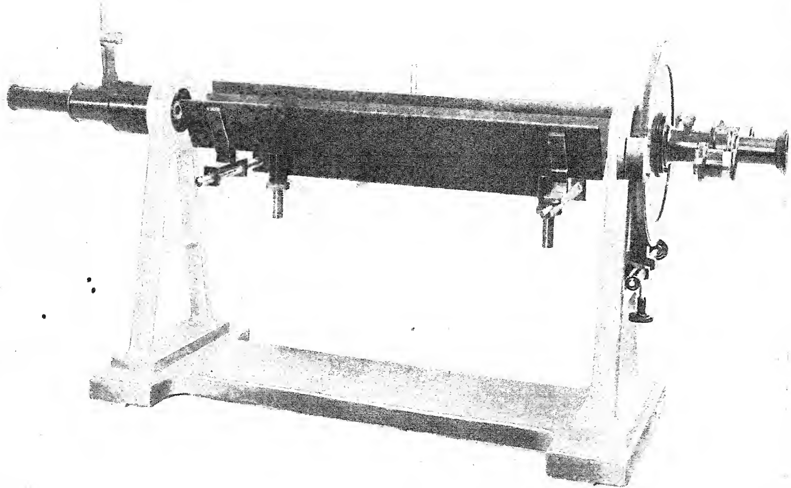

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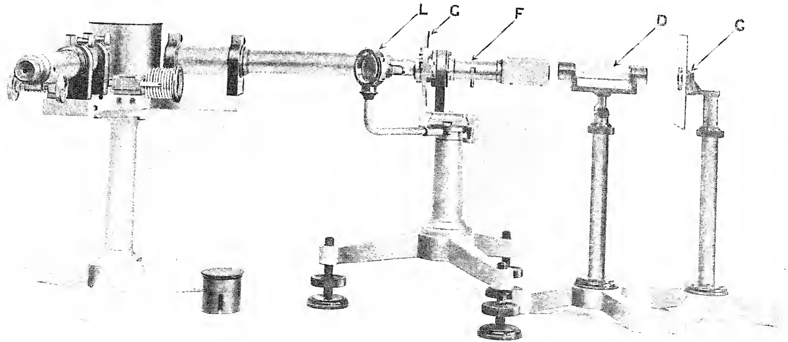

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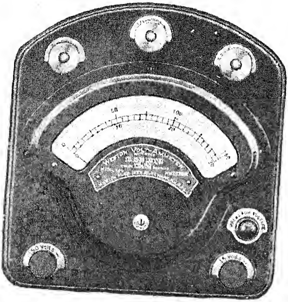

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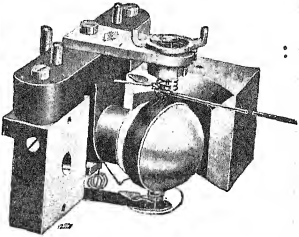

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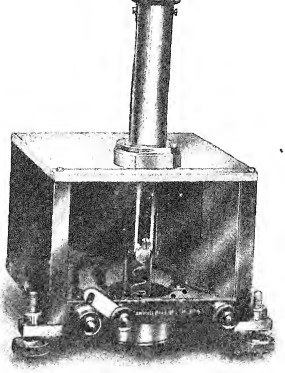

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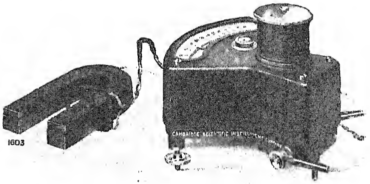

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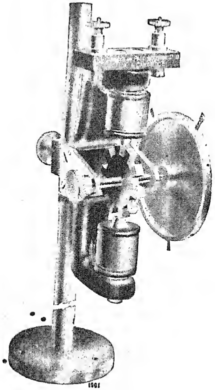

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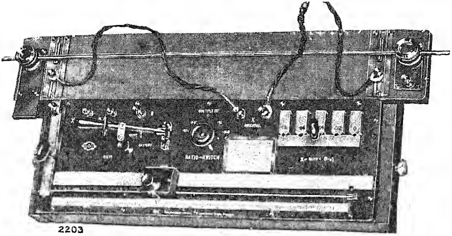

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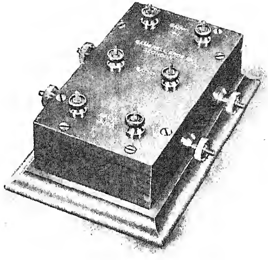
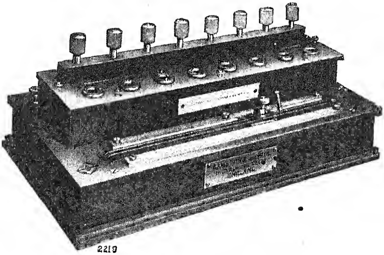


FIG. 350



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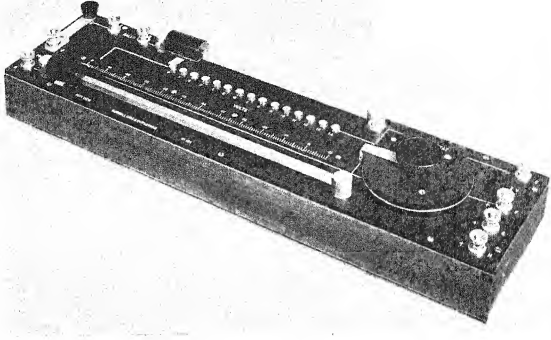

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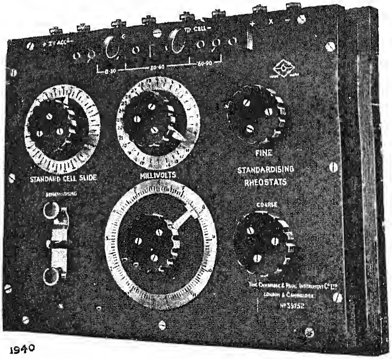

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CHAPTER I

ERRORS OF OBSERVATIONS

In many experiments in the physics laboratory the object is the determination of the numerical value of a quantity. But it is not sufficient merely to record the number which has resulted from the measurements made in the course of the investigation. It is essential to give some indication of the accuracy of the measurement.

It is often the case that the experiment is an exercise set to train the student in the use of apparatus and in the methods of practical physics, so that it frequently happens that the value which should result is known to a high degree of accuracy. In this case an obvious record of the accuracy of the experiment can be obtained by comparison with the known result and by expressing the deviation from it in the form of a percentage error. This is, however, not a record of accuracy such as is required. The observer must place himself in the position of one who is ignorant of the true value of the quantity he is trying to discover, and he must then give a record of the reliance to be placed in the result he has obtained. This reliance will depend upon a number of factors, such as the skill of the observer, the quality of the apparatus he uses, and the constancy of the conditions under which he is working.

What is required is some method of describing the accuracy of the result in a useful and reasonable way. A definition for the estimation of accuracy is therefore required, and as in the case of other definitions the matter is one of convention combined with theoretical considerations. In order to apply certain principles, which the theory of errors brings to light, in a satisfactory way, a large number of determinations of the quantity concerned have to be made. In the majority of the investigations which a student is required to make it is therefore impossible to derive a satisfactory estimation of accuracy in accordance with the definition. There are a number of more obvious and simpler considerations to be examined first, and an observer should pay attention to them in every experiment which leads to the determination of a numerical result.

The Order of Accuracy of Measurements

When an experiment is made in which the measurement of various quantities is involved it is important to note with what accuracy each quantity can be determined with the apparatus available. In the record of the experiment the percentage accuracy with which the different lengths, times, masses, or other quantities are measured should be noted.

The importance of this observation is not merely that it gives a measure of the limit of accuracy which can be attained, but that it directs attention to the degree of accuracy which is required in individual measurements. This point is illustrated in one of the earliest experiments which a student is required to perform in his physics course. This is the experiment for the determination of the density of a solid body by direct measurement and weighing. The body consists of a plate of metal with a thickness of the order of a millimetre and with the other dimensions of a few centimetres. Vernier callipers reading to 0·1 mm. and a micrometer screw gauge reading to 0·01 mm. are usually provided. The idea of the experiment is to stress the fact that, in order to maintain the same order of accuracy in the various measurements of length, the thickness must be measured by the screw gauge and the other dimensions by the callipers.

A balance is provided for the measurement of the mass of the metal which can be determined to a milligram.

Suppose that the length of the sides are a and b and that the thickness is c, the mass being M. As an example, let these have the values 10 cm., 5 cm., 1 mm., and 50 gm. respectively. The percentage accuracies with which these quantities can be measured with the apparatus provided are $\frac{1}{10}$, $\frac{1}{5}$, 1, and $\frac{1}{500}$ respectively. The influence of these upon the final measurement of the density ρ may be determined from the formula

$$\rho = \mathrm{M}/abc. \qquad \ldots(1)$$

The variation of ρ in terms of the variations of the other quantities is derived from the formula

$$\frac{\delta\rho}{\rho} = \frac{\delta\mathrm{M}}{\mathrm{M}} - \frac{\delta a}{a} - \frac{\delta b}{b} - \frac{\delta c}{c}. \qquad \ldots(2)$$

The values of δM, δa, etc., may be positive or negative, and in some cases the terms on the right-hand side of equation (2) may counteract one another. This effect cannot be relied upon, and it is necessary to consider the worst case, which is that in which all the errors tend in the same direction giving for the error $\delta\rho$, the value corresponding to the equation

$$\frac{\delta\rho}{\rho} = \frac{\delta\mathrm{M}}{\mathrm{M}} + \frac{\delta a}{a} + \frac{\delta b}{b} + \frac{\delta c}{c}. \qquad \ldots(3)$$

In the case considered the fraction $\frac{\delta c}{c}$ corresponds to 1 per cent and the other contributions to $\frac{\delta\rho}{\rho}$ are insignificant. This is particularly the case in the determination of the mass. It is not worth while to exercise great care in the determination of the mass of 50 gm. to an accuracy of 1 mg.

The student is recommended, however, to measure carefully the various quantities involved in using calculations of the kind discussed. It is no disadvantage to measure some of the quantities more accurately than is strictly necessary while a definite loss occurs if the unimportance of any term on the right of equation (3) leads to a slipshod determination of the corresponding quantity. The point of this discussion is to suggest the exercise of common sense in making the measurements and to point out how the accuracy of the final result depends on the various individual errors.

A case where much time is often lost and trouble often expended in vain occurs in calorimetric experiments where temperature measurements can be made to an accuracy of $\frac{1}{10}$° C. only, but where masses can be measured easily to $\frac{1}{10000}$. It is clearly useless to waste time in the determination of the masses of calorimeters and contents to this degree when the temperature is observed to $\frac{1}{10}$° in 5° or 6° C. But here again, anything in the nature of careless measurement may lead to large errors; a sense of proportion and the exercise of care are required. A further example of the influence of individual measurement on a final error will be considered from the formula used in deducing the coefficient of the viscosity of a liquid by the method of flow through a tube.

The formula applicable in this case is

$$\eta = \frac{\pi}{8}\frac{R^4}{l}\frac{P}{Q}, \qquad \ldots(4)$$

where R denotes the radius of the capillary tube, P denotes the pressure difference between its ends, l denotes its length, and Q the volume of liquid flowing per sec.

The maximum error $d\eta$ is in this case given by the formula

$$\frac{d\eta}{\eta} = 4\frac{dR}{R} + \frac{dl}{l} + \frac{dP}{P} + \frac{dQ}{Q}. \qquad \ldots(5)$$

It is to be noted that an error in the measurement of R is magnified four times on account of the occurrence of R^4 in the formula. This example illustrates the need for very careful measurement of quantities such as R, which, by the way they occur in the expression and in the calculation, exercise a strong influence on the final result.

In every experiment the possible error in the result must be determined according to the foregoing principles and it must be stated together with the calculated result.

Errors Associated with Observations

In the first place, it is necessary to distinguish between *mistakes* and *errors*. The term *mistake* will be used to denote a fault of measurement or of observation which can be avoided by care on the part of

the observer. An example is the recording of a wrong number. On the other hand, an error may occur in the most careful observation, as in the case of the careful use of an instrument which suffers from an error of graduation.

The correction of mistakes is not now under discussion, although one of the results of the training afforded by a course in practical physics should be the avoidance of carelessness to which mistakes are due. But training in the correction of mistakes is not peculiar to the training of practical physics. On the other hand, the study of the nature of errors and of their elimination is part of the subject of physics. In the first place, types of errors will be classified as constant, systematic, and accidental.

Constant errors are those which affect the results of a series of experiments by the same amount. An example is the case of the faulty graduation of a scale. Thus, if the value of the acceleration due to gravity be determined by the simple pendulum, the length of which is measured by means of a scale in which the intervals marked centimetres are all 0·99 cm., the value obtained from a series of measurements would differ by a constant amount from the true value. Such deviations are difficult to detect, for an examination of the observations may reveal that they have been made carefully and the final result makes clear their mutual agreement.

One reason for making physical measurements by as many different methods as possible is for the purpose of making sure that constant errors are eliminated. In this way it may be discovered that an error is peculiar to one method and its source may then be traced.

Systematic errors are those which occur according to some definite rule, such as would be the case in readings on a circular scale if the pointer were not pivoted at the centre. These can be eliminated once their source is detected, and the rule governing them is known.

One error which falls into these classes is worthy of special mention on account of its special character. It is due to personal peculiarities of the observer, and is known as the personal equation. It is, however, rather indefinite and only approaches the character of a constant or systematic error in the case of experienced observers in their normal state of health. An example in which this type of error occurs is that in which a spot of light passing a zero mark is constantly recorded as passing slightly after the actual instant.

Inexperienced observers or observers not in a normal state make errors of varying magnitude which should strictly be described as mistakes.

When errors of these kinds arising from instruments, external conditions such as temperature variation, or from personal idiosyncrasies are eliminated it is found that there is still a margin of error which requires a further consideration. This is due to *accidental errors*.

From the discussion of these errors it will be evident that the elimination of them is not possible in the experiments performed in the usual laboratory course. But it is important to have some idea of the method of dealing with them, and in the course of this work an experiment will be described in order that it may provide an exercise on accidental errors (p. 390).

The source of accidental or, as they are also called, random errors cannot be traced to any systematic or constant cause of error. They may be defined as errors due to a cause of which the law of action is unknown. Their character can be appreciated from the illustration of the firing of shot at a target. Let it be supposed that the firing is from a rifle by a 'good shot', the target being marked with a bull's-eye with the usual concentric rings. By the term 'good shot' is to be understood a rifleman who makes no mistakes and who has a definite known personal equation. The result to be expected is well known. The target will be marked by a well grouped arrangement of shots. These will consist of a certain number near to a certain point with others grouped round it. The group will consist of points on either side of a central point and of others above and below it. The effect of a constant error such as might arise from a defect in the sights, or from a steady wind blowing across the line of fire, is to cause the shots to fall at a definite distance to one side or other of the centre of the target. But if such an error, as also all systematic errors, be eliminated by making due allowances, the effect is to cause a number of shots to fall close enough to the centre of the target to be registered as 'bulls' with a grouping about the centre. There is no means of avoiding these random shots, for the reason that the law of action of the causes that give rise to them is unknown. They arise as a consequence of errors in taking aim, of small variations in the strength of the wind, and of other similar causes.

An examination of a target at which a large number of shots have been fired will show that the random shots lie with as many to one side of the centre as to the other. They will also show that small deviations from the centre are more numerous than large deviations, and that a large deviation is very rare.

It is assumed as a result of observations of this kind that all attempts at making measurements are associated with random fluctuations which obey the laws that a large number of random errors are present, that positive and negative errors of the same magnitude are equally likely and that a small error is more likely than a large one.

Thus, the problem of determining an accurate measurement consists in making an allowance for constant and systematic errors, and of describing in some way the effect of accidental errors.

Suppose that a target is fired at and then the bull's-eye and other distinguishing marks removed, nothing but the grouping of shots is

to be seen. The physical problem of determining a measurement is like that of finding the bull's-eye from the grouping. The constant and systematic errors can be eliminated, but there still remains the arrangement of shots due to accidental errors. The question is to find the bull's-eye from this arrangement and the answer to the question is that the most likely position is at a certain spot and that the way the shots fell indicates the reliance to be placed upon this determination. Some convention is adopted for the expression of this answer, which will now be studied. In the first place, there are theoretical considerations which help us in studying a law of distribution of these random errors. But any development of a theory must rest upon certain hypotheses which agree closely with the conditions under which the errors occur and which are of as simple a character as possible.

A derivation by Hagen in 1837 of a law of errors known as the normal law, or the Gaussian law, rests upon the assumption that the random error in any measurement is the sum of an infinitely large number of small errors equally likely to be positive or negative. The hypothesis of a very large number is open to criticism, since it must always be applied to a finite number of observations; but the result justifies it as a very good approximation.

Let it be supposed that in a series of observations the number of times the errors have values lying between x and $x+dx$ is $d\mathrm{N}$. This number is dependent upon the value of the error and it is possible to write

$$d\mathrm{N} = f(x)\ dx. \qquad \ldots(6)$$

This means that the number is put equal to an average ordinate $f(x)$ multiplied by the range of the error.

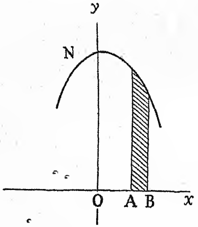


Fig. 1.

If a graph be drawn of N as a function of x, the number of errors lying between the values OA and OB is represented by the shaded area, and $f(x)$ is the average ordinate in the range AB. The curve obtained

in this way is known as the curve of frequency. The determination of the law of error means the determination of the form of $f(x)$ and Hagen's argument leads to the value

$$f(x) = Ae^{-h^2x^2}, \qquad \ldots(7)$$

where A and h are constants.

Thus, equation (6) is

$$dN = Ae^{-h^2x^2}\, dx. \qquad \ldots(8)$$

In any case, all the errors lie between 0 and ∞ on the positive and negative sides, so that the total number of errors is

$$N_0 = A\int_{-\infty}^{\infty} e^{-h^2x^2}\, dx$$

$$= A\frac{\sqrt{\pi}}{h} \qquad \ldots(9)$$

for the value of the integral is $\frac{\sqrt{\pi}}{h}$.

Thus, $$\frac{dN}{N_0} = \frac{h}{\sqrt{\pi}}\, e^{-h^2x^2}\, dx. \qquad \ldots(10)$$

This is the fraction of the total number of errors lying between the limits x and $x+dx$ and, for the sake of brevity, it is described as the probability of the error x.

The graph of the error function:

$$y = \frac{h}{\sqrt{\pi}}\, e^{-h^2x^2} \qquad \ldots(11)$$

is shown in Fig. 2.

It appears from the graph that the probability of zero error is a maximum and that the probability falls off rapidly with the magnitude of the error. There is some probability of very large errors occurring but it is small. This is a point which seems to be at variance with experience, for in a physical measurement it would seem to be quite impossible to incur an error beyond some finite value. Thus, in measuring a length of 10 cm. it seems absurd to suggest that there is any probability of recording 20 cm. as an observed value. It will, however, be noted that the probability curve falls rapidly to very low values, and the curve may be regarded as a close description of the facts of experience. This has been tested in many ways by drawing frequency curves. If a series of observations results in the determination of a value for a certain quantity, it may be assumed for the present purpose that the arithmetic mean is the most accurate value. This assumption is justified when a very large number of observations is made and the law of error is the normal law. The values determined

should then be subtracted from the value of the arithmetic mean and the deviations regarded as errors.

Let a graph be drawn with the deviations as abscissae marking a scale in some convenient unit. Let an ordinate be drawn midway between 0 and 1 to represent the number of observations with deviations lying between these limits, and repeat this procedure for the intervals 1 to 2, and so on. If the tops of these ordinates be joined, a graph will be obtained consisting of a broken line with a maximum

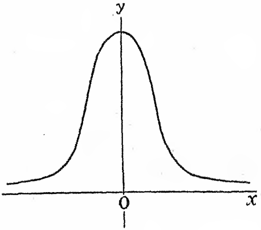


FIG. 2

close to the interval 0 to 1, and as the number increases this interval will be the position of the maximum and the broken line graph will tend towards a curve of a standard shape, whatever be the nature of the measurements, provided that the conditions of normal errors apply. This, at any rate, occurs in a very large number of cases encountered in physics.

The standard shape is that of the curve of errors (11), and experiments of this kind may be regarded as the experimental verification of the theoretical basis upon which the derivation of the equation is founded. An experiment to be described later may be regarded as a verification of this character (p. 390).

Other error distributions than that corresponding to the normal law are possible, but the normal law usually applies when the observations are made with equal care under equally favourable conditions, and it is the only distribution to be considered here.

The Estimation of Errors

In the equation of the curve of error (11) the constant h is characteristic. If a large number of observations has been made and the deviations (x) from the arithmetic mean have been plotted, the accuracy of the result can be judged from the way in which the curve rises to a maximum and it can be expressed quantitatively by means

of the constant h appropriate to the curve. In practice the accuracy of a set of observations is not determined directly by estimating h, but by means of other quantities more suitable from the practical point of view. It will be seen that these depend upon h.

The Average Error

This quantity will be denoted by η, and is the arithmetic mean of all the errors neglecting their sign; the arithmetic mean of all the errors taking account of their sign, in the case of a normal distribution, is zero.

The number of observations having an error x is $N_0\frac{h}{\sqrt{\pi}}e^{-h^2x^2}\,dx$, and the sum of the errors of magnitude x is $\frac{N_0 h}{\sqrt{\pi}}\,x\,e^{-h^2x^2}\,dx$. Thus, the sum of all the errors, when the sign of x is neglected, is twice this quantity integrated from 0 to ∞. The average error is then obtained by division by N_0. After evaluation of the integral it is found that

$$\eta = \frac{2h}{\sqrt{\pi}}\int_0^\infty x\,e^{-h^2x^2}\,dx = \frac{1}{\sqrt{\pi}h}. \qquad \text{...(12)}$$

The Root-Mean-Square Error

Another important quantity is the average of the squares of the errors from which, by taking the square root, the root-mean-square error is obtained. This is sometimes described as the mean square error, or standard deviation, and is denoted by μ. It is evidently given by

$$\mu^2 = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2\,e^{-h^2x^2}\,dx = \frac{1}{2h^2},$$

or

$$\mu = \frac{1}{\sqrt{2}h}. \qquad \text{...(13)}$$

The Probable Error

Finally, an expression for the probable error is often used. This is the value of x such that half the total deviations lie below it and half above it. The value is denoted by r, and this definition is expressed by the equation

$$\frac{1}{2} = \frac{h}{\sqrt{\pi}}\int_{-r}^{r} e^{-h^2x^2}\,dx, \qquad \text{...(14)}$$

which means that half the errors lie between the values $x = -r$ and $x = +r$.

From this equation it may be deduced that

$$r = 0{\cdot}4769/h. \qquad \text{...(15)}$$

These quantities may therefore be regarded as a measure of the accuracy of a set of observations.

The Practical Determination of Error

In making a record of the accuracy of a set of observations it must be noted that the number of individual measurements is finite and that estimates both of the magnitude of the quantity and of the accuracy of the determination are required from this finite number. It might be suggested that the best frequency curve possible should be drawn from the measurements made, and that the value of the maximum should be taken as the best value while the accuracy could be judged from the spread of the curve. The value of h might be determined from the curve, and its value would then serve as an indication of accuracy. As a record of accuracy, it would be possible to introduce values of the average error, the root-mean-square error, or the probable error which, as we have seen, are all connected with h.

This graphical method is inconvenient, and it would be inaccurate in practice. The procedure adopted is to determine the arithmetic mean of the results and to assume that it is the best estimate of the quantity required. This is an assumption that has been accepted as an axiom, and it means that the arithmetic mean is regarded as the best representative of equally trustworthy results. By subtracting the arithmetic mean from each of the values determined, the deviations or residuals are obtained. These may be positive or negative, and the algebraic sum is zero.

The next step is to determine the value of the root mean square of the residuals.

In order to derive an exact description of the error we require an infinite number of values. Such a quantity as the probable error in the case of an infinite number of values is then a characteristic of the conditions in which the random errors occur. It is not a quantity dependent upon the number of observations. The calculation of the root mean square would be made most simply by means of the formula

$$\mu = \pm\sqrt{\frac{\Sigma x^2}{n}}, \qquad \ldots(16)$$

where x denotes the value of a residual and n is the number of determinations. When the number is small, μ will depend on n and a better approximation to the root mean square as representing the conditions of the experiment is provided by Bessel's formula

$$\mu = \pm\sqrt{\frac{\Sigma x^2}{\sqrt{n(n-1)}}}. \qquad \ldots(17)$$

The two formulae become the same for large values of n. Other formulae can be used for the estimation of μ, but Bessel's formula is usually the most accurate. This quantity, μ, as we have seen, is described as the root-mean-square error or mean square deviation.

From equations (13) and (15) μ is related to the probable error by means of the relation

$$r = \pm\, 0{\cdot}6745\, \mu. \qquad \ldots(18)$$

From the definition of the probable error this quantity denotes a small range on either side of the average value, within which it is as likely as not that any measurement chosen at random will lie.

The record of the numerical result of an experiment is made by writing down the arithmetical mean, a, and by placing after it plus or minus the probable error r, thus $(a \pm r)$.

Another method for recording the accuracy of observations is that due to Peters, and it consists in determining the average error. The residuals are again taken to represent the errors, so that for a large number of observations the average error is given by

$$\eta = \pm\, \frac{\Sigma\, |x|}{n}. \qquad \ldots(19)$$

This is only approximately correct for small values of n, and it fails completely in the case of one observation, since in that case η is zero. The symbol $|x|$ means the numerical value of the residual.

A better approximation to the average error is given by

$$\eta = \pm\, \frac{\Sigma\, |x|}{\sqrt{n\,(n-1)}}. \qquad \ldots(20)$$

The probable error may be determined from η for by means of the equations (12) and (15) it follows that

$$r = \pm\, 0{\cdot}8453\, \eta.$$

The conventional character of this determination of error should be noted. It is an attempt to record the influence of accidental errors on the result, and except in so far as the experimenter himself introduces accidental errors, the record is intended to be independent of the observer.

The criticism has sometimes been advanced that such a record does not give information upon which reliance on the accuracy can be placed. It is argued that experience and proved skill of observers and a knowledge of the detail and circumstances of an individual set of observations afford better information. It would, however, be very difficult to formulate any method for the measurement of accuracy on these grounds, and the conventional method has at least the advantage that it is based on considerations which attempt to record the accuracy of the observations from a study of the observations themselves, which thus become a permanent record of the reliance to be placed upon them.

The Rejection of Observations

When a series of values results from a succession of determinations it may happen that one or more are markedly different from a group

which lie close together. There is a strong temptation to neglect these values in the determination of the average on the basis that something must have occurred which has resulted in incorrect determinations. There may be no obvious cause for this belief, and the actual reason for neglecting the values is because they do not agree with the majority of those obtained.

No determination must be neglected merely because of its divergence from other similar determinations. If the experimenter is aware of any cause which indicates that an observation has been made under circumstances differing from those of the rest of the group, it may be neglected; it should in fact, be neglected, for it does not strictly belong to the group. But any result obtained under the same conditions is required to complete the series from which the arithmetic mean and probable error are to be determined. The experimenter may decide to 'repeat' the determination, but he must remember that this is not truly a case of repetition but of addition of a new result to the series. The divergent result must not be discarded and replaced by a new one which may lie closer to the majority of results.

If the curve of frequency (fig. 2) be examined, it will be seen that there is a finite probability of obtaining results widely divergent from the mean value. That is to say, that on account of the existence of random errors a divergent result will occur when every detail of the determination has been carried out with precision on the part of the observer.

The need to examine the question of discarding any result, except those whose inaccuracy can be traced to a definite cause, would not arise if it were possible to obtain a very large number of determinations. They would all be required to give the frequency curve. The difficulty is that in practice a finite number only is obtainable.

In fig. 3, let deviations from the mean be plotted as abscissae. Suppose that four observations result in the determination of values corresponding to the points A, B, A′ and B′. The curve of frequency is, of course, unknown, and in particular the average value is not known. It is simply supposed for the purpose of illustration that the values obtained happen to correspond to these four points. The arithmetic mean of the results in this case would be the correct value. But suppose the four values obtained happened to correspond to A, A′, B′, C′. The arithmetic mean would no longer be correct, because the observation C′ exerts an undue influence on the result. There is nothing incorrect about the determination which led to the value corresponding to C′. The fault lies in the fact that the observations are too few. It is just as if the average age of a class had been taken when most of the older pupils were absent, the ages of the younger then exerting too great a bias on the results. There is thus a reason for avoiding the undue influence of C and, on account of the knowledge of the shape

of the frequency curve, a rule can be stated for discarding or retaining any particular member of a finite group of observations.

In fig. 4, a frequency curve is drawn with a wide spread for the sake of clarity. Draw ordinates PN and P′N′ for deviations $\pm$ON.

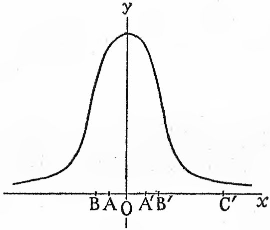


FIG. 3

The area lying under the curve and between the ordinates PN and P′N′ represents the number of observations with deviations up to and less than ON. The area outside these ordinates under the curve represents the number with deviations exceeding this value. The form of this curve is given by equation (11) and when fig. 4 is drawn

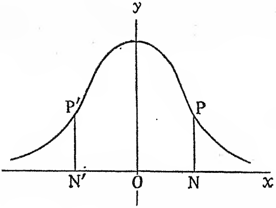


FIG. 4

according to this equation the areas just mentioned give the fraction of the number of observations with deviations less than ON and greater than this value. All that is required theoretically for the determination of these areas is the constant h, which can be found from the probable error by means of equation (15). It is thus possible to calculate how many out of a number of results should lie on one side or the other of a particular deviation.

Suppose that ten determinations have been made and that one of them deviates markedly from the remainder. Let this particular one have a deviation d. The probable error is deduced from the determinations and a calculation can then be made of the fraction which should possess the deviation d or more. Suppose that the fraction is $\frac{1}{20}$. This gives as the number out of ten determinations only $\frac{1}{2}$, which means that one in ten is an undue proportion, and the inclusion of this determination would throw too great a tendency in one direction. It is thus justifiable to reject this result. It may occur that more than one result can with justification be excluded, but the most divergent should be excluded in one step and the process then repeated neglecting this result, and so on until all the remainder lie within the limit set by the greatest remaining deviation.

This process is tedious without the use of tables, and a table of values of the probability integral has been constructed which can be used after the manner of tables of logarithms.

The probability integral is

$$\mathrm{I} = \frac{2}{\sqrt{\pi}}\int_{0}^{z} e^{-z^2}\,dz. \qquad \ldots(21)$$

The area under the curve of fig. 4 lying between the ordinates PN and P′N′ is

$$\mathrm{S} = \frac{h}{\sqrt{\pi}}\int_{-c}^{c} e^{-h^2x^2}\,dx, \qquad \ldots(22)$$

where $c = \mathrm{PN}$.

By writing $z = hx$

$$\mathrm{S} = \frac{2}{\sqrt{\pi}}\int_{0}^{hc} e^{-z^2}\,dz. \qquad \ldots(23)$$

Suppose that the probable error, r, of a set of determinations is 2. The value of h is thus 0·4769/2 by equation (15).

Let it be required to determine the number of results with a deviation greater than 3. Thus, $hc = 0{\cdot}4769 \times 3/2 = 0{\cdot}7153$.

On reference to the tables it is found that the value of the integral from zero to 0·7153 is 0·6883, so that 68·83 per cent of the results have a deviation less than 3 and 31·17 per cent have a greater deviation.

CHAPTER II

MEASUREMENT OF LENGTH, AREA, VOLUME, AND MASS

When the length of a body is required to an order of accuracy greater than that which may be obtained with an ordinary rule, the measurement may be made by using one of the usual vernier or micrometer devices, such as the vernier calliper, the micrometer screw gauge, or the spherometer.

The measurement of small objects may also be carried out by use of the travelling microscope or the micrometer microscope.

In using either of these instruments care must be taken, when viewing the image of the ends of the object, that this image is in the same plane as the cross-hair or the small scale in the eyepiece, otherwise parallax errors may be introduced. The image seen should not show any relative movement with the cross-hairs when the eye is moved across the field of view.

When focusing the cross-hair, as a preliminary adjustment, the eye should be unstrained. The microscope is turned to a bright distant object and the adjustment of the eyepiece should be made so that the distant object viewed by the one eye is in focus when the cross-hair as viewed by the other eye is also clearly focused.

The Comparator

When a length exceeds a few centimetres, the travelling microscope or micrometer microscope are not used individually as measurers, but are replaced by an arrangement of two such instruments arranged at a variable distance apart on a fixed graduated bed. Each microscope may be moved in the usual manner in a direction parallel to the length of the bed. Each microscope is provided with a scale or fine cross-hair in the focal plane of the eyepiece. Each eyepiece is adjusted so that the scale or cross-hair is in sharp focus for normal vision, and is replaced in the carriage.

An object to be measured is fastened rigidly along the bed, in the groove provided for it. The image of one end of the object as seen by the first microscope is brought into coincidence with the image of the intersection of the cross-hairs and the second microscope is moved until the image of the other end on the object coincides with the cross-hair intersection in that microscope.

Thus the images of the two ends of the object as seen by the two microscopes normal to its length are formed at the intersection of the two pairs of cross-hairs.

The object is removed and a standard rule substituted. The readings

of the standard scale seen opposite the two cross-hairs obviously enable the length of the object to be ascertained by subtraction.

As an example of this substitution method we shall consider the experimental details of the following experiment.

The Comparison of the Yard and the Metre

Standard rules, engraved as finely as possible with inch and centimetre graduations, and of lengths one yard and one metre, are employed in this experiment.

Place the yard rule on the bed of the comparator and focus one microscope on a scale division near one end of the yard, moving the latter and the microscope until this is accomplished. Then move the second microscope until a scale division near the other end of the rod is sharply focused, the number of inches included between the two being noted—36 if the scale is sufficiently well graduated to allow of this. Then, taking care not to upset the arrangement of the microscopes in any way, remove the yard scale and substitute the metre, so that the graduations are in good focus. Move the scale so that the image of a division near one end is in coincidence with the cross-hair intersection in one microscope. Under these circumstances the second microscope will not be opposite a division. Coincidence of the cross-hair and the image of a scale division is brought about by a movement of the microscope, parallel to the length of the bed, which is measured on the vernier scale attached to it.

The size of the gap between the two cross-hairs $\pm$ the movement of the microscope is then read off in centimetres.

Care is taken in noting the movement of the microscope to see exactly the unit used in these graduations. In this way we obtain two measurements, one in each system, for the same distance, and may calculate the number of inches to the metre, or centimetres to the yard.

The vernier scale movement on the microscope is often replaced by a micrometer screw capable of a much shorter range of movement. With such a screw traverse, the movement of the microscope may be readily measured to 0·001 cm. Using such a comparator, the values given below were obtained:

(1) $\quad 1 \text{ yard} \doteq 91{\cdot}5 \text{ cm.} + 0{\cdot}008 \text{ inch},$
i.e. $1 \text{ metre} = 39{\cdot}331 \text{ inches}.$

(2) $\quad 50 \text{ cm.} = 19{\cdot}6895 \text{ inches},$
i.e. $1 \text{ metre} = 39{\cdot}379 \text{ inches}.$

(3) $\quad 1 \text{ metre} = 39{\cdot}375 \text{ inches} + 0{\cdot}002 \text{ inch},$
$= 39{\cdot}377 \text{ inches}.$

Mean value 1 *metre* $= 39{\cdot}362$ ***inches.***

The Planimeter

The estimation of the area of a plane figure may be carried out by one of the many geömetrical methods or by the use of a planimeter, an instrument designed to measure such areas directly.

Of this class of instrument the Amsler planimeter is generally used.

It consists, essentially, of two arms AC and EB, hinged at A, fig. 5; AC is of fixed length and is provided with a needle point loaded above by a small weight as shown in the diagram. The second arm may be varied in length by sliding that portion of it which carries the tracer, B, into the slot provided in the other half, EA. By means of fine adjustment, S, the length BE may be set accurately at any division along the graduated face of BA.

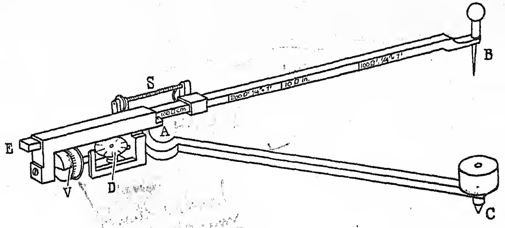


FIG. 5

In addition to the tracer, B, this arm is provided with a small wheel to which is attached a graduated drum which moves past a fixed vernier scale, V. By means of the graduated drum and vernier, the rotation of the wheel may be measured to $\frac{1}{1000}$ of a complete turn. The axle of the wheel and drum is arranged parallel to the length of EB and is provided with a worm gear, which moves a horizontal indicator, D, one division per revolution.

When placed on a plane the instrument is supported at three points: the needle point C, the point of the tracer B, and the point of contact of the wheel with the plane.

To measure an area the needle C is placed at a point outside it, such that B may be moved round the boundary of the area.

For the measurement of large areas this will be impossible, but the details for such a case are given later. Starting at any point on the boundary of the area, the tracer is moved carefully over its contour until it is finally in the starting position. During this operation the wheel will have rotated in general a definite number of revolutions plus a measurable fraction of a revolution. From this observation, the area of the figure may be calculated.

The method of calculating the area will be best understood by first considering the theory of the instrument.

When the needle C is fixed in the plane of the area to be measured, any movement of B along the boundary of the figure will result in a movement of A along the arc of a circle with C as centre and radius CA = a cm. (see fig. 6). Further, the wheel will roll a distance equal to the total displacement when the movement of EB is at right angles to its length: movement parallel to the length causes no rotation, the forces acting on the wheel due to contact with the plane, under these latter circumstances only produce a couple tending to move the axle parallel to its length about the pivots.

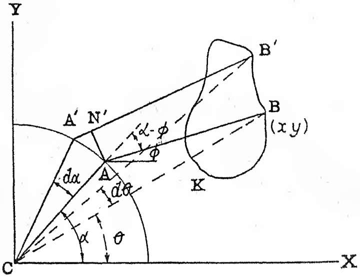


Fig. 6

So for any intermediate form of displacement, the rotation produced in the wheel will correspond to the component of the displacement at right angles to the length EB, and if n revolutions occur, $2\pi n$R will measure this normal displacement, R being the radius of the wheel.

Now it will be shown below that the area to be measured is equal to $2\pi n\text{R} \times \text{AB}$. Since the distance of the wheel from A does not occur in this expression we must first show that this distance has no effect on the number of revolutions the wheel makes.

In fig. 6, let B and B′ be two positions of the tracer a very small distance apart on the boundary of the area to be measured, BB′K; A and A′ being the corresponding positions of the hinge.

Let AB = b, and suppose that the centre of the wheel is at P (fig. 7). Draw AN′ and PN normal to A′B′ from A and P and let AN′ = ds, PN = ds', AP = c.

AL being parallel to A′B′, let the angle LAP = $d\varphi$.

Since BB′ is a small distance, ds, ds', and $d\varphi$ are also small.

Now $\qquad PN = NL + LP$

or $\qquad ds' = ds + cd\varphi.$

The distance moved by P as B traces the boundary of the area BB′K is $\Sigma\, ds'$

or
$$\Sigma\, ds' = \Sigma\, ds + \Sigma\, cd\varphi.$$

B finally returns to the starting-point, and therefore $\Sigma\, d\varphi = 0$,

i.e.
$$\Sigma\, ds' = \Sigma\, ds.$$

Thus a wheel placed at A would indicate the same movement as the one at P, or at any other point along EB. *The position of the wheel on the arm* AB *does not, therefore, affect the reading of the instrument.*

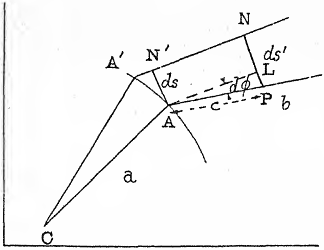


FIG. 7

To show that the area of a plane figure which does not include the needle point C is equal to $(2\pi n\mathrm{R})b$, let us refer the figures to rectangular axes with C as origin, as in fig. 6.

Let CAB be one position of the planimeter and CA′B′ a second position such that BB′ is a small displacement of the tracer. Let the area to be measured be BB′K.

Referred to these axes, let x, y be the co-ordinates of B.

The area BB′K may be conveniently referred to polar co-ordinates r and θ, i.e. $\mathrm{CB} = r$, $\angle\ \mathrm{BCX} = \theta$, then if $\angle\ \mathrm{B'CB} = d\theta$, the area $\mathrm{B'CB} = \frac{1}{2}r \,.\, rd\theta = \frac{1}{2}r^2 d\theta$,

i.e.
$$\Sigma\, \tfrac{1}{2}r^2 d\theta = \text{area of BB'K.} \qquad \ldots(1)$$

For as the radius vector moves round the figure on the boundary remote from C, the small area contains, in turn, each element of the area to be determined, plus the external triangle from the boundary of the figure on the side near to C; this latter area is deducted from the sum as the radius vector travels along this near boundary, for here $\delta\theta$ is negative.

Now $x = r \cos \theta, \quad y = r \sin \theta.$

$\therefore \quad dx = -r \sin \theta \, d\theta + \cos \theta \, dr, \quad dy = r \cos \theta \, d\theta + \sin \theta \, dr,$

whence $x\,dy - y\,dx = r^2\,d\theta$.

Or, the area of the figure, $\frac{1}{2}\Sigma\, r^2 d\theta$ from (1)

$$= \tfrac{1}{2}\Sigma\,(x\,dy - y\,dx). \qquad \ldots(2)$$

Let $\angle$ ACX $= \alpha$; A′CA $= d\alpha$; CA $=$ CA′ $= a$; AB $=$ A′B′ $= b$; and angle between BA and the x-axis be φ.

The co-ordinates x and y of the point B may be expressed as follows:

$$x = a \cos \alpha + b \cos \varphi, \quad y = a \sin \alpha + b \sin \varphi.$$

$$\therefore \; dx = -a \sin \alpha \, d\alpha - b \sin \varphi \, d\varphi, \quad dy = a \cos \alpha \, d\alpha + b \cos \varphi \, d\varphi.$$

$$\begin{aligned}\therefore \; x\,dy - y\,dx &= (a \cos \alpha + b \cos \varphi)\,(a \cos \alpha \, d\alpha + b \cos \varphi \, d\varphi) \\ &\quad + (a \sin \alpha + b \sin \varphi)\,(a \sin \alpha \, d\alpha + b \sin \varphi \, d\varphi) \\ &= a^2\,d\alpha + b^2\,d\varphi + ab \cos(\alpha - \varphi)\,d\,(\alpha + \varphi). \qquad \ldots(3)\end{aligned}$$

But $(\alpha + \varphi) = 2\alpha - (\alpha - \varphi)$ or $d\,(\alpha + \varphi) = 2d\alpha - d\,(\alpha - \varphi)$.

$$\begin{aligned}\therefore \; ab \cos(\alpha - \varphi)\,d\,(\alpha + \varphi) &= ab \cos(\alpha - \varphi)\,\{2dx - d\,(\alpha - \varphi)\} \\ &= 2ab \cos(\alpha - \varphi)\,d\alpha - ab \cos(\alpha - \varphi)\,d\,(\alpha - \varphi)\end{aligned}$$

and $\quad$ AN′ $= ds =$ AA′ $\cos$ A′AN $= a\,d\alpha \cos(\alpha - \varphi)$,

so that

$$ab \cos(\alpha - \varphi)\,d\,(\alpha + \varphi) = 2b\,ds - ab \cos(\alpha - \varphi)\,d\,(\alpha - \varphi),$$

and equation (3) becomes

$$x\,dy - y\,dx = a^2\,d\alpha + b^2\,d\varphi + 2b\,ds - ab \cos(\alpha - \varphi)\,d\,(\alpha - \varphi).$$

Now, when the trace moves round the curve, $\Sigma\, d\varphi = 0$, $\Sigma\, d\alpha = 0$ for the planimeter returns to the exact position of starting: we have further

$$\Sigma \cos(\alpha - \varphi)\,d\,(\alpha - \varphi) = 0, \text{for } \Sigma \cos(\alpha - \varphi)\,d\,(\alpha - \varphi) = \Big[\sin(\alpha - \varphi)\Big]_1^2.$$

But limits (1) and (2) are identical so that this is equal to zero,

i.e. $\quad \Sigma\,(x\,dy - y\,dx) = \Sigma\, 2b\,ds = 2b\Sigma\, ds.$

Now $\Sigma\, ds = 2\pi n$R, as already shown, and we saw in equation (2) above

$$\Sigma\,(x\,dy - y\,dx) = 2 \times \text{area enclosed in BB′K},$$

i.e. $\quad 2$ area of figure $= 2b2\pi n$R.

$$\therefore \; \text{Area of figure BB′K} = b2\pi n\text{R}.$$

Thus, to measure any area sufficiently small, C is fixed in the paper at a point outside the area and B is taken round the boundary. The rotation of the wheel is measured, as is the distance from the hinge to B, and the area is thus equal to the product of BA and $2\pi n$R.

To carry out the calculation, R and AB must be measured. The distance from the tracer point to the hinge may be estimated by holding the instrument parallel to a scale or squared paper, and estimating as nearly as possible the position of the axis of the hinge on

the scale. The difficulty is, of course, in estimating the true position of this axis.

The value of R may be obtained by measurement with a screw gauge, which must be used with great care, as the edge of the wheel is easily damaged by screwing up the gauge unduly.

Another and safer way of finding R is to note the drum-reading on the wheel, and move the wheel along a straight line ruled on paper, until the wheel has made several complete revolutions. The distance moved and the number of revolutions enable the value of $2\pi R$ to be measured directly.

The two measurements described above cannot be made with very great precision, but are performed much more accurately in the construction of the instrument. The graduations on the arm BA, '100 □ cm.', etc., are made in the construction, and signify that when this graduation is adjusted to the fixed mark, one revolution of the wheel corresponds to an area of 100 sq. cm. Thus, if the graduations are to be trusted, i.e. unless the instrument has been subjected to rough handling, the area of the figure is equal to $n \times$ (the number on the graduated arm opposite the fixed mark).

The Case when the Needle is Inside the Figure

Now let us consider the case when large areas are to be measured, e.g. $BE_1E_2E_3 \ldots E_{11}$, fig. 8. The needle support is fixed at a central point C, and the tracer may then be made to trace the boundary line of the area.

But in this case it will be seen that $2\pi n R . b$ does *not* give the true value of the area of the figure; for in this case $\Sigma\, d\varphi = 2\pi$ and is not zero.

Take a point B on the boundary, such that BA is at right angles to CW, the line joining C to the centre of the wheel, W, and draw a circle with C as centre and CB as radius, shown in the broken line in fig. 8. Then if B is moved round this circle the wheel W will move round a second circle of radius CW, the relative positions of the two arms remaining constant so that the axle of wheel is always at right angles to the radius CW. Thus, while B is moved round the broken-line circle, W is moving always parallel to its axis around the second circle, so that during the whole revolution the wheel will not rotate. *That is to say, the circle BF_1F_2 is such that when described by the tracer, the wheel indicates zero movement.* This is called the zero circle or datum circle.

If now we consider our area $E_1E_2E_3 \ldots E_{11}$ and begin with the pointer at B passing from B to E_2 via E_1, B is moved outside the circle. W will move towards C, and a definite rotation in one direction is made by the wheel. We could bring the tracer back to B along the path E_2F_1B without altering the reading of W. This reading corresponds, in the way previously considered, to the area E_2FBE_1.

However, having reached E_2, continue along the boundary $E_2E_3E_4$.

To do this the tracer moves inside the zero circle and W will therefore move outside its circle, i.e. in the opposite direction to the previous movement.

Similar movements occur round the figure. The planimeter therefore adds algebraically the area of the curve outside the zero circle. Having carefully noted the direction of rotation of W when B traverses such a part as E_1, we can tell from the final reading of the dial D whether

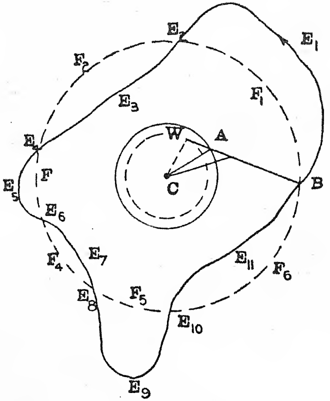


FIG. 8

the figure is of less or greater area of the zero circle. Suppose n revolutions of the wheel are indicated and the arm is set at the '100 □ cm.' mark. If the indication of the n revolutions is in the same direction as the indication of the wheel when moving outwards, e.g. along BE_1E_2, the area of the figure is

$$100\,n + \pi CB^2 \text{ sq. cm. if CB is measured in cm.}$$

The same difficulty as before is met with in finding CB. The instrument may be set along two lines drawn at right angles so that the point of contact of the wheel is at the intersection of the lines, and B and C are each on one line, fig. 9; CB is measured directly or WB and WC are measured and CB calculated. However, the value of this zero circle is inscribed on a second face of the arm BA. Adjacent to the '100 □ cm.' mark is a number which gives a value of the area of the zero circle, not usually in square centimetres, but in revolutions.

Thus, if in the case taken above, there are n revolutions indicated, and the second scale gives m as the equivalent area of the zero circle, the area of the figure is

$$100\ (n + m)\ \text{sq. cm.}$$

To become acquainted with the instrument and familiar with the method of using, draw several small regular figures, calculate the areas and then find them, using the planimeter at different graduated scale settings.

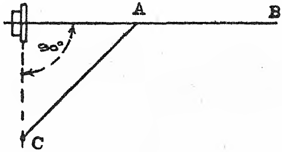


FIG. 9

Draw a circle of known radius, calculate the area. Move the tracer round the circumference when the movable arm is set at various graduations. Note the number of revolutions in each case. From the calculated area and the number of revolutions observed find the area corresponding to one revolution, and so check the graduations. Repeat

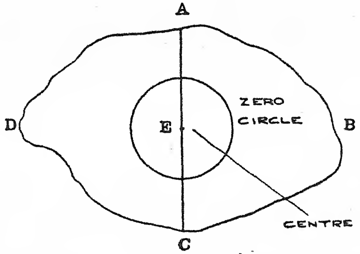


FIG. 10

this process with several measured areas, and obtain a calibration of the instrument.

Measure b and R and again calculate a value for the area corresponding to one revolution. This is not so accurate as the above method.

Draw irregular figures on squared paper and find the areas by adding the squares. Compare these results with those obtained by the calibrated planimeter.

Measure the radius for the zero circle at each setting and check the value of the graduated scale by calculating the area and dividing by $2\pi Rb$. The quotient should agree with the graduations on the scale.

The instrument may subsequently be used for area measurements when required, making use of the calibration if errors are found by these experiments.

Note

When the boundary of the figure does not cut the zero circle the process is identical, for suppose ABCD is such a figure, and the zero circle is completely inside it as shown.

As before, start at any point A, and, with the fixed point inside the figure, trace round the boundary in one direction, say along the path ADC. Join AC. Then, if the tracer be taken along CA the value of the area ADC may be calculated by the foregoing theory. If now the tracer be brought back along AC and thence via B to A, the wheel will indicate precisely the same reading as if the path ADCBA had been taken, for on reversing along AC the record on the wheel for the path CA will be neutralized. The foregoing theory shows that the sum of the readings for the two paths records the number of revolutions corresponding to the two areas and is equal to that for the boundary of the figure. Hence, starting at any point and tracing the complete boundary a record of the number of revolutions n is given which, when added to the zero circle number, enables the area to be evaluated.

The Graduation and Calibration of a Tube

Graduation. The method of graduating a glass capillary tube described below is one which could be employed generally in the etching of scales on glass.

A length of glass capillary tube is coated with a thin layer of paraffin wax by warming the glass and applying a small block of wax to the heated surface. The waxed tube is again gently heated in the Bunsen flame, and rotated so that when cooled it is evenly coated with a thin wax layer.

The tube is then clamped to a board at such a height that the upper surface is approximately on the same plane as a metal scale which is fixed in line with the tube, and at the other end of the board.

A beam compass is arranged so that the two needle-points are from 50 to 100 cm. apart, depending on the length of the glass tube. One needle-point is placed in a centimetre graduation of the scale and the other needle-point is drawn across the wax coating of the tube, removing a straight line of wax.

The beam compass is moved 1 mm., and a short scratch again made on the wax coating. This process is repeated until the required length is marked in this way, the whole centimetre marks and 5 mm. marks being made larger than the rest.

By means of a steel point the centimetre divisions, 0, 1, 2, etc., are scratched on the wax coating.

To etch this scale on the glass, a swab of cotton wool, fastened at the end of a stick, is dipped into hydrofluoric acid, and then applied to the wax-coated tube. Where the scratches of the graduations and numbers have removed the wax, the glass is etched by the acid.

In this process care is taken, of course, to avoid any of the acid touching the skin or clothing.

One end of the tube is also given several scratches, and is covered with acid at the same time as the scale. After ten minutes or so examine one of these scratches near the end of the tube, by scraping away the wax: if the glass is sufficiently etched, the whole process is stopped; if not, leave for a few minutes and then examine a second test scratch near the end, and so on until the etching is complete. Wash off the acid with tap water and remove the wax, the last traces may be removed with turpentine or xylol.

A little rouge, or lamp-black and shellac, rubbed over the etched scale brings out the markings.

Calibration. Linear displacements of a liquid column along a tube are sometimes made use of in the measurement of changes of volume. Thus, in a thermometer a tube is marked at one point 0, at another 100, and equally spaced marks are inserted between the two. An intermediate reading such as 40 should indicate that up to that point the volume is 40/100 of the volume between 0 and 100. Assuming that the linear scale is marked correctly, this will only be the case for all points if the tube is of uniform bore. Calibration of the tube means providing the correction that must be applied to any reading of the scale, in order to give the record that would be obtained if the tube were of uniform bore of cross-section equal to the mean section of the actual tube. This means providing a correction that will give the fraction of the volume of the tube below any point. Thus, if the correction at 40 is $+0{\cdot}5$, the volume between 0 and the mark 40 is $40{\cdot}5/100$ and not 40/100.

The details of the process may be considered by supposing that a thread of mercury of volume v is introduced into the tube and that, by means of end-to-end displacements, it is made to traverse the length of the tube to be calibrated. Unless the tube is uniform, the thread of mercury will not always have the same length. Suppose n displacements are made and that the successive lengths are l_1, l_2, etc., the volume of length $(l_1+l_2+\ldots)$ is nv and, if $\bar{l}$ is the average value of

these lengths, the mean cross-section is $v/\bar{l}$. If the tube were uniform and of this cross-section, the mercury thread would be $\bar{l}$ in every part of it. This means that the length l_1 requires the addition of the correction $(\bar{l}-l_1)$, l_2 that of $(\bar{l}-l_2)$. If l_1 and l_2 are contiguous lengths, the total correction for (l_1+l_2) is $\{2\bar{l}-(l_1+l_2)\}$. Let the deviations from the mean value be denoted by δ_1, δ_2, etc., i.e. $\bar{l}-l_1=\delta_1$, $\bar{l}-l_2=\delta_2$, etc. The corrections to be added to the scale at l_1 and at (l_1+l_2) are δ_1 and $(\delta_1+\delta_2)$ respectively, and the process may be continued.

The inside of the tube is cleaned either by the usual method, using caustic soda, alcohol, ether, and tap water, or by immersing it for about twelve hours in a solution of potassium bichromate and strong sulphuric acid (equal parts). The tube is finally washed well with tap water and dried.

A thread of mercury, about one-third the length of the tube, is drawn into the bore, e.g. 10 cm. for a tube 30 cm. long. By means of a small length of rubber tubing attached to one end of the graduated tube, the position of the thread may be varied at will by altering the pressure within the tube.

The mercury is first adjusted so that one end comes near the zero graduation. By eye estimation the position of the other end may be determined to one-fifth of a subdivision; hence the length of the thread when in this position is obtained. The mercury is then moved, by gently blowing through the rubber tube, until it occupies the central third of the tube, and its length is again estimated in scale divisions. The length taken by the mercury is finally measured in a similar manner in the remaining third of the tube. The volume of mercury is constant, therefore the length indicated will depend on the average cross-section of that part of the tube filled with mercury. Thus this preliminary test will show the general form of the bore.

A short thread of mercury, about 1 cm. long, is next introduced into the tube, replacing the 10 cm. length. This is moved to occupy approximately the space between the 0 and 1 cm. graduation with one edge at 0. The tube is arranged horizontally on a sheet of mirror glass on the platform of a travelling microscope. The readings on the scale engraved on the glass tube, corresponding to the two ends of the thread, are taken by means of this microscope. For this, the cross-hairs in the eyepiece of the microscope are turned so that one is parallel to, and the other at right angles to, the length of the tube. The intersection of the cross-hairs is brought into coincidence with the meniscus, so that one cross-hair appears tangential to it. The vernier reading of the microscope is noted. It is then moved towards the middle of the mercury thread until a graduation on the tube is seen. The difference between the vernier reading under these circumstances and the last readings gives the distance between the end of the thread and the glass tube scale reading. The distance between adjacent scale readings on

the tube is measured in like manner. From these readings the length of the mercury projecting beyond an engraved division on the tube may be calculated in terms of the graduation of the tube, and the length of the thread of mercury may be measured in these units.

The mercury is moved to occupy the space between the 1 and 2 cm. graduations, with one end at 1, and again measured. This is repeated along the length of the tube, and the result entered as in column 2 of the table. The mean value of this length is obtained, and in the third column the difference between the observed and the mean value is tabulated for each part of the scale.

From these observations we may calculate the correction to be applied at each part of the scale to convert the scale readings to the corresponding volume readings. Thus, if a tube is divided into 20 cm. the mean value of the thread length as calculated from column 2 gives the reading for all parts in a uniform tube which is 20 cm. long. The difference between the length of the thread between 0 and 1 as observed, and the mean value gives the correction to be applied to the scale reading to correct it to the true volume of thread equal to $\frac{1}{20}$ of the total volume of the bore.

Position of the thread in the tube	Length of thread	Difference from mean	Correction
0— 1	1·040	—0·044	—0·044
1— 2	1·000	—0·004	—0·048
2— 3	1·000	—0·004	—0·052
3— 4	0·975	+0·021	—0·031
4— 5	0·950	+0·046	+0·015
5— 6	1·000	—0·004	+0·011
6— 7	0·980	+0·016	+0·027
7— 8	1·000	—0·004	+0·023
8— 9	1·000	—0·004	+0·019
9—10	1·000	—0·004	+0·016
10—11	1·000	—0·004	+0·011
11—12	1·020	—0·024	—0·013
12—13	1·000	—0·004	—0·017
13—14	0·980	+0·016	—0·001
14—15	1·010	—0·014	—0·015
15—16	0·975	+0·021	+0·006
16—17	1·000	—0·004	+0·002
17—18	1·000	—0·004	—0·002
18—19	1·000	—0·004	—0·006
19—20	0·990	+0·006	0·000
mean	0·996		

For the bore between 1 and 2 cm. graduations, column 3 again gives the correction to be applied to this part of the tube. To correct the total length from 0 to 2, to give $\frac{2}{20}$ of the volume, the sum of the first two terms in column 3 must be added algebraically. So, to find the correction for any scale reading, the sum of the third column must be taken up to and including the difference term for that reading.

This is done with the figures for a tube tested in this manner and the correction entered in the last column.

With the corrections so obtained a correction curve should be drawn, plotting the correction as ordinate and the scale-reading as abscissa.

It should be noted that the corrections apply to the actual positions of the mercury thread, but if the length of thread is chosen to be near the 1, 2, etc. graduations the corrections may be taken to apply to these points.

In graduating a thermometer the maker fixes a few points by comparison with a standard thermometer and thereby automatically makes some allowance for change in cross-section of the tube.

The procedure for calibration of a thermometer is given on p. 169.

The Balance

The balance, as seen in fig. 11, consists of two pans suspended by knife-edge supports, KK, from the ends of equal arms of the beam ST, which is pivoted on a central pair of knife-edges.

The central knife-edge is made of agate and rests on small plates of the same material and KK support these plates. The free and sensitive movement of the beam depends upon the sharpness of the knife-edges. It is therefore important that they should support weight only when in use. To release the knife-edges the central pillar Q supports an 'arrestment' A. The fixed arm A carries at each end two points which fit into a pair of cups on the upper agate plates at K. When the beam is lowered the weight is taken from the knife-edges by this means.

When releasing the beam, the latter should be in the horizontal position, i.e. the beam is only arrested when two outer knife-edges are opposite the supporting point of the arrestment. In such circumstances all three knife-edges are released with an absence of jolting.

In the ordinary way the centre of gravity of the beam, etc., is just under the line of support. The lower the centre of gravity the less sensitive the instrument. The position of the centre of gravity may be varied by an adjustment of the 'gravity bob' B.

The two masses W may be adjusted to change the rest positions of the pointer which moves over the small scale at the base of the pillar Q.

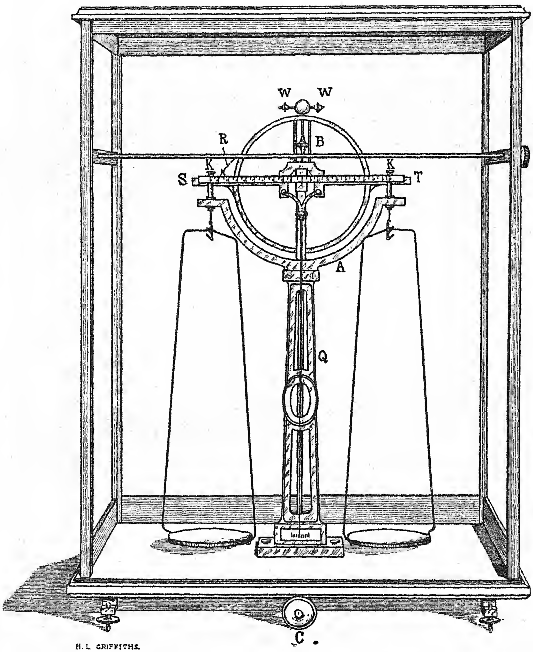


FIG. 11

A fuller account of the balance will be found in *The Theory of the Physical Balance,* by J. Walker.

In using the balance, several simple things should be remembered, viz., before using, dust the pans with a camel-hair or similar soft brush:

arrest the beam before changing masses in the pans; use the rider to start oscillation. Never touch the pans or the weights with the fingers, or place chemicals or wet vessels on the pans! Final observations should be performed with the case closed.

METHODS OF WEIGHING

The Oscillation Method of finding the Rest Position of the Pointer

The arrangement of knife-edge supports ensures that friction is reduced to a minimum. Consequently, when the beam is oscillating it will have a long period of swing which is but slightly damped, i.e. the pointer moving just clear of the scale will take a long time in which to come to its final position of rest.

This rest position may be estimated by the method of oscillation as follows. Suppose the scale be graduated from left to right into, say, 20 graduations. The end of the pointer, being arranged as near to the scale as possible, is viewed by the eye or by the aid of a lens, taking care to avoid any parallax error. The position of the turning point should be estimated to at least $\frac{1}{5}$ of a small division—if possible to $\frac{1}{10}$ of a division. Seven such turning points should be obtained—three on one side and four on the other side of the zero position. The mean value of the three on the one side and also the mean of the other four should be obtained. Then the mean value of the two means gives the position of rest. (See example in the table below.)

For such damped oscillations the mean of an *even* number of observations on each side of the zero would give a value biased towards the side of the zero of the first observed turning point; so an *odd* number is taken as stated. For if we assume that the damping is small and that the first swing through the rest position is to the left, the angular deflection, θ'_L, which occurs after a time, $\frac{T}{4}$, is given by

$$\theta'_L = \theta_0 \, e^{-A\frac{T}{4}},$$

where A is a constant and θ_0 is the undamped amplitude.

Under the conditions of very small damping,

$$\theta'_L = \theta_0 \left(1 - A\frac{T}{4}\right) \text{ to the left.}$$

The deflections are therefore:

To the left of zero	To the right of zero
$\theta_0\left(1 - A\frac{T}{4}\right)$	$\theta_0\left(1 - A\frac{3T}{4}\right)$
$\theta_0\left(1 - A\frac{5T}{4}\right)$	$\theta_0\left(1 - A\frac{7T}{4}\right)$
$\theta_0\left(1 - A\frac{9T}{4}\right)$	$\theta_0\left(1 - A\frac{11T}{4}\right)$
$\theta_0\left(1 - A\frac{13T}{4}\right)$	
mean $\theta_0\left(1 - A\frac{7T}{4}\right)$	$\theta_0\left(1 - A\frac{7T}{4}\right)$

The zero position is thus midway between the average deflections to the left and to the right.

Zero Position of the Unloaded Balance

As an example of the method of oscillation, the following determination of the zero position for the unloaded balance is taken.

The beam is released from the arrestment and given a slight oscillation over about five divisions of the scale on each side of the zero. After a few swings the oscillation should be steady, and seven readings are taken, as shown below.

Reading to the left		Reading to the right	
1.	8·8	2.	12·0
3.	9·2	4.	11·16
5.	9·5	6.	11·2
7.	9·9		

$$\text{Mean of left-hand reading} = 9{\cdot}35$$
$$\text{Mean of right-hand reading} = 11{\cdot}6$$
$$\text{Zero position} = \frac{9{\cdot}35 + 11{\cdot}6}{2} = 10{\cdot}475$$

A mean value of three such determinations gives the rest-point. In general, call this rest position for no load x.

Sensitivity of the Balance

The sensitivity of the balance is defined as 'the angle through which the beam will turn for 1 mg. difference in load in the two pans'

It is usual in practice to measure the sensitivity in terms of the movement of the pointer over the scale.

This should be measured for *no load*, by adding 1 mg., by means of the rider, to one pan and noting the position of rest by the oscillation method as before.

Load each pan with increasing equal masses up to the limits specified for the balance, and for each mass find the sensitivity.

If the beam were rigid and the knife-edges truly in the same line the sensitivity-load curve would be a straight line parallel to the load axis.

For increasing loads there is, however, a slight depression of the knife-edges at the pan supports, and a change in sensitivity as a result.

Find the sensitivity-load curve for the balance and note the region, if any, of maximum sensitivity. The value of the load at which there is maximum sensitivity depends upon the use for which the balance is designed.

Method of Gauss or Double Weighing

As an example of this method we will consider a determination of the number of grammes equivalent to one ounce troy; the process can naturally be repeated with any other unknown mass. Place the ounce troy in the left-hand pan and add weights (grammes) to the right-hand pan until the pointer remains on the scale when the beam is released; the rest position of the oscillating pointer is estimated in the manner previously used. Let this be y on the scale.

Now find the sensitivity of the balance at this load by adding 1 mg. and proceeding as before. Suppose the sensitivity for this load is s.

The mass of the ounce troy in grammes is the mass (to the nearest centigram) in the scale pan $+ \left(\frac{y - x}{s}\right)$ mg. $= M_1$, say, where x is the zero reading for no load.

The ounce troy is then transferred to the right-hand pan and the process repeated by adding weights to the left until, to the nearest milligram, M_2, is obtained.

Suppose that the two arms of the balance are of slightly different length, the left-hand arm being a cm. and the right b cm., then W being the true mass of the ounce troy, neglecting buoyancy,

$$aW = M_1 b,$$
$$aM_2 = Wb.$$

Hence
$$W = \sqrt{M_1 M_2}.$$

The difference between M_1, M_2 and W will be very small, so that we may take as an approximation

$$W = \tfrac{1}{2}(M_1 + M_2).$$

Borda's Method or the Method of Substitution

A second method, quite as accurate as the double weighing, is a simple method of substitution. It eliminates equally well the errors due to unequal length of the arms, etc.

The ounce troy (the unknown mass) is placed on a scale pan and lead shot is used to counterpoise it. The position of rest when the counterpoise is complete is noted by the method of oscillation.

The ounce troy is now removed and replaced by standard masses until balance is again obtained. From the sensitivity of the balance for this load we may estimate to a milligram the mass which has exactly substituted the ounce troy. Thus the mass is obtained, avoiding errors due to faulty construction of the balance, etc.

Buoyancy Correction

When discussing methods of weighing no account was taken of the buoyancy of the air on the weights and the mass to be compared with them.

Suppose that W is the mass of the body determined by one of the methods above, the true mass allowing for buoyancy being denoted by M.

Suppose that in the determination of W we used copies of standard masses made of a substance of density D.

Let ρ be the density of the unknown mass and σ the density of the air.

We have really compared the true mass of the body M—the buoyancy on the mass, with the mass W of the weight—the buoyancy on the weights. These two quantities are equal,

i.e. $$\left(M - \frac{M}{\rho}\sigma\right) = W - \frac{W}{D}\sigma,$$

i.e. $$M = \frac{W\left(1 - \frac{\sigma}{D}\right)}{1 - \frac{\sigma}{\rho}}.$$

$$= W\left(1 + \frac{\sigma}{\rho} - \frac{\sigma}{D} \cdots\right),$$

neglecting $\frac{\sigma^2}{\rho D}$ $$M = W + W\left(\frac{1}{\rho} - \frac{1}{D}\right)\sigma.$$

The observed value of W has therefore to be corrected by the factor

$$W\left(\frac{1}{\rho} - \frac{1}{D}\right)\sigma.$$

This correction depends on the density of the 'weights' and the substance and the density of the air.

For most purposes the density of the air may be taken as 0·0012 gm. per c.c. and for general use a table may be calculated giving the value of the correcting factor $\left(\frac{1}{\rho} - \frac{1}{D}\right)\sigma$ for the two common materials used in the manufacture of weights, brass and aluminium, taking the density of brass = 8·40 gm. per c.c., aluminium = 2·65 gm. per c.c. Thus:

Density of substance weighed (ρ)	Correction for buoyancy	
	Brass 'weights' D = 8·4 $\sigma\left(\frac{1}{\rho} - \frac{1}{8 \cdot 4}\right)$	Aluminium 'weights' D = 2·65 $\sigma\left(\frac{1}{\rho} - \frac{1}{2 \cdot 65}\right)$
0·5	0·00226	0·00195
0·55	etc.	etc.
0·60	etc.	etc.
etc.		

CHAPTER III

MOMENTS OF INERTIA AND DETERMINATION OF 'g'

Kinetic Energy of a Body Rotating about an Axis

LET ABC (fig. 12) be a section of a body by a plane at right angles to the axis about which it is rotating, O being the point of intersection of this plane and the axis.

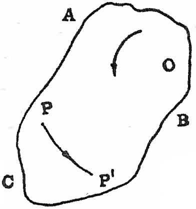


FIG. 12

If we imagine the body to be subdivided into a large number of very small particles of mass m_1, m_2, m_3, etc., distant r_1, r_2, r_3 cm. from the axis, it is evident that when the body rotates each particle will move with a velocity which depends on the distance r from the axis.

Consider one such particle at P, of mass m and distant r cm. from O. If the body rotates with a uniform angular velocity w, in the direction of the arrow, and if v is the velocity of P in the path, we have

$$\frac{v}{r} = w.$$

The kinetic energy of this particle is $\frac{1}{2}mv^2 = \frac{1}{2}mw^2r^2$. For all such particles the total kinetic energy of the body is therefore

$$\tfrac{1}{2}m_1w^2r^2 + \tfrac{1}{2}m_2w^2r_2{}^2 + \tfrac{1}{2}m_3w^2r_3{}^2 + \ldots$$

i.e.

$$\begin{aligned}\text{kinetic energy} &= \Sigma\tfrac{1}{2}w^2mr^2\\ &= \tfrac{1}{2}w^2\Sigma mr^2.\end{aligned}$$

The sum of such quantities as mr^2, taking every particle throughout the body, is defined as the *moment of inertia* of the body *about the axis through* O. If we denote this by I_0, then the kinetic energy of the body is

$$\tfrac{1}{2}I_0w^2, \qquad \ldots(1)$$

thus I_0 replaces the mass and the angular velocity replaces linear velocity in the corresponding case for linear motion where K.E. $= \frac{1}{2}mv^2$.

In the same way, if the axis passes through the centre of gravity, I_G being the moment of inertia about an axis through the centre of gravity, K.E. $= \frac{1}{2}I_G w^2$, I_G being of a different magnitude from I_0.

To express the moment of inertia of a body about an axis in terms of the moment of inertia about a parallel axis passing through the centre of gravity, we proceed in the following manner.

In fig. 13 let ABC be a section of the body at right angles to either axis. Let G be a section of the axis passing through the centre of gravity and O be the corresponding point of intersection for a parallel axis, the distance OG being fixed and equal to a cm.

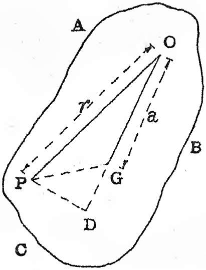


FIG. 13

Consider at any point P a small particle of mass m gm., OP being r cm.

The contribution of this particle to I_0 is mr^2. Now produce OG to D, and from P drop a line perpendicular to OD meeting it at D.

$$r^2 = OP^2 = OD^2 + PD^2 = PD^2 + DG^2 + GO^2 + 2OG \,.\, GD$$
$$= PG^2 + a^2 + 2a \,.\, GD.$$

Thus
$$I_0 = \Sigma m\,(PG^2 + a^2 + 2a \,.\, GD)$$
$$= \Sigma mPG^2 + \Sigma ma^2 + \Sigma 2ma \,.\, GD.$$

Now $\Sigma mPG^2 = I_G$, the moment of inertia about a parallel axis through the centre of gravity.

$$\Sigma ma^2 = a^2\Sigma m = a^2M,$$

where M is the total mass of the body. Further, the expression $2a\Sigma GD \,.\, m = 0$, for by the definition of the centre of gravity the sum of the moments (GD . m) throughout the body, about an axis through G, is zero.

Thus
$$I_0 = I_G + Ma^2. \qquad \ldots(2)$$

Radius of Gyration

We have defined the moment of inertia of a body about an axis as Σmr^2. Now if the whole of the mass of the body were concentrated at one point distant k from the axis, we should have

$$I = k^2M.$$

If the distance k is so chosen that

$$k^2M = \Sigma mr^2 = I,$$

it is called the 'radius of gyration' of the body about the axis of rotation taken.

As with the moment of inertia k has different values depending upon the axis chosen.

Moment of External Forces

Considering a rotating body as before, we may readily deduce an expression for the moment of the external forces applied to the body to impart a definite angular acceleration.

Imagine a force applied to such a body as shown in fig. 12. Suppose the body to be subdivided into small particles as before, of which one at P has mass m and is r cm. from O. Then v, the velocity of the particles, is given by

$$v = r\frac{d\theta}{dt} \quad \text{or} \quad r\dot{\theta};$$

the acceleration of the particle in its path is

$$\frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r\ddot{\theta},$$

this is occasioned by a force $mr\ddot{\theta}$, whose moment about O is $mr^2\ddot{\theta}$. For the whole body to rotate with this angular acceleration the total external couple applied is thus equal to

$$\Sigma mr^2\ddot{\theta} = \ddot{\theta}\Sigma mr^2 = I\ddot{\theta}. \qquad \ldots(3)$$

Thus the moment of external forces applied to the body is equal to $I\ddot{\theta}$.

Calculation of the Moment of Inertia for a Solid about any Axis

The numerical value of the moment of inertia of a solid about any axis may be readily obtained by integration. Having calculated this value for an axis passing through the centre of gravity, the corresponding value for the case of the body suspended through a parallel axis may be obtained by adding the term Ma^2, as shown on p. 36.

We will consider an example of such calculations which is often employed, especially in magnetism, and which illustrates the points already considered.

Calculation of the Moment of Inertia of a Rectangular Rod about an Axis at Right Angles to its Length and passing through the Centre of Gravity

Let ABCD be the rectangular bar, with centre of gravity at G, and supported through the axis KK′ passing through G, normal to the face AD, fig. 14. Let

M be the mass of bar (assumed to be uniform),
ρ the density of the material of the bar,
$2l$ the length of the bar,
$2b$ the breadth of the bar,
$2d$ the depth of the bar.

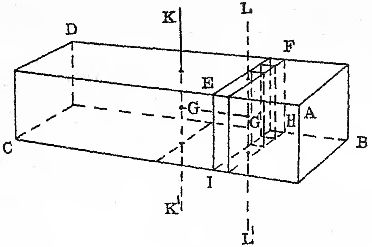


FIG. 14

Let G be the origin of the system of co-ordinates, the z axis coinciding with the axis of rotation, the x and y axes being at right angles to this; the x axis being parallel to the length, the y parallel to the breadth.

The most convenient method of finding I_G the moment of inertia about KK′ is to consider firstly a very thin section of the bar cut at right angles to the x axis, and of thickness δx. Through the centre of mass G′ of this section imagine an axis, LL′, parallel to KK′. The moment of inertia of this section about KK′ is equal to the moment of inertia about LL′ plus the product of the mass of the section and x^2, where x is the distance between G and G′. Imagine a very small rectangular portion PQRS of EFHI, y cm. from the axis and of width dy (fig. 15).

The mass of this parallelepiped is $[dx\, dy\, 2d]\,\rho$. The moment of inertia of the section EFHI about LL′ is therefore

$$\int_{-b}^{+b} 2d\, dx\, dy\, \rho y^2 = \frac{4b^3 d\rho}{3}\,.\, dx.$$

About KK′ the moment of inertia is therefore

$$\frac{4db^3\rho dx}{3} + 2d2b\rho x^2dx.$$

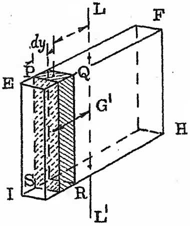


FIG. 15

Hence the total moment of inertia of the whole body about the axis KK′ is

$$\int_{-l}^{+l}\left(\frac{4b^3d}{3}\rho dx + 4bdx^2\rho dx\right)$$
$$= \frac{8bdl\rho}{3}(b^2 + l^2)$$
$$= \frac{M}{3}(b^2 + l^2).$$

EXPERIMENTAL MEASUREMENT OF MOMENT OF INERTIA

When the moment of inertia of a body cannot be conveniently calculated it may be found experimentally by imparting to it a known amount of energy and observing the resulting rotation, or, if the body is small, the method of the moment of inertia table may be employed, whereby the change in moment of inertia in a given system, due to the body, may be directly calculated.

The following typical experiments will make these methods evident.

Moment of Inertia of a Fly-wheel

To find experimentally the moment of inertia of a fly-wheel about the fixed axis of rotation, a mass is attached to the axle of the fly-wheel by a cord which is wrapped several times round the axle. When the mass descends, it causes a rotation of the wheel. The mass in its descent loses a definite amount of potential energy. Neglecting friction for the moment, this loss is equated to the gain of kinetic energy of the mass and the fly-wheel, and an equation results in which all the

terms are known or measurable except I, the moment of inertia of the wheel and axle about the fixed axis.

The wheel may be supported on a horizontal or vertical axis. The usual types met with are seen in fig. 16. The method of finding I is the same, so we will consider one of them—the vertical axis type.

The mass m is attached to the axle at a point where there is either a hole or a pin. If there is a small hole in the axle as at P, then to the end of the cord a small 'pin' is attached. This can be made from a short length of suitable-sized brass wire to fit easily in the hole, or if the axle has a pin projecting, a loop is made at one end of the string. The length of the string is so adjusted that when m is on the floor, or whatever solid object is to arrest it in its descent, the other end of the string may be just attached to the axle. So that when the mass descends, the moment it is arrested the string leaves the axle.

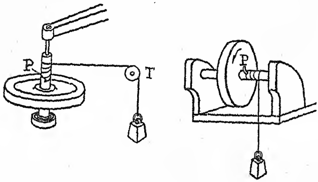


FIG. 16

If w be the angular velocity imparted to the wheel, and r be the radius of the axle, the velocity of the mass m just before striking the floor is rw. So that, neglecting friction, we have

$$mgh = \tfrac{1}{2}mr^2w^2 + \tfrac{1}{2}\mathrm{I}w^2, \qquad \ldots(4)$$

where h is the distance through which m has fallen.

To measure the angular velocity, a chalk-mark is made on the circumference of the wheel, in a position which can be seen the moment the mass touches the floor. The number of revolutions, n, made by the wheel after the mass becomes detached is counted by observing the chalk-mark. The time taken for the wheel to come to rest whilst completing the n revolutions is also observed (t sec.).

The wheel is finally brought to rest by the frictional forces acting against it. If this frictional force is constant, the wheel is uniformly retarded. It begins with a definite angular velocity and ends with zero angular velocity, so that the initial velocity is double the average velocity.

Now the average angular velocity $= \dfrac{2\pi n}{t}$ radians per second,

i.e. $$w = \frac{4\pi n}{t}.$$

The linear velocity of the mass m is $\frac{4n\pi r}{t}$.

Another method, though inferior to the above, is to observe v directly by timing the descent of the mass. If the mass descends the distance h in t' secs., the average velocity is $\frac{h}{t'}$, and the final velocity is $\frac{2h}{t'}$.

As we have noticed above, the frictional forces are not always negligible, so that, for a more accurate determination of I, allowance must be made for the energy lost in overcoming friction.

Let there be n_1 revolutions of the wheel during the descent of the mass, and let f ergs be the energy per revolution used in overcoming the frictional forces, then the total energy expended in this way is $n_1 f$.

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\mathrm{I}w^2 + n_1 f.$$

We already know that the energy possessed by the rotating wheel, $\frac{1}{2}\mathrm{I}w^2$, is used up in overcoming friction in n revolutions.

Thus $$fn = \tfrac{1}{2}\mathrm{I}w^2,$$

$$f = \frac{1}{n}(\tfrac{1}{2}\mathrm{I}w^2),$$

and $$mgh = \tfrac{1}{2}mr^2w^2 + \tfrac{1}{2}\mathrm{I}w^2\left(1 + \frac{n_1}{n}\right). \qquad \ldots(5)$$

Experimental Details

Arrange the cord round the axle so that throughout the whole of the unwinding, the cord from the axle to the pulley, T, is practically horizontal, or at right angles to the axle.

Wind up the mass m so that its base is level with a fixed point, and the string of such a length that it fulfils the conditions already stated.

The distance h from the fixed point to the floor is directly measured. The number of revolutions the wheel makes whilst the mass is descending may be determined by making a chalk-mark on the axle and allowing the mass to descend slowly, counting the number of revolutions (n_1) during the descent.

The mass is once more wound up and allowed to fall freely. When it is heard to strike the floor a stop-clock is started and the number of revolutions of the wheel before being brought finally to rest is counted, i.e. n and t are observed: m the mass is known and $w = \frac{4\pi n}{t}$, hence the value of I is calculated by the aid of equation (5).

The experiment is repeated two or three times with the same mass and the mean value of $\frac{n}{t}$ taken.

I is further checked by repeating with two other masses ...

The cord used should be of small diameter compared with the diameter of the axle, otherwise the value of r in equation (5) is the sum of the radii of the axle and cord.

Rolling Bodies

The two following experimental methods of finding the moment of inertia of a body about a given axis depend upon observations of rolling bodies.

The energy of a rolling body may be very simply obtained. Consider for example, a cylinder rolling with a uniform linear velocity v cm. per sec. (fig. 17).

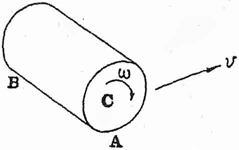


FIG. 17

AB, the line of contact of the cylinder and the plane on which it rolls, is the momentary axis of rotation of the cylinder.

The kinetic energy of the body is therefore given by $\frac{1}{2}w^2$ (moment of inertia of the cylinder about the axis AB) where w is the angular velocity of rotation.

Now, if a is the radius of the cylinder, m its mass, and I the moment of inertia about a parallel axis through the centre of gravity, the kinetic energy is

$$\begin{aligned} &\tfrac{1}{2}w^2\,(\mathrm{I} + ma^2) \\ &= \tfrac{1}{2}\mathrm{I}w^2 + \tfrac{1}{2}mv^2. \qquad \ldots(6) \end{aligned}$$

That is, the kinetic energy is equal to the sum of the kinetic energies of rotation and translation.

Wheel and Axle on an Inclined Plane

The moment of inertia of a wheel and axle about an axis passing through the centre of gravity and parallel to the axle may be obtained by observing its descent down an inclined plane, and applying equation (6).

In fig. 18, R and R' are rails supported on a hollowed inclined plane. The axle of the wheel rests upon the rails. The whole plane may be inclined at any angle to the horizontal. For each inclination the wheel and axle is allowed to roll down a measured length, l cm., of the plane, in a time which is measured by means of a stop-clock (t sec.).

If the vertical distance between the original position and the final position be h cm., fig. 19, and v be the final velocity acquired in the descent, we have, equating potential energy lost to kinetic energy gained,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\mathrm{I}\left(\frac{v}{a}\right)^2, \qquad \ldots(7)$$

where m is the mass of the wheel and axle, and a the radius of the axle.

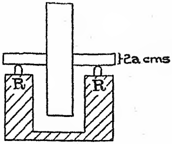


FIG. 18

The body starts from rest and moves with a constant acceleration; the final velocity is therefore twice the average velocity.

Thus $$v = \frac{2l}{t}.$$

The plane is adjusted by suitable means to one fixed inclination. The wheel and axle is placed at a convenient marked starting-point on the rails and the position of the centre of the axle is noted by means of a cathetometer. The position of the centre of the axle is also noted

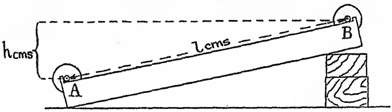


FIG. 19

when the wheel is against the stop at the other end of the plane. The length l along the plane between these two points is measured directly. The value of the mean time of descent for at least three experiments is obtained. The mass m is also obtained by means of a spring balance or an ordinary balance which is capable of weighing such a mass.

a is obtained in the usual way by means of vernier callipers.

Hence, from (7), substituting the value $\frac{2l}{t}$ for v,

$$\mathrm{I} = ma^2\left(\frac{ght^2}{2l^2} - 1\right).$$

The experiment is repeated for several values of h and the mean value of I is obtained.

Moment of Inertia of a Disk Supported on Strings

A disk, usually made of wood, is suspended by means of a metal axle on two strings, as shown in fig. 20. The string is wound evenly on the axle AB on both sides until as much string as possible is wound up. If now the axle and disk, of mass m grammes, is released, it will descend until the whole of the cord is unwound; it will then rise again, the string being wound on the axle in the other direction.

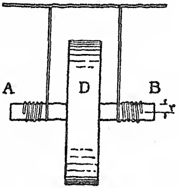


FIG. 20

Suppose that from the starting-point to the lowest point reached, the distance the centre of the axle moves is h cm., and that the linear velocity at the moment when all the string is just unwound is v cm. per sec., then, if r is the radius of the axle, we have as the energy equation (equating potential energy lost to kinetic energy gained)

$$mgh = \tfrac{1}{2}\mathrm{I}w^2 + \tfrac{1}{2}mv^2,$$

I being the moment of inertia about an axis passing through the centre of gravity and parallel to the axle. Since w the angular velocity at the lowest point is $\frac{v}{r}$, we have

$$mgh = \frac{\tfrac{1}{2}\mathrm{I}v^2}{r^2} + \tfrac{1}{2}mv^2.$$

$$\mathrm{I} = (mgh - \tfrac{1}{2}mv^2)\,\frac{2r^2}{v^2}.$$

If the time of descent of the disk is t seconds, the average velocity is $\frac{h}{t}$ cm. per sec.

v, the final velocity, is therefore $\frac{2h}{t}$.

$$\mathrm{I} = mr^2\left(\frac{gt^2}{2h} - 1\right).$$

Experimental Details

Weigh the disk, then measure the distance between the position of the centre of the axle in the starting position and the final lower position.

The value of r is equal to the sum of the radii of the axle and the cord which supports it. These radii are measured by means of a micrometer screw.

The cord is wound evenly on the axle until the disk is at the starting-point. Care is taken to ensure that the axle is horizontal, otherwise the disk fouls the cords in descent.

The time of descent is measured several times by means of a stop-clock, and the mean value taken.

The Bifilar Suspension

In order to determine the moment of inertia of a body about an axis passing through its centre of gravity, we may make use of a bifilar suspension. The body is suspended with the axis of rotation vertical, and the time of vibration of the system, T, obtained by observing the time of 40 or 50 complete swings.

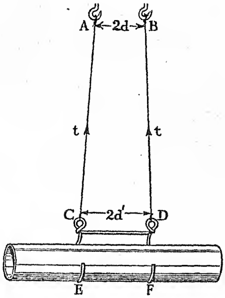


FIG. 21

If, for example, we wish to determine the moment of inertia of a cylinder about an axis through the centre of gravity, the cylinder is supported in two wire stirrups CE and DF (fig. 21), which hang at the ends of two very thin wires AC and BD, which are fixed at A and B, AB being $2d$ cm. The distance between C and D remains fixed and equal to $2d'$ cm.

When the body is displaced slightly in the horizontal plane it will perform oscillations of periodic time T.

If m is the mass of the cylinder and l is the length of the wire AC or BD, we may readily see that

$$T = 2\pi\sqrt{\frac{Il}{mgdd'}}.$$

For, let the tension in the string be t dynes and let O be the mid-point between A and B.

When viewed from above, fig. 22 (a) represents the relative positions of the four ends of the wires. When displaced, the state of affairs is seen in fig. 22 (*b*). Where A′ and B′ are projections of A, B, fig. 21, on the horizontal plane through CD, and C′, D′ are the displaced positions of C, D.

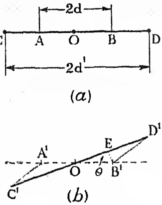


Fig. 22

Consider the forces at D. Due to the tension on the string there is a force t, which has a horizontal value $t \cos \alpha$, where α is the angle between the string BD′ and the horizontal. Now $\cos \alpha = \frac{B'D'}{l}$, for the point of suspension, B, is above B′, BB′D′ being a right-angled triangle.

So that along D′B′ in the horizontal plane there is a force

$$f = t\frac{B'D'}{l}. \qquad \ldots(8)$$

Of this force the component at right angles to the displaced body, i.e. to C′D′, is effective in restoring the body to its original position, the component along the direction OC having no turning moment.

From B′ draw B′E at right angles to C′D′. The component normal to C′D′ is $f \sin ED'B' = f\frac{EB'}{B'D'} = f'$, say;

from (8) $$f' = EB'\frac{t}{l}. \qquad \ldots(9)$$

A similar force acts as C′, constituting a restoring couple of moment $f' \cdot C'D'$.

The restoring couple is, substituting value of f' from (9),

$$\frac{\mathrm{EB}'t}{l} \cdot \mathrm{C'D'}.$$

If the small angle of displacement, $\mathrm{D'OB'} = \theta$,

$$\mathrm{EB'} = \mathrm{OB} \sin \theta = d \sin \theta,$$
$$\mathrm{C'D'} = 2d'.$$

$$\therefore \text{ Moment of the couple } = \frac{2dd'}{l} t \sin \theta.$$

θ is usually made very small, so we have for the value of the restoring couple:

$$\frac{2dd'}{l}\frac{mg}{2} \cdot \theta = \frac{dd'mg}{l} \cdot \theta \qquad \ldots(10)$$

for *when d is not very different from d'*, $t \simeq \frac{mg}{2}$.

We have already seen (p. 37) that the moment of external forces acting on a suspended system is $\mathrm{I}\frac{d^2\theta}{dt^2}$.

Since the couple acts in a direction tending to diminish the angular velocity, we have

$$\mathrm{I}\frac{d^2\theta}{dt^2} = -\frac{mgdd'}{l} \cdot \theta.$$

An equation of this form occurs repeatedly in experiments in physics. It is of the form

$$\frac{d^2\theta}{dt^2} + n^2\theta = 0,$$

where n^2 is placed equal to the ratio of the constant multipliers of θ and $\frac{d^2\theta}{dt^2}$.

This is the equation of simple harmonic motion and the solution is

$$\theta = \mathrm{A} \sin nt + \mathrm{B} \cos nt,$$

as can be seen by substitution.

Another way of writing this is

$$\theta = \mathrm{C} \sin (nt + \alpha),$$

the two constants being replaced by C and α, where

$$\mathrm{C}^2 = \mathrm{A}^2 + \mathrm{B}^2 \quad \text{and} \quad \tan \alpha = \mathrm{B}/\mathrm{A}.$$

The curve showing the relation between θ and t is illustrated in fig. 23. In the figure the intercept $\mathrm{OA} = \mathrm{C} \sin \alpha$ and $\mathrm{OB} = -\alpha/n$. θ regains the same value at intervals π/n, e.g. at C, D, and E. The complete cycle of values occurs in the range CE, i.e. $2\pi/n$, and this is the period, T, of the motion

$$\mathrm{T} = \frac{2\pi}{n}.$$

If this be applied to the equation of the problem now considered it appears that the period is

$$T = 2\pi\sqrt{\frac{Il}{mgdd'}}.$$

It is interesting to note that for bodies of the same dimensions and of uniform density the value of T is the same. For, let ρ be the uniform

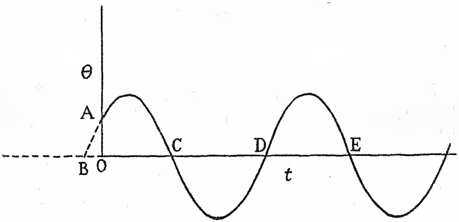


FIG. 23

density, then $I = \Sigma mr^2$. Consider the body divided into small volumes v, then $\Sigma mr^2 = \Sigma v\rho r^2 = \rho\Sigma vr^2$.

But $$m = \Sigma\rho v = \rho\Sigma v,$$

thus $T = 2\pi\sqrt{\frac{l}{gdd'}\cdot\frac{\Sigma vr^2}{\Sigma v}}$, i.e. independent of the density.

The above experiment may be carried out using a metal and a wooden cylinder as the supported body. If the dimensions of the wood and metal cylinder are identical, the time T will be found to be the same. Suspend each in turn and find T (by timing 50 swings) and calculate I for each cylinder.

For such regular solids, the value of I should also be obtained by calculation.

For a cylinder, about the axis taken, $I = M\left(\frac{L^2}{3} + \frac{r^2}{4}\right)$, where 2L is the length of the cylinder, r the radius.

The formula may be further tested by varying d, d', and l. It will be found that T^2 is proportional to $\frac{l}{dd'}$.

Moment of Inertia Table

Fig. 24 shows the essential features of the moment of inertia table. AB is the table suspended by a fairly stout wire, K, from the overhead frame. The circular table supports three or more masses which just fit into a groove, concentric with the circumference. When the wire

support is vertically above the centre of gravity the masses, W, may be moved round the groove into any position without altering the moment of inertia of the whole. The masses are arranged so that the table lies horizontally. In that case the axis is through the centre of gravity and is normal to the surface of the table. If now the table is given a slight twist, a restoring couple is called into play in the wire, equal to, say, τ per unit angular displacement, and the table oscillates about the axis of support, with periodic time

$$T = 2\pi\sqrt{\frac{I}{\tau}},$$

where I is the moment of inertia of the system about the axis of rotation; for suppose the table be twisted through an angle θ, the

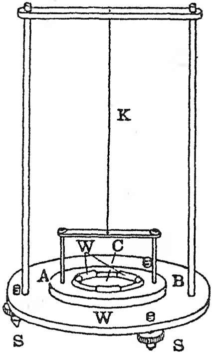


FIG. 24

restoring couple due to torsion is $\tau\theta$. We have already seen, p. 37, that the moment of external forces on such a rotating body is $I\ddot{\theta}$, where $\ddot{\theta}$ is $\frac{d^2\theta}{dt^2}$. This couple is opposed by $\tau\theta$: $\tau\theta$ and $I\ddot{\theta}$ being equal and opposite; i.e.

$$I\ddot{\theta} = -\tau\theta$$

or

$$\ddot{\theta} = -\frac{\tau\theta}{I}.$$

Thus the motion is simple harmonic with a periodic time, T, given by

$$T = 2\pi\sqrt{\frac{I}{\tau}}.$$

If now a regularly shaped body is placed symmetrically on the table at C, so that its centre of gravity is vertically above the centre of gravity of the table and therefore in the previous axis of rotation, the time of oscillation for the loaded table is

$$T_1 = 2\pi\sqrt{\frac{(I+K)}{\tau}},$$

where K is the moment of inertia of the regular body about the axis of oscillation: K may be calculated directly from the mass and dimensions of the body.

We have, therefore, from the two equations above:

$$T_1^2 = \frac{4\pi^2(I+K)}{\tau}; \quad T^2 = \frac{4\pi^2 I}{\tau}.$$

Hence

$$\frac{T_1^2}{T^2} = \frac{I+K}{I} = 1 + \frac{K}{I},$$

or

$$I = K\frac{T^2}{T_1^2 - T^2}. \qquad \ldots(12)$$

If, then, a body of unknown moment of inertia about a given axis is placed centrally on the table with the centre of gravity in the axis of oscillation, we may find the value of I_1, its moment of inertia about this vertical axis passing through the centre of gravity, by timing the oscillations of the table when loaded by the body. If T_2 is the time of complete swing,

$$T_2 = 2\pi\sqrt{\frac{I+I_1}{\tau}},$$

and we have

$$T = 2\pi\sqrt{\frac{I}{\tau}}.$$

Thus, from these two equations

$$I_1 = \frac{T_2^2 - T^2}{T^2}\,.\,I. \qquad \ldots(13)$$

Substituting, from equation (12) above,

$$I_1 = K\,.\,\frac{T_2^2 - T^2}{T_1^2 - T^2}. \qquad \ldots(14)$$

Experimental Details

The time T for a complete swing of the table is obtained by timing as many swings as possible, or by the method of p. 101.

To find the value of I, the moment of inertia of the unloaded table about the wire as axis, a regular solid, such as a plain cylindrical 'weight', is employed. The mass should be fairly heavy so as to cause as big an alteration in T as possible. A two-kilogram 'weight' is of the right order to use with the apparatus described.

To ensure that this standardizing mass is arranged with its centre of gravity over that of the table, the lead weights W should be adjusted so that the table swings horizontally. Any alteration in the position of the masses, W, will not alter I so long as the table is horizontal and the axis of oscillation is vertically through the centre of gravity.

Having obtained T and T_1, using a cylindrical regular 'weight', the value of K should be calculated. For such a flat cylindrical object, $K = \frac{Ma^2}{2}$, where a is the radius and M the mass of the 'weight'. I may be calculated using this value of K in equation (12) and the table may then be used to find the moment of inertia of other solid bodies.

The following results were obtained in the above manner:

Table unloaded	Table loaded with 2000 gm.	Table loaded with unknown body
9·375 sec.	9·7 sec.	15·5 sec.

radius of 2000 gm. 'weight' = 6·75 cm.

$$K = \frac{2000 \times 6{\cdot}75^2}{2} = 45{\cdot}562 \text{ gm. cm.}^2$$

$$I_1 = 45{\cdot}562 \times \frac{15{\cdot}5^2 - 9{\cdot}375^2}{9{\cdot}7^2 - 9{\cdot}375^2}$$

$$= 1{\cdot}11 \times 10^6 \text{ gm. cm.}^2.$$

I_1, by approximate calculation, assuming a regular shape of the body $= 9{\cdot}86 \times 10^5$ gm. cm.^{2}

The Compound Pendulum

Let ABC, fig. 25, be a section passing through the centre of gravity of a body, and at right angles to an axis about which it may turn, the point O being the intersection of the axis with this plane section.

The body is at rest when the centre of gravity, G, is vertically under O.

If the body is given a *small* displacement so that GO makes a small angle θ with the vertical, then, m being the mass of the solid, the restoring force has a moment mg OG $\sin\theta = mga \sin\theta$ (where OG $= a$).

We saw (p. 37) that in such a case the moment of the forces is equal to

$$I_0 \frac{d^2\theta}{dt^2},$$

i.e. $I_0 \frac{d^2\theta}{dt^2} = -mga \sin\theta = -mga\theta$ for small angular displacements.

This represents a simple harmonic motion (p. 47), whose periodic time $T = 2\pi\sqrt{\frac{I_0}{mga}}$.

$I_0 = I + ma^2$, where I is the moment of inertia about a parallel axis through centre of gravity and is equal to mk^2, where k is the radius of gyration about this axis.

Thus $$T = 2\pi\sqrt{\frac{k^2m + a^2m}{mag}} = 2\pi\sqrt{\frac{\frac{k^2}{a} + a}{g}}. \qquad \ldots(15)$$

This result is similar to that obtained for a simple pendulum, in fact a simple pendulum of length $l = \frac{k^2}{a} + a$ would have the same periodic time, T. Such a simple pendulum is called the *equivalent simple pendulum.*

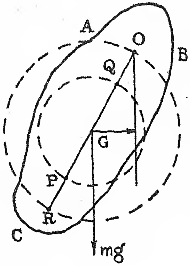


FIG. 25

In the case taken, if all the mass of the body were concentrated at a point P, along OG produced such that $OP = \frac{k^2}{a} + a$, we should have a simple pendulum with the same periodic time.

The point, P, is called the *centre of oscillation*, O being called the *centre of suspension.*

Now since

$$l = \frac{k^2}{a} + a$$

or $$a^2 - al + k^2 = 0,$$

the length a is not the only value for OG, which has l as the length of the equivalent simple pendulum, for the above equation has two roots, α_1 and α_2, such that,

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= l \\ \alpha_1\alpha_2 &= k^2 \end{aligned}\right\}. \qquad \ldots(16)$$

Since a is one value, α_1 say,

we have $$a + \alpha_2 = l \quad \text{or} \quad \alpha_2 = l - a = \frac{k^2}{a}.$$

Thus, if the body were supported on a parallel axis through the former centre of oscillation, P, it would oscillate with the same time T as when supported at O.

From what has been seen above it is evident that there are an infinite number of points distant a and $\frac{k^2}{a}$ from G, for any point on a circle drawn from G as centre and radius a or $\frac{k^2}{a}$ will satisfy the condition given; so that any axis parallel to the normal at G and on the curved surface of two cylinders, of which the dotted circles are sections, will be axes of suspension, about which the body will have the same periodic time.

If the body were supported by an axis through G, the time of oscillation would be infinite. From any other axis in the body the time is
$$T = 2\pi\sqrt{\frac{a^2 + k^2}{ag}}.$$

This has a minimum value when $\frac{a^2 + k^2}{a}$ is minimum.

Now $$\frac{a^2 + k^2}{a} = \frac{a^2 - 2ak + k^2 + 2ak}{a} = \frac{(a - k)^2 + 2ak}{a}.$$

This is a minimum when $a = k$.

The corresponding minimum
$$T_1 = 2\pi\sqrt{\frac{2k}{g}} \qquad \ldots(17)$$

and will occur for a series of axes parallel to that through G, and on the surface of a cylinder whose axis is the axis through the centre of gravity and of radius, k.

An experiment which brings out these facts may be performed by using as the body a rectangular rod of brass about 1 m. long. This may be suspended on a knife-edge at various points along its length. To facilitate the suspension it is convenient to have a series of holes drilled along the bar at about 2 cm. intervals (fig. 26).

Level the knife-edge, and suspend the bar at, say, every other hole in turn, and time fifty swings at each hole, which is at a measured distance from the centre of gravity of the bar, the position of which may be obtained by simple balancing on a knife-edge. Alternatively, the position of the holes may be measured from one end of the bar.

Having obtained a set of values for T, and the corresponding distances from the centre of gravity, plot a curve with the periodic times

as ordinates and the distances on either side of the centre of gravity as abscissae. A curve such as shown in fig. 27 will be obtained.

The values of T near the minimum points, MM′, should be further investigated by taking the time for vibrations in *every* hole, three each side of the approximate position, and the graph completed. The line CG, fig. 27, is drawn from C, which represents the centre of gravity

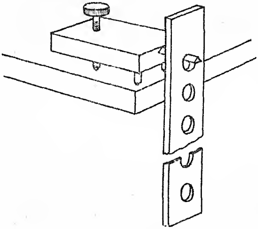


Fig. 26

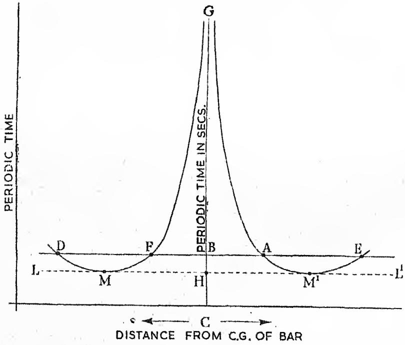


Fig. 27

of the bar. It will be found that the curve is symmetrical about GC. Draw any line EABFD parallel to the axis. This cuts the curve in four points, which have the same periodic time, T = BC. It will be found that the lengths, FB, BA, and BE, BD, are equal; i.e. FB and

BA correspond to radii GQ, GP, and BE and BD to radii GO and GR, in fig. 25.

Take either set in pairs, say BA, BD, these are lengths corresponding to α_1 and α_2 in equation (16), i.e. $AB + BD = AD = l$, the length of the equivalent simple pendulum. Its periodic time T is numerically equal to BC.

Hence, in the equation, $T = 2\pi\sqrt{\frac{l}{g}}$

all factors except g are known, whence g may be calculated.

If the experimental results permit, it is preferable to draw several lines parallel to DE (fig. 27) and for each to obtain the corresponding values of l and T. The mean value of $\frac{4\pi^2 l}{T^2}$ is used to calculate the value of g.

If now a tangent is drawn to the curve, such as line LMM′L′, $HM = HM'$ = radius of gyration about an axis through the centre of gravity: this may be measured directly.

Further, by equation (16), $k = \sqrt{\alpha_1\alpha_2} = \sqrt{AB \, . \, BD}$, so a second value of k may be found. The corresponding periodic time is, numerically, the length of $HC = T_1$, say.

Hence, once more, by equation (17),

$$T_1 = 2\pi\sqrt{\frac{2k}{g}}.$$

g may be evaluated or, alternatively, assuming the value of g, the formula be used to calculate a third value of k,

$$k = \frac{T_1^{\,2}}{8\pi^2}\,g.$$

The mean value of k may be taken and the moment of inertia about a parallel axis through the centre of gravity calculated, for

$$I = k^2M,$$

where M is obtained by direct weighing.

Owen's Method

A variation of this experiment, using a solid bar without holes for support, has been described by Dr. D. Owen (*Proc. Phys. Soc.*, Vol. 51, p. 456, 1939). Use is made of a sliding carriage whose knife-edges (see fig. 28) may be supported on sheets of glass mounted on either side of the platform of a wall bracket, the platform between the plates being cut away to allow the free passage of the pendulum. This removes the difficulty of the short length of bearing in the holes of the previous method, and also the uncertain effects of the holes themselves. It introduces two possible complications due to the carriage: (a) the

gravitational couple, and (*b*) the moment of inertia of the carriage. The first (*a*) may be reduced to a negligible amount if the centre of gravity of the carriage is in the line of the knife-edges, and the second (*b*) may be reduced to 1 in 10,000 in a suitably designed apparatus.* In this case the formula $T = 2\pi\sqrt{\frac{a^2 + k^2}{ag}}$ may be used and k may be calculated for the solid bar, using the formula $k^2 = \frac{L^2 + b^2}{12}$, when L is the length and b the breadth of the bar. a, the distance from the knife-edge to the centre of gravity of the bar, may be measured as in

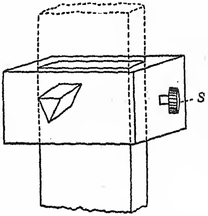


FIG. 28.—Bar pendulum with sliding carriage supporting knife-edge. S, clamping screw.

the last experiment, and the curve of fig. 27 may be obtained by experiment, from which the value of g may be obtained using the various methods described, or we may plot a^2 against aT^2, as in fig. 29. Where the graph, when produced, cuts the axis at D, we have OD $= k^2$ and the slope of the graph $= \frac{4\pi^2}{g}$.

The most accurate value of g may be obtained by choosing the minimum T_1 when $a = k$ and the value of

$$T_1 = 2\pi\sqrt{\frac{2k}{g}} \quad \text{or} \quad g = \frac{8\pi^2 k}{T_1^2}.$$

In the case of a pendulum 1 m. long, a variation of 1 mm. in a only affects T by 3 parts in one million. The value of $a = k$ is obtained from the preliminary results of fig. 27 and the pendulum is set at this length. The timing of fifty complete swings is done with a stop-watch and the observation of the swings is made with a telescope which is first focused on the pendulum when at rest. The passage of the zero

* The pendulum as described is made by Griffin and Tatlock, Kingsway, London.

line is observed when the amplitude of the swings is about 2 cm. Having obtained T as an average of three sets of timings, a further refinement in observing T may be carried out as follows.

Find a good approximation by the method already described. Set the pendulum swinging and start the stop-watch at a transit across zero. Leave the pendulum swinging for about ten minutes without making any counts. At the end of the period stop the watch at a similar transit across zero. Note the reading, t, in seconds. A whole number, n, of oscillations has occurred in this interval, and if T_1 is the approximate period already found, this number is the integer nearest t/T_1. Thus n is known and a more accurate value of the period is $T = t/n$.

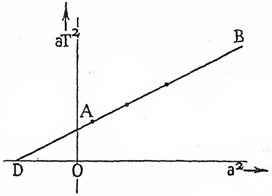


FIG. 29

Record the estimated accuracy of your experiment as illustrated by the following example.

$t = 600$ sec. Estimate of error in this time $\frac{1}{20}$ sec. With the stop-watches provided the error will be greater, but make a careful estimate of its magnitude. Assume a period of about 1 sec.

The error in T is thus $\frac{1}{20} \div 600$, i.e. in round figures $\frac{1}{10000}$.

$l = 100{\cdot}00$ cm. With an error $\pm$ 0·02.

$b = 2{\cdot}54$ cm. With an error $\pm$ 0·02.

$k = 28{\cdot}876$ cm.

It is sufficient to set the knife-edge at 28·9 cm.

A preliminary estimate gave $T_1 = 1{\cdot}5248$ sec.

Over a longer period $t = 599{\cdot}17$ sec.

Thus $$n \simeq \frac{599{\cdot}17}{1{\cdot}5248} \simeq 392{\cdot}95.$$

Thus $$n = 393,\ T = \frac{599{\cdot}17}{393} = 1{\cdot}5246 \text{ sec.}$$

The error in T depends upon the error in 599·17.

From this example $g = 980{\cdot}6$ cm. per sec.

(For further details, see Owen. *Proc. Phys. Soc.* Vol. 51 ... 1939.)

Kater's Pendulum

In the preceding discussions it is clear that if two axes on opposite sides of the centre of gravity and at right angles to a line passing through it are found experimentally to be such that the periodic time, T, about each is the same (as, for example, those through O and P in fig. 25), then the distance between them, l, is the equivalent simple pendulum, and

$$T = 2\pi\sqrt{\frac{l}{g}},$$

from which g may be calculated. In Kater's pendulum these conditions can be very nearly realized.

It consists of a long rod which is provided with two fixed knife-edge supports, K and K_1, and terminates at each end in a bob, B and B. Usually one bob, B, is made of brass and the other of wood.

M and m are two adjustable masses which may be fixed in any position between the knife-edges. Their adjustment serves to move the centre of gravity to such a position that the time of swing is approximately the same from either knife-edge.

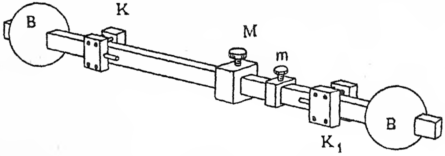


FIG. 30

The pendulum is supported on the knife-edges, K and K_1, in turn, and the approximate periodic time, T and T_1, is obtained by counting swings, timed by means of a stop-watch.

The large mass M is moved until these times are approximately the same. The small mass m is then used as a fine adjustment.

In adjusting the masses so that the time for a complete vibration is very nearly the same for both knife-edges, it will be realized that to obtain exact agreement for T and T_1 would be most tedious.

However, we can see in the following way that such exact agreement is not essential.

Let a and a_1 be the distance from the centre of gravity of the pendulum to K and K_1.

$$T = 2\pi\sqrt{\frac{a^2 + k^2}{ag}}, \qquad T_1 = 2\pi\sqrt{\frac{a_1{}^2 + k^2}{a_1 g}},$$

$$T^2 ag = 4\pi^2 (a^2 + k^2) \qquad T_1{}^2 a_1 g = 4\pi^2 (a_1{}^2 + k^2).$$

Subtracting

$$(T^2a - T_1^2a_1)\,g = 4\pi^2(a^2 - a_1^2)$$

$$\frac{4\pi^2}{g} = \frac{aT^2 - a_1T_1^2}{a^2 - a_1^2} = \frac{1}{2}\left(\frac{T^2 + T_1^2}{a + a_1} + \frac{T^2 - T_1^2}{a - a_1}\right). \quad \ldots(18)$$

a and a_1 may be made to differ by a fairly large amount by suitable adjustment of the masses, M and m. With a little care T and T_1 may be very nearly equated, and so the term $\frac{T^2 - T_1^2}{a - a_1}$ becomes small. T and T_1 may be measured very accurately by the method of coincidences, $(a + a_1)$, the distance between the knife-edges may be measured directly by comparison with a metal metre scale, using a comparator (p. 15). The important first term on the right-hand side of equation (18) is thus carefully evaluated.

The second term is small and no serious error is involved if a and a_1 are measured from the knife-edges to a point at which the pendulum may be balanced horizontally on a knife-edge.

By this method a very reliable value of g may be obtained.

Method of Coincidences

This method of timing a pendulum consists in hanging it by a knife-edge from a rigid support, in front of the pendulum of a standard clock, the height of the support being so arranged that the tails of both pendulums are on the same level. When at rest and viewed by a telescope from in front, the tail of the Kater pendulum covers that of the seconds pendulum.

If both pendulums have the same periodic time and start oscillating together, they appear to move as one when viewed through the telescope. If the periods are not the same, they will be seen to get out of step, and at one instant both will pass a fixed reference point together going in the same direction. This will not occur again until one pendulum has gained or lost a whole swing.

Suppose the seconds pendulum makes n complete vibrations, each of period T_0 (2 sec.), and the experimental pendulum makes $(n + 1)$ complete swings, of period T_1.

Then

$$T_0 n = T_1(n + 1); \quad \ldots(19)$$

$$\frac{T_1}{T_0} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} = 1 - \frac{1}{n} + \frac{1}{n^2}. \quad \ldots(20)$$

Suppose $n = 500$,

$$\frac{T_1}{T_0} = 1 - \frac{1}{500} + \frac{1}{250000},$$

further terms are negligible.

Hence T_1 is obtained in terms of T_0, which in the case taken is 2 sec.

In the coincidence method it is difficult to be certain within a few (say m) passages of the pendulum which is the correct coincidence. We can easily see that the error introduced by this cause is not appreciable when n is fairly large. Thus we know that in equation (19) instead of n we may put $(n \pm m)$,

i.e. $$T_0 (n \pm m) = T_1 (n \pm m + 1).$$

$$\frac{T_1}{T_0} = \frac{1}{1 + \frac{1}{n \pm m}}$$

$$= 1 - \frac{1}{n \pm m} \text{ approx.}$$

$$= 1 - \frac{1}{n} \frac{1}{\left(1 \pm \frac{m}{n}\right)}$$

$$= 1 - \frac{1}{n} \pm \frac{m}{n^2} \ldots.$$

In practice the coincidence may usually be limited to one in about 6, i.e. $m = 3$. If $n = 500$, we have

$$\frac{T_1}{T_0} = 1 - \frac{1}{500} \pm \frac{3}{500^2};$$

i.e. an error of about 1 in 100,000 is introduced, due to the uncertainty.

In practice a cross-hair in the focal plane of eyepiece of the telescope is a useful reference point against which to estimate coincidence.

Sphere on a Concave Mirror. An Approximate Method of Determining 'g', the Acceleration due to Gravity

A concave mirror is arranged horizontally, facing upwards, so that a small steel ball may be allowed to perform oscillations on its surface, in a vertical plane through the lowest point.

The time of oscillation of the steel ball is obtained by timing as many oscillations as possible on the surface. The observation is repeated, and from these results a mean value of the periodic time T is calculated.

Then if

R is the radius of curvature of the upper face of the concave mirror, as measured by a spherometer,

m the mass of the sphere,

r its radius,

g the accleration due to gravity,

it will be shown that

$$T = 2\pi\sqrt{\frac{\frac{7}{5}(R - r)}{g}}, \qquad \ldots(21)$$

whence

$$g = \frac{28}{5} \cdot \frac{\pi^2}{T^2} \cdot (R - r). \qquad \ldots(22)$$

Consider the sphere in its position of equilibrium to be with its centre at B (fig. 31), and when displaced to the extreme position, with the centre at C.

We will consider the case of a mirror of large radius of curvature and the displacement BC to be small compared with R.

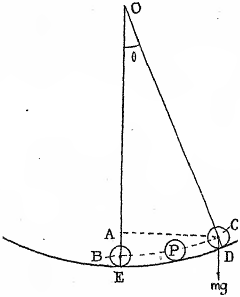


FIG. 31

The potential energy of the sphere at C is mg . AB.

Now $\qquad AB = OB - OA = (R - r)(1 - \cos\theta),$

i.e. the potential energy is $(R - r)\ 2 \sin^2 \frac{\theta}{2} . mg$, or since θ is small

$$P.E. = 2(R - r)\left(\frac{\theta}{2}\right)^2 mg = \frac{1}{2}(R - r)\,\theta^2\, mg.$$

The centre of gravity describes a circular path, BC in the diagram, so that $\frac{BC}{R - r} = \theta.$

Hence the potential energy at C is

$$\frac{1}{2}(R - r)\frac{BC^2}{(R - r)^2}mg = \frac{1}{2}\frac{mg}{R - r} . BC^2.$$

At this point there is no kinetic energy.

At B the whole of the energy is kinetic, and equal to

$$\frac{1}{2} mv_m^2 + \frac{1}{2} \mathrm{I}w_m^2,$$

where v_m is the maximum linear velocity of the centre of gravity and w_m the maximum angular velocity of rotation. I is the moment of inertia of the ball about an axis through the centre of gravity, at right angles to the plane of the paper,

i.e. $$\text{K.E. at B is } \frac{1}{2} mv_m^2 + \frac{1}{2} \mathrm{I} \frac{v_m^2}{r^2}.$$

At any intermediate point, P, distant x cm. from B along the arc, the total energy is equal to either of these quantities and is therefore a constant,

i.e. $$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} \frac{\mathrm{I}}{r^2} \dot{x}^2 + \frac{1}{2} mg \frac{x^2}{(\mathrm{R} - r)} = \text{constant},$$

$\dot{x}$ being the velocity at that P along the path.

Differentiating we have

$$m\dot{x}\ddot{x} + \frac{\mathrm{I}}{r^2} \ddot{x} \,.\, \dot{x} + \frac{mg}{(\mathrm{R} - r)} x \,.\, \dot{x} = 0.$$

Dividing by $\dot{x}$ and rearranging

$$\ddot{x} = -\frac{mg}{(\mathrm{R} - r)\left(m + \dfrac{\mathrm{I}}{r^2}\right)} \cdot x,$$

i.e. the acceleration is a constant times the displacement. The motion is thus simple harmonic with a periodic time T, where

$$\mathrm{T} = \frac{2\pi}{\sqrt{\dfrac{mg}{(\mathrm{R} - r)\left(m + \dfrac{\mathrm{I}}{r^2}\right)}}} = 2\pi \sqrt{\frac{(\mathrm{R} - r)\left(m + \dfrac{\mathrm{I}}{r^2}\right)}{mg}}.$$

Now, I, the moment of inertia about the axis described, is equal to $\frac{2}{5} mr^2$.

Hence $$\mathrm{T} = 2\pi \sqrt{\frac{(\mathrm{R} - r) \,.\, \dfrac{7}{5}}{g}}. \qquad \text{...(21)}$$

T is obtained from observations as indicated above, r is measured by means of a screw gauge, R, by means of a spherometer, *not* by an

optical method, unless the front surface is silvered. Hence, all the terms in (21) are known except g, which may be calculated. The above method does not yield accurate values for g, but provides a useful exercise in mechanics.

Atwood's Machine

A modern form of Atwood's machine is illustrated in fig. 32.* The two masses, M_1 and M_2, are equal, and are connected over the pulley, N, by a strip of white paper in the form of tape, while an equally long

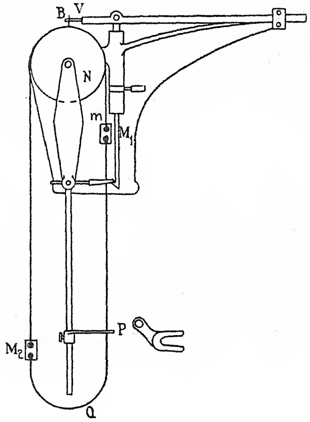


FIG. 32

strip, M_1QM_2, connects the other ends of the weights, so that for all positions of M_1 and M_2 there is an equal mass of paper at each side of the pulley.

The vibrator, V, carries an inked brush, B, and as the paper passes below it a trace, somewhat in the form of a sine curve, is drawn on the tape. The complete period of the vibrator is usually about one-fifth of a second; but this should be carefully tested by means of a good stop-watch before beginning the experiment.

The curve then provides a time record. The apparatus is used to provide an exercise in the determination of the acceleration due to

* An apparatus of this type is manufactured by M[illegible]

gravity; but even in its best form the experiment has no claim to great accuracy; it does, however, provide an instructive exercise in Mechanics.

M_1 stands on a platform, as shown in the diagram, and the mechanism of the apparatus provides for the release of the vibrator and of M_1 simultaneously. Small weights are provided, which rest on the top of M_1, and on release of the platform cause an acceleration of the masses. The rider can be removed by a second platform, P after a velocity has been acquired.

The trace on the tape records the acceleration and velocity beyond P, and from the former of these the value of g may be determined.

There is always a frictional resistance to be accounted for, although this is reduced by making the pulley light and mounting it on ball-bearings.

It is best to get rid of the retardation due to friction by making loops of wire, which may be placed on the top of M_1 and remain when M_1 passes through P.

The necessary addition can be judged approximately by observing the fall of M_1 after it has been given a small velocity. If M_1 moves down uniformly without appreciable loss of speed, the frictional error is nearly corrected. A finer observation may then be made by allowing the brush to make a trace. If the line consists of uniformly spaced waves the velocity is uniform. When this has been adjusted the wire loops are left in position and are not taken into account in the calculations.

Suppose a rider of mass, m, lies on M_1 (in addition to the wire loops).

When M_1 is allowed to fall, suppose it does so with an acceleration, f.

Let t_1 denote the tension in the paper above M_1. The lower paper strip is loose and is not supposed to exert any force on the two masses.

Let I denote the moment of inertia of the pulley, and let t_2 denote the tension in the paper on the left of the pulley, i.e. the tension acting on M_2. Denote the radius of the pulley by a and its angular velocity at any instant by w. Then $\frac{dw}{dt}$ is the angular acceleration, and we have

$$a\frac{dw}{dt} = f.$$

From the forces on $(M_1 + m)$ we have

$$(m + M_1)f = (m + M_1)g - t_1$$

and from considering M_2

while the motion of the pulley is expressed by

$$\mathrm{I}\,\frac{dw}{dt} = (t_1 - t_2)\,a,$$

from which we have

$$\mathrm{I}f = (t_1 - t_2)\,a^2.$$

We may therefore eliminate t_1 and t_2 and find

$$f = \frac{gm}{m + \mathrm{M}_1 + \mathrm{M}_2 + \dfrac{\mathrm{I}}{a^2}}.$$

The quantity $\dfrac{\mathrm{I}}{a^2}$ is called the 'equivalent mass' of the pulley, and its magnitude in grammes is engraved on the pulley.

We may therefore find the value of g from this equation from observations which give the value of f.

This may be determined from the trace. By removing P, the trace may be made long, M_1 being allowed an extended fall.

The line drawn by the brush will consist of waves which open out uniformly. Mark these off in groups of five, as at A, B, C, etc., beginning at a point A, where the trace is opened out sufficiently to be distinct. Measure carefully the distances AC, BD, CE, etc., and divide by the time interval which elapses between these points. This will give the average velocity over the strips measured, and this velocity is the velocity at the points B, C, D, etc. The differences between these velocities should all be the same, or very approximately so. Take the average of all the determinations and so obtain the average increase in velocity during five periods of the vibrator. Hence, deduce the acceleration by dividing by the time of five vibrations.

This is the value of f.

Repeat the experiment with the various riders provided.

FIG. 33

CHAPTER IV

ELASTICITY

ALL bodies, when acted upon by forces, are deformed a certain amount. The magnitude of the deformation produced by a definite applied force enables a value of the elastic constant of the material used to be calculated.

We may, in a general manner, call the forces applied 'stresses', and the deformations produced 'strains'. However, these two terms have, more often, a more precise meaning, depending on the mode of application of the forces. We shall recognize three ways of producing a deformation: (1) by uniform compression or extension, (2) by applying equal and opposite forces in one direction, e.g. stretching, (3) uniform shear. Deformation may be produced in any of these ways or by a combination of them.

(1) *Uniform Compression or Extension*

If a body of volume V be subjected to a uniform pressure of p dynes per sq. cm., a contraction will ensue. This corresponds to a change in volume of δV, say. The fractional increase in volume is $\frac{\delta V}{V}$.

In this case the *stress* applied is p dynes per sq. cm., and the *strain* is $\frac{\delta V}{V}$ numerically.

(2) *Stretching*

The most direct example of this type of deformation is seen in the case of a wire fixed at one end and supporting masses at the other end. In this case the force acting on the wire is the weight of the suspended masses and the reaction at the point of support. These are equal and opposite, acting in a direction which coincides with the length of the wire. Due to their action the wire will increase in length and at the same time will be reduced a very small amount in cross-section. The reduction in cross-section for a wire will not be of a sufficiently large amount to be readily measured directly.

If δl is the increase in length of a specimen whose original length is l, the fractional increase in length, the *strain*, is $\frac{\delta l}{l}$.

The stress producing this strain is defined as the force acting on each unit of area normal to it, i.e. if a is the area of cross-section of the wire and the total mass applied is m gm., the stress is $\frac{mg}{a}$ dynes per sq. cm.

(3) *Shear*

Consider a cube of material ABCDEFGH (fig. 34) fixed at the lower face and acted upon by a tangential force, F, at the upper face. As a result of this force the cube will take up a position shown in an exaggerated manner by the broken lines in the figure, the vertical sides

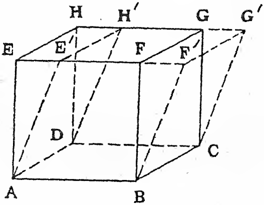


Fig. 34

being sheared through an angle of φ radians from AH and BG to AH′ and BG′.

For such a shear $\frac{F}{\text{area EFGH}}$ is the stress, i.e. the tangential force per unit area.

The strain is measured by φ, i.e. the ratio $\frac{EE'}{AE}$ if EE′ is small compared with AE′.

In all the above cases the stress is measured as a force per unit area; in the c.g.s. system in dynes per sq. cm. The strain in each case is a ratio of like quantities, and has therefore no dimensions.

Hooke's Law

If the stresses are below a certain limiting value which depends on the material of the body to which they are applied, the strain disappears when the stresses are removed. If the limiting value is exceeded, the material is strained beyond the elastic limit, and such strain is permanent; as the stresses are still further increased the result is a fracture of the material.

For stresses below the elastic limit, it was established by Hooke that the strain produced is proportional to the stress applied, i.e. under such conditions we have

$$\frac{\text{stress}}{\text{strain}} = \text{constant.}$$

The constant has a definite value which depends on the material, and which, in the three cases taken, is called (1) the 'bulk modulus of elasticity', (2) 'Young's modulus', and (3) the 'modulus of rigidity'.

Young's modulus is the most readily obtained directly by experiment.

The following notation will be used throughout in dealing with these elastic constants:

$$\text{(1) bulk modulus, K} = \frac{p}{\frac{\delta V}{V}};$$

$$\text{(2) Young's modulus, Y} = \frac{\frac{F}{A}}{\frac{\delta l}{l}};$$

$$\text{(3) modulus of rigidity, } n = \frac{\frac{F}{A}}{\varphi}.$$

In addition to the above three elastic constants, we may add a fourth, which is concerned with stretching. We noticed that during stretching there is a lateral contraction of the specimen. The *fractional* lateral contraction produced is proportional to the longitudinal stress applied and the ratio of

$$\frac{\text{fractional lateral contraction}}{\text{fractional longitudinal extension}}$$

is called 'Poisson's Ratio' (σ). Thus, if the specimen is a cylinder of radius r and length l, and the changes produced in these dimensions are δr and δl, we have

$$\sigma = \frac{\frac{\delta r}{r}}{\frac{\delta l}{l}}.$$

The following relations between the elastic constants may be readily deduced (see, for example, Poynting and Thomson's *Properties of Matter*):

$$Y = \frac{9nK}{3K + n}; \qquad \text{...(1)}$$

$$\sigma = \frac{(3K - 2n)}{2\,(3K + n)}. \qquad \text{...(2)}$$

Thus, if any two of the constants are found experimentally, the remaining two may be calculated from the above equations.

Determination of Young's Modulus for the Material of a Wire

A direct method of finding Y is to support, vertically, a long length of the wire, load it with definite masses, and observe the extension produced.

It will usually be most convenient to obtain such a length that, when supported at the ceiling, the wire extends almost to the floor. A second wire, C, is supported in like manner from the *same* support, and carries a fixed load of sufficient magnitude to keep the wire taut. The wire D carries a platform P. At V a vernier scale is attached to the wires, one half fixed to one wire, the other half to the second wire.

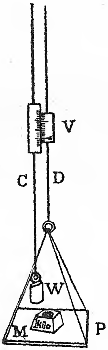


Fig. 35

The wire D, for which Young's modulus is to be determined, should be free from kinks and should carry sufficient load to make it taut, so that any further load added causes a stretching of the wire, and does not simply straighten out bends and kinks. If a heavy platform is employed at P, this weight may be sufficient. Read the vernier; place one kilogram on the platform and notice the extension. If this is due solely to the stretching of the taut wire, the vernier reading, on removing the kilogram load, should be once more the same as at the beginning. If this is not so, increase the load until this condition is satisfied.

Having obtained satisfactory repeats for this adjustment, the load on the scale pan P should be increased by equal increments and the vernier reading for each load noted. Having arrived at a safe maximum load, the latter is reduced by the same increments and the vernier readings again noted.

The values obtained may be tabulated as shown on the following page.

Load	Vernier Readings			Extension for 6 kg.
	Load increasing	Load decreasing	Mean	
0				
2				
4				
6				
8				
10				
12				
			mean extension for 6 kg. =	

The mean vernier reading for each load being taken and tabulated as shown, we may obtain several values of the extension of the wire for a definite load.

Thus, in the case taken, the loads were 0, 2, 4, 6, 8, 10, 12 kg. The difference between the vernier readings for 0 and 6 kg. load gives the extension for 6 kg. In the same way the difference between the readings for 2 and 8 kg. load, 4 and 10, 6 and 12, also gives the extension for 6 kg. increase in load. This is entered in the last column, the mean value, l say, of which is used in the calculation of Y.

The radius of the wire is measured in at least six places, using a micrometer screw, and the mean value taken, r cm., say. The original length of the wire, about 7 m., is measured directly (L cm.).

Hence $$Y = \frac{\dfrac{6000 \times 981}{\pi r^2}}{\dfrac{l}{L}} \text{ in the case taken.}$$

An alternative way (due to G. F. C. Searle) of measuring the extension of the wire is illustrated in fig. 36.

The standard wire terminates in a frame A which supports a mass M, sufficiently large to maintain the wire in a stretched state. The wire to be investigated is also fastened to a similar framework B. The two are fastened by cross-pieces C and D, which prevent relative rotation of the frames, but allow the frame B to be depressed relative to A, when masses are added to the scale pan S.

A spirit-level L is supported at one end on a rigid cross-bar of the frame A, and at the other on the point of a micrometer screw V, which moves vertically through a rigid cross-bar. The micrometer screw has the usual circular division, which enables the movement of the head to be estimated to $\frac{1}{50}$ or $\frac{1}{100}$ of a complete turn, enabling the movement

of the point of the screw (and hence the end of the spirit-level) to be measured to $\frac{1}{200}$ or $\frac{1}{100}$ of a millimetre.

The level is first adjusted, when the wire is suitably stretched free from kinks, so that it is truly horizontal. The load of, say, 1 kg. is added to the scale pan S. The micrometer screw is moved a suitable distance over the scale G, so that the spirit-level is once more horizontal.

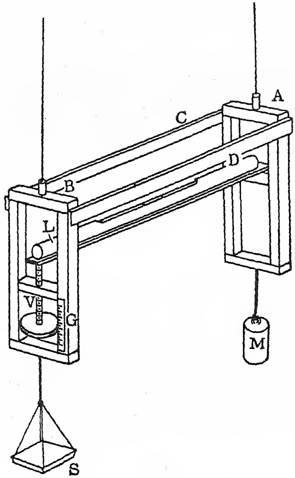


FIG. 36

The amount of movement required to bring this about is obviously equal to the elongation of the wire by the load added.

The results may be tabulated as in the former method and the value of Y calculated from the mean of a set of observations.

Bending of Beams

The value of Young's modulus may be found by less direct measurements for substances not in the form of a wire.

Consider a rod of any uniform cross-section, say rectangular, bent into the form of a circular arc of fairly large radius. Take a section of the rod by a plane passing through the long axis of symm...

parallel to it, and passing through the centre of curvature (i.e. the plane of bending). The layers of the material of the bar in the lower half will be compressed and those in the upper half extended. There will be one plane, therefore, at right angles to the plane of bending, whose dimensions remain unaltered. This plane is called the neutral surface, and it will be shown to pass through the centre of gravity of the bar. It is represented in fig. 37 by NS.

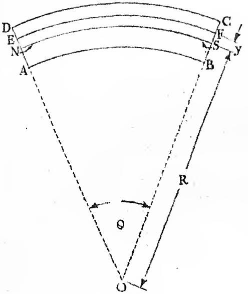


FIG. 37

If we imagine the bar to be made up of a number of filaments along the length, then such filaments, as stated above, will be extended or compressed according to their position above or below the neutral surface. One, such as is shown at EF (fig. 37), or at P in the section diagram (fig. 38) a distance y above the neutral surface is extended.

The strain in such a filament depends upon y, for, if R is the radius of curvature of the neutral surface, and θ the angle subtended at the centre O by NS, the unstretched length of the filament EF is the same as the present length of NS $= \mathrm{R}\theta$. Also EF $= (\mathrm{R} + y)\theta$.

Hence the elongation is $y\theta$ and the strain is therefore $\dfrac{y}{\mathrm{R}}$.

If f is the force acting on the filament of cross-section a, we have from the definition of Young's modulus

$$\mathrm{Y} = \frac{\dfrac{f}{a}}{\dfrac{y}{\mathrm{R}}}$$

or

$$f = \frac{\mathrm{Y}a}{\mathrm{R}} \cdot y, \quad \text{i.e. } f \propto y.$$

Thus, the arrows in the lower part of fig. 38 show the type of forces acting on all such filaments into which we have subdivided the bar.

This system of forces on the bar must have an algebraic sum of zero.

i.e. $$\Sigma f = 0 \quad \text{or} \quad \frac{Y}{R} \Sigma ay = 0.$$

Since $\frac{Y}{R}$ is not equal to zero, $\Sigma ay = 0$.

Thus, *the neutral surface passes through the centre of area of the cross-sections.*

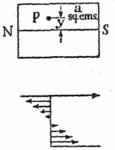


FIG. 38

The forces have a moment about the neutral surface. The moment for the single force is $$fy = \frac{Y}{R} \cdot ay^2.$$

For equilibrium the sum of such moments is equal and opposite to the external couple which set up the internal forces. If C is the external couple, we have

$$C = \Sigma \frac{Y}{R} \cdot ay^2 = \frac{Y}{R} i,$$

where $i = \Sigma ay^2$. From the similarity between this case and the corresponding sum in considering masses in connexion with moment of inertia, i is sometimes called 'the moment of inertia of cross-section'. It may be calculated in the same way as the moment of inertia if area replaces mass.

We have, therefore, for a bent beam

$$iY = CR. \qquad \ldots(3)$$

Cantilever

Consider a light beam fixed horizontally at one end and loaded with a mass m at the other. If the mass of the beam is small compared with the load m, the whole depression may be taken as due to the load.

In fig. 39, let AC′ be the unloaded position for the neutral surface, and AC the position taken when the load is applied at C.

To obtain an expression for the depression of the end in terms of the dimensions of the bar, m and Y, it is convenient to refer to a system of axes with the end A as origin; AC′ being the x axis, and a line at right angles to AC′ from A in the plane of the paper the ...

We will assume that the curvature of the loaded rod is small, i.e. the total depression at C is small. Consider a section at B (fig. 39), x cm. from A. As already seen, across the face of such a section a system of forces exists. These forces on the segment, BC, cause extensions above the neutral surface, and compressions below, constituting a counter-clockwise couple $\mathrm{C} = \dfrac{i\mathrm{Y}}{\mathrm{R}}$ on BC.

At the same time the force mg at C has a clockwise moment equal to $mg\ (l - x)$ on BC.

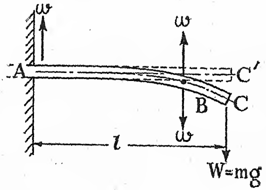


Fig. 39

For equilibrium these two opposite couples are equal in magnitude,

i.e. $$\frac{i\mathrm{Y}}{\mathrm{R}} = mg\ (l - x), \qquad \ldots(4)$$

where R is the radius of curvature at B.

Now $$\mathrm{R} = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

But in this case $\dfrac{dy}{dx}$ is small—for the total depression is assumed to be small—so that $\left(\dfrac{dy}{dx}\right)^2$ is negligible compared with unity, and therefore we have

$$\mathrm{R} = \frac{1}{\dfrac{d^2y}{dx^2}},$$

i.e. $$\frac{d^2y}{dx^2} = \frac{\mathrm{W}}{i\mathrm{Y}}\ (l - x) \quad \text{from equation (4) above.}$$

The value of the total depression at the end of the bar may be obtained by integration, and is the value of y when $x = l$.

Integrating, we have

$$\left[\frac{dy}{dx}\right]=\frac{W}{iY}\left(lx-\frac{x^2}{2}\right), \qquad ...(5)$$

the constant of integration being zero, for when $x=0$, $\frac{dy}{dx}=0$.

A second integration between the limits 0 and l gives

$$\left[y\right]_0^{y_0}=\frac{W}{iY}\left[\frac{lx^2}{2}-\frac{x^3}{6}\right]_0^l,$$

where y_0 is the end depression,

i.e.
$$y_0=\frac{W}{iY}\frac{l^3}{3}. \qquad ...(6)$$

For a bar of rectangular cross-section, i about the neutral axis NS (fig. 38) is $\frac{bd^3}{12}$, where b is the breadth and d the depth of the bar.

Hence
$$y_0=\frac{12W}{bd^3Y}\cdot\frac{l^3}{3}$$

or
$$Y=\frac{4mgl^3}{bd^3y_0}. \qquad ...(6a)$$

The case when the mass of the beam is not negligible is treated on p. 79.

Beam Supported at Two Knife-edges and Loaded in the Middle

If the beam is now supported at the two ends and loaded with a mass m at the mid-point, we have a reaction $\frac{mg}{2}$ at each knife-edge. If l is the total length of the bar, the depression produced at the centre will be exactly the same as the depression at the end of a similar bar of length $\frac{l}{2}$ and loaded at the end with a mass $\frac{m}{2}$, i.e. if we imagine the bar clamped at the mid-point and a force $\frac{mg}{2}$ applied at one end. The movement of the end is precisely the same as the depression in the middle in the actual case taken.

Such a depression may be obtained by substituting the length $\frac{l}{2}$ and force $\frac{mg}{2}$ in equation (6*a*), giving for the depression

$$y_0=\frac{W}{iY}\cdot\frac{l^3}{48}$$

or
$$Y=\frac{mgl^3}{4bd^3y_0}. \qquad ...(7)$$

The value of Young's modulus for the material of a beam may be obtained as follows. The beam, of about 1 cm. square cross-section, and about 1 m. long is supported on two knife-edges near its ends. A load is applied at the mid-point by placing masses on a pan which is suspended from a knife-edge which rests on the bar at this place. The depression is measured on a vernier scale, one scale of which is fixed, the other moves with the beam.

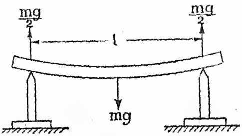


FIG. 40

The load is increased, and the vernier reading for each load is tabulated as below. The value for $\frac{m}{y_0}$ is obtained in each case, as in the table.

Load m	Vernier Reading	Depression (y_0)	$\frac{m}{y_0}$
0		0	0
2000 gm.			
3000 gm.			
4000 gm.			
5000 gm.			

mean $\frac{m}{y_0}$

A mean value of $\frac{m}{y_0}$ from the series of observations is taken.

The distance l between the knife-edges is measured directly, being of the order of 1 m.

Now d occurs in the third power, and is only a small quantity, therefore many observations must be taken and the mean value used. For d approx. 1 cm., an error of 0·1 mm. means 1 per cent error, and this is magnified in d^3 to 3 per cent.

Substitute the values found in equation (7)

$$Y = \left(\frac{m}{y_0}\right)\frac{gl^3}{4bd^3}.$$

Koenig's Method

Another method, due to Koenig, of determining the value of Y for the material of a beam, is by means of the type of apparatus shown in fig. 41.

The bar carries a knife-edge which supports the load on a pan P.

At the ends of the bar are two mirrors, M_1 and M_2, almost normal to the bar, but slightly displaced to enable a scale S to be seen in the telescope T, the light from S having suffered two reflections.

The telescope carries a cross-hair in the eyepiece, and the apparatus is arranged so that a scale division, as seen in the telescope, coincides with the cross-hair.

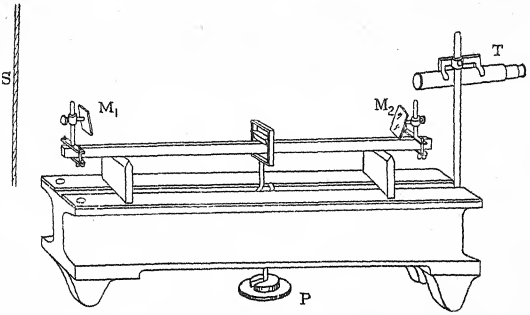


FIG. 41

If now the bar is loaded with, say, 1 or 2 kg., the mirrors will be turned towards each other as a result of the depression produced, and the scale division viewed in the telescope will be altered. This difference is noted $= x$ divisions, say.

It will be shown that

$$Y = \frac{3Wl^2(2D + \alpha)}{2bd^3x}, \qquad \ldots(8)$$

where $W = Mg$, M being the load in gm.,
l = distance between knife-edges,
D = distance between scale and the more remote mirror, M_2,
α = the distance between the mirrors,
b and d having the same values as before (breadth and depth of the bar).

For a bent cantilever we saw, equation (5),

$$\frac{dy}{dx} = \frac{W}{iY}\left(lx - \frac{x^2}{2}\right),$$

or between limits 0 and l,

$$\left[\frac{dy}{dx}\right]_0^l = \frac{W}{iY} \cdot \frac{l^2}{2}.$$

Now $\left[\frac{dy}{dx}\right]_0^l$ is tangent of the angle through which the beam has been bent. Let φ be this angle.

Then $$\tan \varphi = \frac{W}{iY} \cdot \frac{l^2}{2}.$$

Now for in the present case, l being the whole length of the bar supported by *two* knife-edges, we obtain the angle through which each end is turned by substituting $\frac{l}{2}$ for l and $\frac{W}{2}$ for W in the last equation.

i.e. in the present case $$\tan \varphi = \frac{WY^2}{16iY}.$$

For a rectangular bar $$i = \frac{bd^3}{12},$$

$$\tan \varphi = \frac{3Wl^2}{4bd^3Y}.$$

For small depressions the angle φ is very small and so

$$\tan \varphi = \varphi.$$

$$\varphi = \frac{3Wl^2}{4bd^3Y}. \qquad \text{...(9)}$$

Now a value of φ may be obtained from a consideration of the movement of the mirrors.

Let m_1 and m_2 be the original positions of the mirror (fig. 42). In the first case the image of the division at D is in coincidence with the cross-hair; when the mirrors move, each through an angle φ, F is then seen in coincidence with the cross-hair.

Let us imagine the rays of light reversed, ABCD being the original path: when m_1 moves through φ and takes position m_1', BC is moved through 2φ striking m_2 at E.

Obviously α being the distance between the mirrors $CE = \alpha \cdot 2\varphi$ very nearly.

The ray EF is swung round through an angle 4φ, for in addition to BE having moved through 2φ, m_2 itself has rotated through φ.

EG is a line drawn parallel to CD.

i.e. since $GEF = 4\varphi$, $\quad FG = 4\varphi \cdot D$, very nearly.

But $\quad DG = CE = 2\alpha\varphi$.

$$\therefore \quad x = DF = 2\varphi(\alpha + 2D),$$

$$\therefore \quad \varphi = \frac{x}{2(\alpha + 2D)},$$

from (8) and (9)

$$\frac{x}{2(\alpha + 2D)} = \frac{3Wl^2}{4bd^3Y},$$

$$Y = \frac{3Wl^2(\alpha + 2D)}{2bd^3x}.$$

In performing this experiment a mean value of $\frac{W}{x}$ is obtained as in the last experiment, from observation of x corresponding to several loads and the other terms measured as before. Hence, by substituting in (8) Y is obtained.

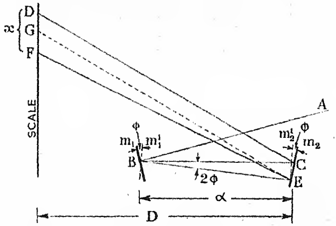


FIG. 42

Determination of Young's Modulus of the Material of a Bar by the Vibration Method

It has been shown (p. 75, equation (6)) that when a light beam is loaded with a mass m at one end, the other end being rigidly fixed horizontally, the depression is y_0, where

$$mg = W = \frac{3iY}{l^3} y_0.$$

The internal stress of the beam thus produces a restoring force of magnitude W proportional to y_0.

Suppose that the mass m is further displaced by an amount y, the total displacement from the unloaded position being $(y + y_0)$. The beam now produces a restoring force upward of magnitude

$$\frac{3iY}{l^3}(y + y_0),$$

and when the weight is let go the force upward is

$$\frac{3iY}{l^3}(y + y_0) - mg = \frac{3iYy}{l^3}.$$

Thus there is a force tending to restore the mass to its original position of equilibrium, which is proportional to the displacement, y, from it.

The equation of motion is thus

$$m\frac{d^2y}{dt^2} = -\frac{3i\mathrm{Y}}{l^3}y.$$

This represents simple harmonic motion with a period

$$\mathrm{T} = 2\pi\sqrt{\frac{ml^3}{3i\mathrm{Y}}}.$$

If we wish to consider the effect of the mass of the beam we can do so conveniently by considering the energy of the system.

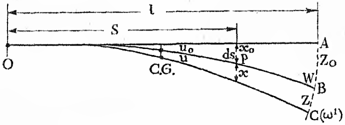


FIG. 43

Let OB (fig. 43) represent the position of equilibrium, and let OC be a position slightly displaced from it.

At any point, P, of the bar let the displacement in the position of equilibrium be y_0.

Let the corresponding displacements at the end be z_0, and at the c.g., u.

The potential energy of the system with reference to the position of equilibrium as a standard is

$$-\mathrm{W}z - \mathrm{M}gu + \text{energy stored in the beam,}$$

where M denotes the total mass of the beam and where z and u denote further displacements beyond equilibrium.

In order to calculate the energy stored we can determine the work done in straining the beam into the new position.

We require first the depression, y, at any point of the bar loaded at one end, taking into consideration its mass. Let the mass per unit length be μ, so that $\mathrm{M} = \mu l$.

Let x denote the distance of the point P from the clamped end. The moment of the internal forces across the vertical section of the bar at P is $\mathrm{Y}i\frac{d^2y}{dx^2}$, and this balances the moment of the external forces. These arise from the load at the end and from the weight of the part of the bar beyond P. This is of magnitude $\mu(l-x)$ and can be regarded as concentrated at the middle point which lies at $(l-x)/2$ from P.

Thus $$\mathrm{Y}i\frac{d^2y}{dx^2} = \mathrm{W}(l-x) + \tfrac{1}{2}\mu g(l-x)^2.$$

On integration we obtain

$$Yi\frac{dy}{dx} = W\,(lx - \tfrac{1}{2}x^2) + \tfrac{1}{2}\mu g\,(l^2x - lx^2 + \tfrac{1}{3}x^3).$$

Since $\dfrac{dy}{dx}$ vanishes when $x = 0$ on account of the fact that the bar is clamped at the end, the integration constant is zero.

Integrating again,

$$Yiy = W\,(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3) + \tfrac{1}{2}\mu g\,(\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4). \quad \ldots(10)$$

The constant vanishes again since there is no depression at the origin.

At the end of the bar, corresponding to $x = l$, the depression is z_0, given by

$$Yiz_0 = \tfrac{1}{3}Wl^3 + \tfrac{1}{8}\mu gl^4 = l^3\left(\frac{W}{3} + \frac{Mg}{8}\right).$$

Suppose a further displacement z is made by applying a force F at the end.

Then
$$Yi\,(z_0 + z) = l^3\left(\frac{W + F}{3} + \frac{Mg}{8}\right)$$

from which
$$F = \frac{3Yi}{l^3}z. \quad \ldots(11)$$

The force F balances the additional force set up in the bar by the internal stresses due to the additional displacement z. Thus the potential energy associated with these stresses reckoned from the position of equilibrium to the displacement z is

$$\int_0^z Fdz = \frac{3}{2}\frac{Yi}{l^3}z^2.$$

Thus the potential energy referred to the standard position is

$$\frac{3}{2}\frac{Yiz^2}{l^3} - Wz - Mgu.$$

By calculating u_0 from the equation (10) for the value $x = \dfrac{l}{2}$, we obtain

$$Yiu_0 = l^3\left(\frac{5}{48}W + \frac{17}{384}Mg\right).$$

Let u denote the additional displacement which results from increasing W by F. It follows that

$$u = \frac{5Fl^3}{48Yi}.$$

Thus
$$u = \tfrac{5}{16}z,$$

and we can express the potential energy in terms of z as follows:

$$\text{P.E.} = \frac{3}{2}\frac{Yiz^2}{l^3} - Wz - \frac{5Mgz}{16}.$$

Since the rod is considered to be vibrating, it is necessary to determine its kinetic energy before its total energy can be found.

In equation (10) y denotes the total displacement ...

co-ordinate x. In the case when W is increased by F, y increases by an amount s, where

$$s = \frac{F}{Yi}\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right) = \frac{3}{l^3}\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right)z$$

by equation (11).

Thus the velocity at any point P is given by

$$\dot{s} = \frac{3}{l^3}\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right)\dot{z}$$

and the kinetic energy of an element at P is

$$\frac{1}{2}\mu\left\{\frac{3}{l^3}\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right)\dot{z}\right\}^2 dx.$$

The total kinetic energy of the whole bar is obtained by integration between the limits of 0 and l for x, giving the result

$$\text{K.E.} = \frac{33}{280}\mu l\dot{z}^2 = \frac{33}{280}\text{M}\dot{z}^2.$$

In addition, the kinetic energy of the load at the end of the bar is

$$\frac{1}{2}\frac{W}{g}\dot{z}^2.$$

Thus the total energy of the bar and load is

$$\text{P.E.} + \text{K.E.} = \frac{1}{2}\left(\frac{W}{g} + \frac{33}{140}\text{M}\right)\dot{z}^2 + \frac{3Yiz^2}{2l^3} - Wz - \frac{5Mgz}{16}.$$

This is a constant, so that, differentiating and dividing by $\dot{z}$, we obtain

$$\left(\frac{W}{g} + \frac{33}{140}\text{M}\right)\ddot{z} + \frac{3Yi}{l^3}z - \left(W + \frac{5Mg}{16}\right) = 0.$$

This equation describes the changes in displacement at the loaded end of the bar when it is in vibration after an initial disturbance from the position of equilibrium. It can be put into the familiar form by

$$\text{writing } q = z - \frac{l^3\left(W + \frac{5Mg}{16}\right)}{3Yi},$$

when we obtain $$\left(\frac{W}{g} + \frac{33}{140}\text{M}\right)\ddot{q} + \frac{3Yi}{l^3}q = 0.$$

This represents simple harmonic motion (p. 47) with the period

$$T = 2\pi\sqrt{\frac{\left(\frac{W}{g} + \frac{33}{140}\text{M}\right)l^3}{3Yi}} \qquad \ldots(12)$$

about the centre $q = 0$ or $z = l^3\left(W + \frac{5Mg}{16}\right)\Big/3Yi$.

The value of i depends on the shape and size of the cross-section of the bar. For a rectangular rod $i = bd^3/12$, where b = breadth and d = depth.

The experiment may very well be carried out using an ordinary boxwood metre rule.

A definite length, l, of the rod is projected from the top of a table, to which it is rigidly clamped, as seen in fig. 44.

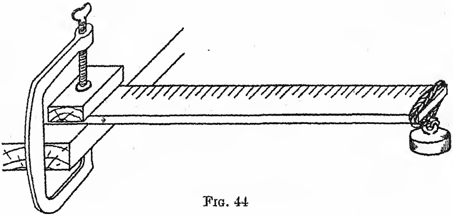


FIG. 44

A mass is rigidly attached to the end of the metre scale, so that it has no play, i.e. there is only one vibration, namely, that of the scale itself. The mass should be such as to cause only a small depression. The rod is made to vibrate, and the time T is obtained by timing fifty vibrations in the usual manner. The length of the vibrating rod is altered, and the periodic time again determined. This is carried out for several lengths of rod, and also for several masses at the end of the scale.

It must be remembered that M in the expression (21) for T is the mass of the vibrating part of the scale, and must be obtained for each length l employed.

Of course, if m' is the total mass of the scale and L the total length, then $M = \frac{m'l}{L}$ for the uniform rod.

The results of the various experiments may be tabulated as follows:

l	l^3	$\frac{W}{g}$	M	$\frac{W}{g} + \frac{33}{140}M$	T	T^2	$\frac{l^3}{T^2}\left(\frac{W}{g} + \frac{33M}{140}\right)$

$$\text{Mean } \frac{l^3}{T^2}\left(\frac{W}{g} + \frac{33M}{140}\right) =$$

$$Y = \frac{16\pi^2}{bd^3}\left\{\frac{l^3}{T^2}\left(\frac{W}{g} + \frac{33M}{140}\right)\right\},$$

$\frac{W}{g}$ denotes the *mass* attached.

The method works equally well for such substances as wood, where the correction for the mass of the beam is small, and for brass, etc., where $\frac{33}{140}M$ is comparable with the values of $\frac{W}{g}$.

An interesting way of interpreting the observations is to plot T^2 against the load in grammes $\left(\frac{W}{g}\right)$. The graph should be linear cutting the axis of the load with a negative intercept of magnitude $\frac{33M}{140}$. This result may be verified by weighing the bar and finding the mass of the vibrating portion.

The slope of the graph has a value $\frac{4\pi^2 l^3}{3Yi}$, but the value of Y is better obtained by the table method as described above.

RIGIDITY

The modulus of rigidity is determined by observation of the twist produced in a wire by a definite couple, either statically, or less directly by torsional oscillation.

Measurement of the Rigidity of a Wire by the Static Method

Consider a wire fixed at the upper end and subjected to a couple C about the vertical axis, as shown in fig. 45, causing the lower end to twist through an angle θ.

If we consider a section of the wire (fig. 46) it will be seen that the couple applied sets up a shear in the section and the twist produced is governed by the rigidity of the material of the wire.

Fig. 46 shows an enlarged section of the wire, of radius R. If an annular ring of radius r and width δr be taken in this section it will be clear that an elementary cube in the ring, such as EL, will shear so that the vertical sides move through some angle φ. The cubes above and below it throughout the length of the wire behave in a similar manner, and so a line such as MN (fig. 45), when under the action of the couple, takes up a position such as MP indiced at φ with MN.

It is clear that $NP = l\varphi = R\theta$, where θ and φ are measured in

radians and therefore φ may be expressed in terms of the more easily measured $\frac{R\theta}{l}$.

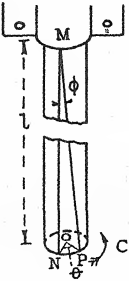


FIG. 45

If we consider that the force set up over the upper and lower faces of the annular ring is f, we have from the definition of rigidity n,

$$n = \frac{\frac{f}{2\pi r\ .\ \delta r}}{\varphi}$$

or

$$f = 2\pi n\varphi r dr.$$

Since this force has a moment fr about the axis of the wire, we have, replacing φ by the more easily measured term, θ.

$$fr = \frac{2\pi n\theta}{l} r^3 dr.$$

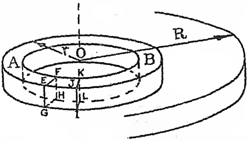


FIG. 46

The total couple throughout the solid section is therefore

$$\int_0^R \frac{2\pi\theta n}{l} r^3 dr = \frac{\pi n\theta R^4}{2l},$$

a couple which is equal to the applied couple C for equilibrium,

hence $$C = \frac{\pi n\theta R^4}{2l}. \quad (12)$$

This may be rewritten

$$\frac{C}{\theta} \cdot l = \frac{\pi R^4}{2} \cdot n.$$

$\frac{C}{\theta}$, the couple required to produce unit angular twist, is called the 'coefficient of torsion' $= \tau$, say.

We thus have
$$\tau l = n \cdot \frac{\pi R^4}{2} \qquad \ldots(14)$$

or
$$\tau l = i \cdot n.$$

$\frac{\pi R^4}{2} = i$ the moment of inertia of cross-section for the circular wire, and θ in equation (13) is measured in radians.

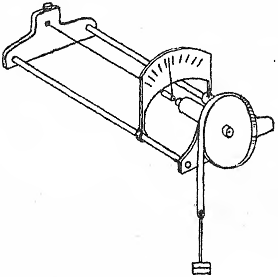


Fig. 47

Experimental Details

The method of finding the coefficient is to fix a wire specimen rigidly at one end, apply a measured couple at the other, and measure the twist produced at a given distance l from the fixed end.

The two types of apparatus usually employed are shown in figs. 47 and 48.

Using the vertical wire type, pointers are fixed to the wires at different distances from the fixed end, and a couple is applied to the free end by adding weights to the scale pans S, S^1. The amounts of twist, θ_1, θ_2, θ_3, at the distances, l_1, l_2, l_3, from the clamped ends are observed on the circular scales shown. It will be found that $\frac{\theta_1}{l_1} = \frac{\theta_2}{l_2} = \frac{\theta_3}{l_3}$ as we may expect from equation (13).

Now consider a fixed length of wire l cm. from the support. If M is the sum of the masses applied in S and S^1, and the diameter of the

wheel at which the couple is applied is D, then $Mg\frac{D}{2} = C$ in equation (13).

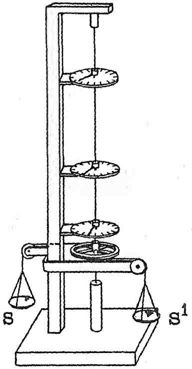


FIG. 48

A series of values for θ, corresponding to various masses on the pans, is obtained. These results may be tabulated as shown.

Mass on pans M	Angle of twist $\theta°$	$\frac{M}{\theta}$
0	0	—
100	15	6·6
200	30	6·6
300	46	6·52
400	61	6·55
500	77	6·50

Rewriting (13), we have

$$Mg \cdot \frac{D}{2} = \frac{\pi n R^4}{2l} \cdot \theta, \quad \theta \text{ in radians},$$

or

$$n = \frac{M}{\theta} \frac{gDl}{\pi R^4}.$$

The length of the wire is measured by a metre rule. The arm of the couple, D, is measured by means of callipers

As R, the radius of the wire, occurs in the fourth power, care is taken to obtain an accurate value. R is measured with a micrometer screw gauge at several points along the length of the wire, and a mean value taken.

Substituting the mean value of $\frac{M}{\theta}$ from the table of results, all the unknowns of the equation are ascertained.

Assuming that θ is measured in degrees, the value of n is given by

$$n = \frac{M}{\theta}\frac{l}{R^4} \cdot \frac{g\,(180)\,D}{\pi^2}. \qquad \ldots(15)$$

The same type of observations are necessary with the horizontal apparatus, which usually makes use of a shorter length of wire specimen.

Measurement of the Rigidity of a Wire by the Dynamic Method. Maxwell's Needle

Suppose that a body such as a horizontal bar is supported at the end of a vertical wire. A small displacement from the position of equilibrium causes simple harmonic oscillations about this position. When the displacement is measured by the angle θ the stresses produce a restoring couple of magnitude $\tau\theta$ and, if I denotes the moment of inertia about the vertical axis of rotation, the equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\tau\theta.$$

Thus the period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{\tau}}.$$

This equation for T can be used to determine τ, and hence the modulus of rigidity, n, of the material of the wire. The period, T, can be measured with considerable accuracy. In the static method the angle, θ, is measured and a high degree of accuracy is difficult to attain. In the dynamic method the period, T, is measured so that this method is in general to be preferred.

In Maxwell's needle (fig. 49) the bar is replaced by a hollow tube, of length D cm. Fitting in the tube are four equal-sized cylinders, each of length $\left(\frac{D}{4}\right)$, two of solid brass, and two hollow.

Let I_0 be the moment of inertia of the hollow cylinder of length D, about the wire as axis,

I_1 be the moment of inertia of the solid brass cylinder about a parallel axis through its centre of gravity,

I_2 be the similar moment of inertia for the hollow brass cylinder,

m_1 the mass of each solid brass cylinder,

m_2 the mass of each short hollow brass cylinder,

l the length of the wire in cm.

In the first case, place the cylinders in the order shown in fig. 49 (*a*) and find the periodic time T_1. Then arrange the cylinders as in (*b*), once more obtaining the periodic time, T_2, by timing, say, fifty complete swings.

Suppose that the moment of inertia of the complex bar in the first case is I′, and I″ in the second.

We have
$$T_1 = 2\pi\sqrt{\frac{I'}{\tau}};$$
$$T_2 = 2\pi\sqrt{\frac{I''}{\tau}};$$
$$T_1^2 - T_2^2 = \frac{4\pi^2}{\tau}\left(I' - I''\right).$$

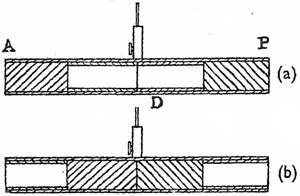


FIG. 49

Now we have
$$I'' = I_0 + 2I_1 + 2m_1\left(\frac{D}{8}\right)^2 + 2I_2 + 2m_2\left(\frac{3D}{8}\right)^2$$
for the moment of inertia of each of the four units is given by the 'law of parallel axes',
$$I' = I_0 + 2I_2 + 2m_2\left(\frac{D}{8}\right)^2 + 2I_1 + 2m_1\left(\frac{3D}{8}\right)^2,$$
whence
$$I' - I'' = 2m_2\left\{\left(\frac{D}{8}\right)^2 - \left(\frac{3D}{8}\right)^2\right\} + 2m_1\left\{\left(\frac{3D}{8}\right)^2 - \left(\frac{D}{8}\right)^2\right\}$$
$$= 2m_1\frac{D^2}{8} - 2m_2\frac{D^2}{8} = \left(m_1 - m_2\right)\frac{D^2}{4}.$$

By such an arrangement I′ − I″ may be evaluated and hence τ may be calculated.

To enable an accurate measurement of T in both cases, the hollow frame carries a small mirror. A beam of light from a lamp is focused on the mirror, and the reflected beam is directed on a scale, and the position of equilibrium is noted. As the ...

spot of light on the scale, passes this mark in one direction, a stop-clock is started, and time for fifty complete oscillations is measured, when the four short cylinders are arranged (as in fig. 49 (*a*) and (*b*)), whence T_1 and T_2 are obtained. The length l of the wire from the rigid support, and the length D are measured directly. m_1 and m_2 are obtained by weighing, and therefore all the factors for the determination of n are available. Now

$$\tau = 4\pi^2 \frac{I' - I''}{T_1{}^2 - T_2{}^2} = 4\pi^2 \frac{m_1 - m_2}{T_1{}^2 - T_2{}^2} \cdot \frac{D^2}{4},$$

whence n may be calculated, since it is given by

$$n = \frac{2\tau l}{\pi R^4}.$$

Determination of the Modulus of Rigidity of the Material of a Flat Spiral Spring

In this experiment the value of n for the material of the wire of a flat spiral spring is deduced from a knowledge of the periodic time of vertical oscillations of the spring, when loaded with a known mass, and the dimensions of the spring.

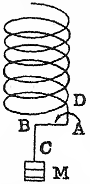


FIG. 50

Let us consider a flat spiral spring made of closely wound wire whose radius r is small compared with the radius of the spring itself. Such a spring may be made by winding the wire on a wooden cylinder of suitable diameter.

Let the ends of the spring be bent twice at right angles so that the free end, BC (fig. 50), is along the axis of the cylinder.

If the spring is clamped vertically at one end, and loaded with a mass, M, at the other end, as shown in the figure, the force, Mg, along the axis exerts a couple tending to twist the wire in the direction of the arrow.

It was shown on p. 91 that, under such circumstances, if l is the length of the wire from the fixed end, and r the radius of the wire,

$$C = \frac{\pi r^4}{2} \frac{n\theta}{l}. \qquad \ldots(16)$$

In the case of the spring we may apply this formula when l represents the total length of wire, and is equal to $2\pi RN$, where N is the number of turns.

If r = radius of the wire,
R = radius of the cylinder on which the spring measured to centre of the wire,
$l = 2\pi RN$,
M = the load in grammes,

then
$$C = \frac{\pi r^4}{2} \cdot \frac{n\theta}{2\pi RN}. \qquad \dots(16a)$$

Take a section of the spring at A, shown enlarged in fig. 51. When the couple is applied the arm AB is twisted through the angle θ, given by equation (16a) above, and takes up the position AB′.

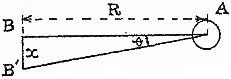


FIG. 51

The depression x, of the end B, is approximately equal to $R\theta$, when R is large,

i.e.
$$x = R \cdot \frac{4\pi RNC}{\pi r^4 n} = \frac{4NR^2C}{r^4 n}.$$

If f be the restoring force on M due to the wire, the couple $C = fR$,

i.e.
$$x\frac{r^4 n}{4NR^3} = f. \qquad \dots(17)$$

The equation of motion of the moving mass is therefore

$$M\frac{d^2x}{dt^2} = -\frac{r^4 n}{4NR^3} \cdot x.$$

This represents S.H.M., whose period is

$$T = 2\pi\sqrt{\frac{4MNR^3}{r^4 n}}. \qquad \dots(18)$$

From observation of the periodic time of the spring oscillating vertically, n, the rigidity coefficient of the material of the spring, can therefore be obtained.

However, in the foregoing we have neglected the mass of the spring itself, and have also assumed that the total depression produced by M was due to twist alone.

We must now consider these factors

The Effect of Shear in the Spring

We can very easily see that the shear effect in the type of spring taken above is negligible when the radius of the wire is small compared with the radius of the spring. For this purpose again neglect the mass of the spring.

If f is the force acting along the axis, at any point A in the spring there is an equal and opposite force, f, for equilibrium.

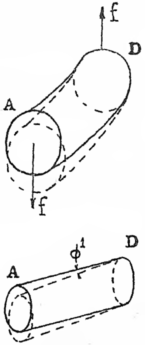


FIG. 52

Thus for the small segment AD shown enlarged in fig. 52, there is a downward force f at A and an upward force f at D, constituting a shearing couple which would cause a depression of A to some lower position A^1, due to the shear φ' produced. From the definition of rigidity

$$\varphi' = \frac{f}{An}.$$

The area A is the area over which the forces f act, which is the area of cross-section of the wire, πr^2,

i.e. $$\varphi' = \frac{f}{\pi r^2 n}.$$

The total depression due to this shear for a length l of the wire is

$$\varphi' l = \frac{fl}{\pi r^2 n}. \qquad \ldots(19)$$

Now, using the expression for the depression due to twist, given in equation (17) and substituting $l = 2\pi RN$ in the expression (19) above,

we have

$$\frac{\text{Depression due to shear}}{\text{Depression due to twist}} = \frac{\dfrac{f2\pi RN}{\pi r^2 n}}{\dfrac{f4NR^3}{r^4 n}} = \frac{r^2}{2R^2}.$$

Take an average case, $r = 0{\cdot}05$ cm., $R = 0{\cdot}8$ cm.,

$$\frac{r^2}{2R^2} = \frac{1}{512},$$

i.e. depression due to shear is about 0·2 per cent of the depression due to twist.

Thus, when r is small compared with R we may neglect the shearing effect.

The Effect of the Mass of the Spring

The expression for the period of oscillation is modified if the mass of the spring is taken into account. The effect of this mass is to cause an increase in the depression of the lower end, but since the mass of the spring is distributed this increase will not be obtained simply by adding the mass of the spring to the load. It seems evident that the increase will be proportional to the mass, m, of the spring, so that the formula will be changed by adding a quantity cm to M, where c will be a fraction. In order to calculate the value of c it is necessary to consider the problem in detail.

But we may anticipate that the formula for the period will be

$$T = 2\pi\sqrt{\frac{(M + cm)\, 4NR^2}{nr^4}}.$$

In the case of the light spring the twist at the end is given by

$$\theta = \frac{2lC}{\pi r^4 n} = \frac{2lMgR}{\pi r^4 n},$$

where l = total length of the wire of the spring $= 2\pi RN$ and $C = MgR$.

The depression $$x = R\theta = \frac{2lMgR^2}{\pi r^4 n}.$$

Thus, taking into account the mass of the spring, we anticipate a corrected formula for the depression

$$x = \frac{2lgR^2}{\pi r^4 n}\left(M + cm\right).$$

In order to increase x by an amount z we require to increase the load by a mass m', where

$$z = \frac{2lgR^2}{\pi r^4 n} m'.$$

This means that the restoring force in the wire due to the displacement z is

$$F = m'g = \frac{\pi r^4 n}{} z.$$

Thus, if we regard the position of equilibrium as the place of zero potential energy, in the position of further displacement in which the load is depressed a distance z, we have

$$\text{P.E.} = -\text{M}gz - \tfrac{1}{2}mgz + \text{energy stored in straining the wire.}$$

The second term of this expression is due to the depression of the c.g. of the spring by the amount $z/2$.

The energy stored in the spring may be determined by the work done in overcoming the force F in the extension from 0 to z. This energy is

$$\int_0^z \text{F}dz = \frac{\pi r^4 nz^2}{4l\text{R}^2}.$$

Thus
$$\text{P.E.} = \frac{\pi r^4 nz^2}{4l\text{R}^2} - \text{M}gz - \tfrac{1}{2}mgz.$$

We now require the kinetic of the spring when the displacement of the load is z.

The spring is assumed to be uniformly stretched so that at any point s cm., measured along the wire from the support, the velocity is $\frac{s\dot{z}}{l}$, $\dot{z}$ denoting the velocity at the end. The mass of an element at the point is $\frac{mds}{l}$, so that the kinetic energy of the element is

$$\frac{1}{2}\frac{mds}{l}\left(\frac{s\dot{z}}{l}\right)^2.$$

The total K.E. of the spring

$$\int_0^l \frac{1}{2}\frac{m\dot{z}^2}{l^3}s^2ds = \tfrac{1}{6}m\dot{z}^2.$$

To this must be added the K.E. of the load, $\tfrac{1}{2}\text{M}\dot{z}^2$.

Thus, the total energy of the system is

$$\frac{1}{2}\left(\text{M}+\frac{m}{3}\right)\dot{z}^2 + \frac{\pi r^4 n\dot{z}^2}{4l\text{R}^2} - \left(\text{M}+\frac{m}{2}\right)g\dot{z} = \text{const.}$$

Differentiating and dividing by $\dot{z}$, we obtain

$$\left(\text{M}+\frac{m}{3}\right)\ddot{z} + \frac{\pi r^4 n\dot{z}}{2l\text{R}^2} - \left(\text{M}+\frac{m}{2}\right)g = 0.$$

If we substitute* $q = \dot{z} - \frac{2l\text{R}^2}{\pi r^4 n}\left(\text{M}+\frac{m}{2}\right)g,$

the equation reduces to

$$\left(\text{M}+\frac{m}{3}\right)\ddot{q} + \frac{\pi r^4 n}{2l\text{R}^2}q = 0,$$

and thus the period of oscillation is (p. 47)

$$T = 2\pi\sqrt{\frac{\left(M + \frac{m}{3}\right) . 2lR^2}{\pi r^4 n}}. \qquad ...(20)$$

The effect of the mass of the spring is therefore the same as though M were at the end of a massless spring together with a load equal to one-third the mass of the spring.

Experimental Details

A flat spiral spring is chosen, having a radius R which is fairly large compared with the radius of the wire. One end is clamped firmly in a heavy retort stand; a mass is attached to the lower end of the spring, and the time of vibration of the vertical oscillations is obtained by timing fifty vibrations with a stop-watch. This is repeated for various loads.

l, the length of the wire in the spring, may be obtained from a knowledge of N, the total number of turns in the spiral. If there are exactly N turns, the length is $2\pi NR$, where R is the average value of several observations of the mean radius of the spiral, i.e. the value of the outside radius of the wire spring, minus the radius of the wire of which it is made.

r occurs in the fourth power, so should be measured with extreme care. A number of values are obtained with a screw gauge at points along the length of the spring, and the mean value taken.

l being equal to $2\pi RN$, we have from equation (20)

$$n = \frac{16\pi^2 R^3 N}{r^4} \cdot \frac{M + \frac{m}{3}}{T^2}. \qquad ...(21)$$

The results of the several experiments may be conveniently tabulated in the form shown.

M	$\left(M + \frac{m}{3}\right)$	T	T^2	$\frac{M + m/3}{T^2}$

$$\text{Mean value of } \frac{M + \frac{m}{3}}{T^2} =$$

The mean value of $\dfrac{M + \dfrac{m}{3}}{T^2}$ for the series of loads (M) taken is obtained and substituted in equation (20), thus giving the mean value of n for the complete set of observations.

A graph should be constructed of T^2 against M. The intercept on the axis of M is equal to $\dfrac{m}{3}$.

Determination of Young's Modulus of the Material of a Spring

The value of Young's modulus for the material of the wire of which a spring is made may be obtained by supporting the spring vertically and allowing a bar, which is firmly fastened to the lower end, to perform horizontal swings as in the case of Maxwell's needle (p. 88).

It is shown below that in such a case the time of a complete horizontal oscillation is

$$T = 2\pi\sqrt{\frac{8RNI}{r^4 Y}},$$

where Y is Young's modulus for the wire,
R is the radius of the spiral, measured to the centre of the wire,
r the radius of the wire,
I the moment of inertia of the bar about the axis of suspension,
N the total number of turns in the spring.

The above assumes, as in the last experiment, that the spring is a flat spiral; the layers are very close together, and each may therefore be regarded as horizontal.

Let fig. 53 represent a horizontal section of a small length s of the spring, with NS the section of the neutral surface by the plane of the diagram. This neutral surface will be normal to the diagram. ABCD is the section of the element of the spring when the bar is in an undisturbed position.

When the bar is turned in a horizontal plane through any angle ψ, the section taken, in common with every other element of the spring, will become more curved as shown by the broken lines.

Consider a single filament of the material, as was done in the case of the bending of beams (p. 72). If a is the area of cross-section of the filament, and f is the force required to bend the filament through a small angle

$$Y = \frac{\dfrac{f}{a}}{\text{Strain of the filament}}.$$

If R_0 is the normal radius of curvature of the neutral surface, and PQ is a distance x cm. beyond NS, the original length of $PQ = (R_0 + x)\theta_0$, where θ_0 is the angle between the radii from the original centre of curvature O to the ends AC and BD of the element considered.

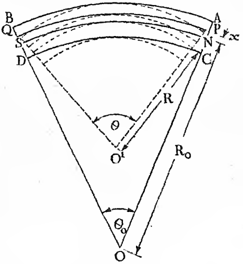


Fig. 53

Now, due to stresses similar to f, the element has a greater curvature, and consequently the neutral surface has a smaller radius of curvature R. If O^1 is the new centre of curvature, the new length of PQ is $(R + x)\theta$.

However, the neutral surface is of the same length as before, so that $R_0\theta_0 = R\theta = s$.

The strain is therefore

$$(R + x)\,\theta - (R_0 + x)\,\theta_0 \div \text{original length} = \frac{x\,(\theta - \theta_0)}{s}$$

$$= x\left(\frac{\theta}{R\theta} - \frac{\theta_0}{R_0\theta_0}\right) = x\left(\frac{1}{R} - \frac{1}{R_0}\right).$$

$$\therefore\ Y = \frac{\dfrac{f}{a}}{x\left(\dfrac{1}{R} - \dfrac{1}{R_0}\right)},$$

$$f = Yax\left(\frac{1}{R} - \frac{1}{R_0}\right).$$

The moment of this force on the filament about the neutral surface is fx. The total moment of the couple acting on the element considered is therefore

$$\Sigma fx = \mathrm{Y}ax^2\left(\frac{1}{\mathrm{R}} - \frac{1}{\mathrm{R}_0}\right)$$

$$= \mathrm{Y}i\left(\frac{1}{\mathrm{R}} - \frac{1}{\mathrm{R}_0}\right).$$

If the angles subtended at the centre by 1 cm. of the wire be φ and φ_0, respectively, the above couple is

$$\mathrm{C} = \mathrm{Y}i\,(\varphi - \varphi_0).$$

If l is the length of the wire in the spring, the total change of angle, i.e. the angle through which the inertia bar moves is $\psi = l\,(\varphi - \varphi_0)$,

therefore
$$\mathrm{C} = \mathrm{Y}i\,\frac{\psi}{l}.$$

This is the value of the couple due to the internal stresses, called into play by the strains in the wire, so that

$$\mathrm{I}\ddot{\psi} = -\frac{\mathrm{Y}i}{l}\,\psi.$$

The periodic time T is given by (p. 47)

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}l}{\mathrm{Y}i}},$$

and
$$i = \frac{\pi r^4}{4}, \quad l = 2\pi\mathrm{RN}.$$

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{8\mathrm{INR}}{r^4\mathrm{Y}}}$$

or
$$\mathrm{Y} = \frac{32\pi^2\mathrm{INR}}{r^4\mathrm{T}^2}.$$

Experimental Details

The spring is set up as in the last experiment, and a rod, say, of rectangular cross-section is clamped to the end of it, so that there is no free play between the end of the spring and the rod, and the centre of gravity of the rod is under the suspension.

The rod is then given a displacement in the horizontal plane, and the subsequent horizontal oscillations are timed. T is obtained by timing fifty complete swings in the usual way.

r and R are carefully measured as previously described, and the total number of turns of wire in the spring, N, is counted.

The value of I may be calculated from a knowledge of the dimensions and the mass of the bar (see p. 38).

Determination of the Modulus of Rigidity and Young's Modulus for the Material of a Wire by Searle's Method

To find Y, Young's modulus, and n, the coefficient of rigidity, of the material of a wire specimen by this method, the wire is fastened to two identical rods, A_1B_1, A_2B_2, at their mid-points, as shown in the diagram at C_1, C_2. These rods are usually square or circular in cross-section, and are supported by threads from points T_1 and T_2, such that the axis of suspension and the axis of the wire intersect at the centre of gravity, and the suspended rods are a distance apart, which allows the wire to be stretched in a straight line, the whole assuming an H formation.

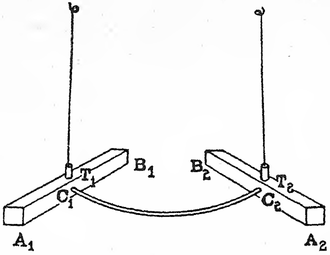


FIG. 54

If now the ends, B_1 and B_2, of the rods are drawn together and fastened by a loop of thin cotton, the wire will be bent into the arc of a circle subtending an angle 2φ at the centre; each rod will make an angle φ with its original direction.

The suspension of the two rods being such that the torsion is negligible, the only couple acting on the bars is that due to the bending of the specimen, C_1C_2. With this method of suspension we have a couple produced equal to $\frac{iY}{R}$ in the wire (p. 73), where i is the 'moment of inertia of cross-section', and R is the radius of curvature of the arc C_1C_2.

If l is the length of the wire between the rods,

$$R = \frac{l}{2\varphi}.$$

If r is the radius of the wire (assumed circular),

$$i = \frac{\pi r^4}{4},$$

whence the couple acting is

$$Y\frac{\pi r^4}{4}\cdot\frac{2\varphi}{l}=\frac{\pi Y r^4\varphi}{2l}.$$

Let I_1 be the moment of inertia of the rod, A_1B_1, about the axis of suspension, $\frac{d^2\varphi}{dt^2}$ the angular acceleration. The moment of the external force is, by the theorem given on p. 37, $I\frac{d^2\varphi}{dt^2}$.

This is equal and opposite to the restoring couple exerted by the bent wire. The equation of motion for the rod is therefore

$$I\frac{d^2\varphi}{dt^2}=-\frac{\pi Y r^4}{2l}\cdot\varphi.$$

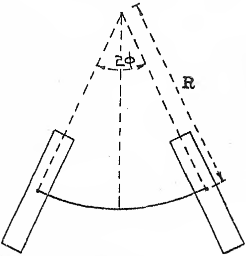


FIG. 55

This is simple harmonic motion (p. 47), and

$$T=2\pi\sqrt{\frac{2lI}{\pi Y r^4}}.$$

If the rod A_1B_1 has a length 2L and width $2a$ and mass M gm.,

$$I=M\frac{L^2+a^2}{3},$$

whence

$$Y=\frac{8\pi l}{r^4T^2}\,\frac{L^2+a^2}{3}\,M.$$

To evaluate Y for a specimen of wire, the arrangement described is fitted up. The thin cotton loop holding the rods together at B_1B_2 is burnt, and the resulting oscillations are timed with a stop-watch, T being obtained by timing fifty complete oscillations. An alternative method of finding T, and which may be used in many experiments where T is required, is illustrated in the table below.

A reference point having been chosen—say, a chalk-mark under the position of rest of the oscillating bar—the time is noted at which the counting of the swing is commenced. After five swings the time is again noted, and the times at the end of every five are recorded in a table, as seen below.

(1) No. of oscillations	(2) Time	(3) No. of oscillations	(4) Time	(5) Time for 30 oscillations
0	u	30	a	$a - u$
5	v	35	b	$b - v$
10	w	40	c	$c - w$
15	x	45	d	$d - x$
20	y	50	e	$e - y$
25	z	55	f	$f - z$

Mean time for 30 oscillations =

Tabulating in the second column the time for the starting-point, 5, 10, 15, 20, 25 oscillations, and in the fourth column the time for the 30, 35, 40, 45, 50, 55 oscillations, the difference in any line between the fourth and second column value will give the time for 30 complete oscillations. The mean value of the last column being taken as the time for 30, T may be evaluated.

The radius of the wire occurs in the fourth power. The value of the diameter is therefore measured in several places, say, six, and the mean radius, r, calculated. The other measurements for the evaluation of Y are straightforward.

It will be noticed that in this method the value of Y is obtained by timing, and not by observing deflections as in some of the other methods. Another advantage of the method is that it requires only a small specimen of the material.

To find the modulus of rigidity of the specimen, the rod A_2B_2 is clamped rigidly in a horizontal plane and the wire acts as a support for the rod A_1B_1, which may therefore be made to perform horizontal torsional oscillations. The rod is displaced slightly from its position of rest, causing a restoring couple to act on it, due to the twist of the wire.

For a displacement θ, the restoring couple is

$$\frac{\pi n r^4}{2l}\,\theta$$

as established on p. 86.

We therefore have as the equation of motion of the rod

$$\mathrm{I}\frac{d^2\theta}{dt^2} = -\frac{\pi n r^4}{2l}\cdot\theta.$$

This is once more a case of simple harmonic motion, whose periodic time, T_1, is given by (p. 47)

$$T_1 = 2\pi\sqrt{\frac{2lI}{\pi n^4 r}}.$$

The moment of inertia may be obtained by calculation as before.

T_1 may be found as described for the first experiment; the length l and the radius r are already known.

Hence n may be found by substitution in the formula:

$$n = \frac{8\pi l I}{T_1{}^2 r^4}.$$

Determination of the Bulk Modulus for Glass

The bulk modulus, as shown on p. 68, may be expressed as

$$K = \frac{p}{\frac{\delta V}{V}},$$

where p is an increase in pressure causing the small volume change δV.

In general, it will not be convenient to apply a uniform pressure to a body, but it is a simple matter in most cases to apply an extending force per unit area, or a pressure, in one direction only. If, as a result of such a pressure, the change in volume is $\delta V'$, then $\delta V' = \frac{1}{3}\delta V$, where δV is the change in volume when the same pressure is applied uniformly over the body.

If a cylinder be supported at one end, and an extending force be applied to the other, in a direction coincident with the axis of the cylinder, we have the extending force at one end and the equal reaction at the other. The change in volume in this case is of the type $\delta V'$ above. So that if the change in volume of a body be observed in such a case, the value of K may be calculated.

The apparatus used to determine the bulk modulus of, say, glass is seen in fig. 56. A glass tube g is cemented to two caps of brass, A and B. The upper brass cap A is provided with two pegs, which act as a support for the whole apparatus. The lower cap B carries a hook to which a pan S is attached.

Fitting in the upper end of the tube is a rubber cork, carrying a capillary tube, CD, which is graduated and calibrated in the manner described on p. 24.

The tube g is filled with water. Care is taken to avoid trapping air, and the cork, etc., are placed in position, resulting in a little water rising in the capillary tube, i.e. the whole of the glass tube and part of the capillary are filled with water to a definite level. If now the pan S is loaded with, say, 5 kg. the volume of the glass will increase by a small amount. The difference in the levels on the calibrated tube

enables the value of the volume change to be determined if the temperature remains constant throughout the observation.

However, the apparatus is, by the construction, very much affected by small temperature changes; it is a water thermometer with a very large bulb. To obtain a good approximation to the value of $\delta V'$, the

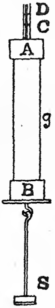


FIG. 56

volume change, the reading on the tube is noted when S has no load; 5 kg. are applied and the reading noted, the load is removed, and once more the scale reading is taken and the process repeated with 5 kg. and no load for ten observations, each one being taken at a regular time interval as below.

Time from commencement	Load	Reading	Time from commencement	Load	Reading
0 min.	0 kg.		5 min.	0 kg.	
$\frac{1}{2}$	5		$5\frac{1}{2}$	10	
1	0		6	0	
$1\frac{1}{2}$	5		$6\frac{1}{2}$	10	
2	0		7	0	
$2\frac{1}{2}$	5		$7\frac{1}{2}$	10	
3	0		8	0	
$3\frac{1}{2}$	5		$8\frac{1}{2}$	10	
4	0		9	0	
$4\frac{1}{2}$	5		$9\frac{1}{2}$	10	
			etc.		

CHAPTER V

SURFACE TENSION

The surface of a liquid acts in many respects in a manner analogous to a stretched membrane. The well-known example of mercury resting on a clean wooden surface shows the effect to a marked degree. The mercury takes the form of a globule, as if it were surrounded by a membrane supporting it in this form.

The examination of a water drop slowly formed at the end of a glass tube or tap from which it emerges, provides another example of this phenomenon. The water in this case accumulates, as though it were collected in an invisible membrane, until of a definite size, when it is detached as a spherical drop.

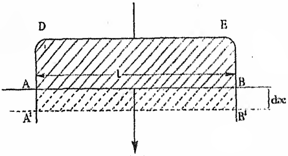


FIG. 58

These effects are due to forces existing in the surface of separation of the liquid from the air or other media in contact with it. The effect is generally known as *surface tension*.

If we imagine the surface of the liquid to be cut by a plane, there is a definite force per cm. acting on the line of intersection, at right angles to its length and parallel to the surface. This force per cm., is defined as the surface tension of the liquid. The value of the surface tension depends on the liquid and the surrounding medium, but, unless otherwise stated, will be taken throughout this book to represent the force per cm. when air is the medium.

For example, consider a frame ADEB, fig. 58, of width l cm., across which a film of liquid, whose surface tension is T dynes per cm., is stretched.

If the film terminates at the lower end on a light rod AB, since the liquid film has two surfaces, it will exert an upward force of $2Tl$ dynes on the rod. If the rod has a mass of m gm. it will be in equilibrium when $2Tl = mg$.

When a liquid is placed on a horizontal plane surface, the form it

takes depends, for a given liquid, on the material of which the plane surface is made. Thus, if water is placed on a clean glass surface it spreads over it, whereas if the glass is greasy the water takes the form of globules.

The angle contained between the plane surface and the liquid surface is different in each case. If we measure this angle in the liquid we have a measure of the *angle of contact.* Thus, in fig. 73, showing a section of a mercury drop on a glass surface, θ is the angle of contact, whereas for a liquid like water which 'wets' the glass the angle of contact is zero—the water spreads over the surface.

METHODS OF MEASURING THE SURFACE TENSION OF A LIQUID

(1) Wilhelmy's Method

To determine the approximate value of the surface tension of such liquids as water, paraffin oil, or turpentine, the following method may be employed.

A clean wire, preferably platinum, is bent into the form shown in fig. 59, making three sides of a rectangle of breadth l cm.

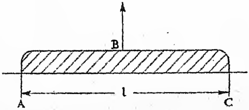


FIG. 59

It is then suspended from a beam of a balance by means of a thin wire, and counterpoised when the upper horizontal arm, B, of the frame is almost immersed in the liquid.

When a balance is obtained the frame is dipped under the surface of the liquid by lowering the beam of the balance, and then withdrawn again. A film of the liquid will be formed in the frame as seen in the shaded part of fig. 59.

Due to the downward surface tension on each side of the film there will be an apparent increase in weight $= 2l\mathrm{T}$. This can be found experimentally by adding weights to the other scale pan, until on raising the beam it remains horizontal.

Clean the frame by holding it in a Bunsen flame until red-hot, and repeat the experiment as described above several times. Take the mean value, m, of the added mass, whence

$$2\mathrm{T}l = mg,$$

$$\mathrm{T} = \frac{mg}{2l} \text{ dynes per cm.}$$

Take care that the horizontal part of the frame is at the same height above the water surface when the balance is made, before and after immersing in water, to eliminate buoyancy errors, and the effects of surface tension on the vertical limbs.

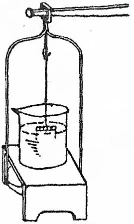


FIG. 60

Another way of carrying out this measurement is by means of the apparatus illustrated in fig. 61 (*a*).

A spring AB is stretched across a wooden stand and passes through a light lever LL to which it is soldered. In this way the lever, when carrying a frame F and a weight W, is approximately horizontal. One end of the lever is pointed and a pin-point carried on a stand is brought near to this end so that its position may be marked or the point is focused by means of a microscope provided with cross wires.

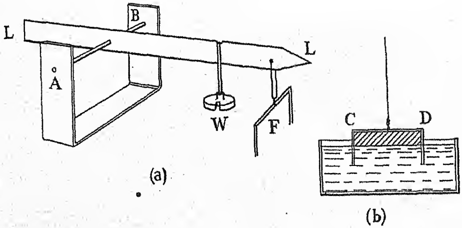


FIG. 61

The liquid of which the surface tension is required is brought up below the frame F and a film is made across it. The substance may conveniently be a soap solution which easily forms a film when the

frame is immersed to the horizontal bar and the dish containing the solution is lowered (fig. 61 (*b*)). The effect of the surface tension is to draw the lever down. The weight W, usually about 1 gm., is attached to a loop of cotton and can slide to various positions along the lever. It is placed so that with film formed, the lever is nearly horizontal. The film is then broken by placing a hot needle through it when the equilibrium is disturbed and it is necessary to slide W into a new position until the former position of the pointer is regained. This is observed to be the case by means of the pin-point or microscope.

In this way the movement of the weight exactly makes up for the disappearance of the film, the depth of immersion of the sides of the frame and the effect of surface tension on the outside of the vertical portions remaining the same.

Mechanically, this means that the moment on the lever due to surface tension in the film is equal to the change in moment due to the movement of the weight.

Let the distance between the point of suspension of the frame and the point where the wire passes through the lever be d. Let the width of the frame be l, i.e. $CD = l$, and let the surface tension be T. Then, accounting for both sides of the film, the moment due to the film is $2Tld$. If the weight is displaced through a distance a, the change in moment is Wa, so that

$$2Tld = Wa$$

and T can be found from the formula.

(2) By Weighing Drops

The liquid whose surface tension is to be measured is allowed to form drops at the end of a narrow tube, C, fig. 62. If m is the mass of the drop, we have

$$mg = KT,$$

where K is a constant.

From a simple approximate investigation of the case given below we see that $mg = \pi r T$.

When the drop is about to break away from the tube we will assume it has the cylindrical form shown in the firm lines, D. The broken line indicates one of the subsequent forms.

If r is the radius of the orifice, we have, at this level inside the drop an excess of pressure equal to $\frac{T}{r}$ due to the cylindrical curvature of the liquid surface, producing a downward *force* $\frac{T}{r} \cdot \pi r^2 = T\pi r$.

The weight of the drop being mg, the total downward force is

This is equal to the upward force due to surface tension over the circle of contact, i.e. $T \cdot 2\pi r$,

or $$T 2\pi r = \pi T r + mg,$$

or $$T\pi r = mg.$$

This is deduced, assuming static conditions to hold in the actual case. To a closer approximation Lord Rayleigh showed that

$$mg = 3{\cdot}8 r T.$$

In practice the expression $T = \frac{mg}{K}$ should be assumed and the value of K found for the particular tube used.

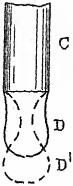


FIG. 62

A tube is drawn out to a fine capillary of about 0·5 mm. at the end. It is connected to a burette by means of a short length of rubber tubing. The burette and the tube are of course first made thoroughly clean, as in all determinations of surface tension, by the method given on p. 114. A liquid of known surface tension, say water, is placed in the burette and the tap is opened to an extent which allows the drops to form at the rate of about one every second. A large number, say 100 or more, are collected in a weighed vessel and the mass per drop is obtained, hence, knowing T, K may be calculated. The same tube is now used for the liquid of unknown surface tension. Some of the liquid is run through the apparatus to remove all traces of water, and the burette is then filled with an uncontaminated sample of oil. The collecting and weighing is repeated, and knowing K, T is calculated.

Find by the above means the surface tension of alcohol, paraffin oil, benzine, or any similar liquid.

(3) Determination of the Value of the Surface Tension of a Solution in the Form of a Film

The value of the surface tension can be obtained for a soap solution film in the following way.

Two pieces of copper wire, ABC and DEF, are bent as seen in figs.

63 and 64, so that AC = DF = about 4 or 5 cm. At the points A, C, D, and F a length of cotton thread is fastened, so that when the whole arrangement is suspended at B the thread takes the form of a rectangle about 5 cm. by 10 cm. (fig. 63).

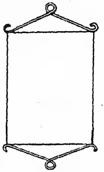


FIG. 63

If now a film of soap solution be stretched across the thread, the latter will be pulled into the form shown in fig. 64, AD and CF taking the form of arcs of circles.

Let AC = DF = $2a$ cm., GH = $2b$ cm., AD = CF = $2h$ cm., and the mass of the lower copper wire, DEF = m gm.

Let T be the value of the surface tension in dynes per cm., f the tension in dynes in the thread, and α the angle included between the horizontal and the threads as shown in fig. 64.

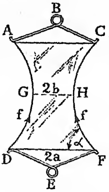


FIG. 64

The film being two-sided, the vertical force upwards due to surface tension on the length $2a$ at the bottom of the film $= 4a$T. The resolved part of the tension f at D is f in α upwards, i.e. total upward force of $2f \sin \alpha$.

The downward force is mg, the weight of the lower copper wire. We thus have for equilibrium:

$$mg = 2f \sin \alpha + 4a\text{T}$$

Considering the equilibrium of half of the film as obtained by a vertical dividing line through the mid-point of the film, we have the following equal and opposite forces acting:

(a) a force $f \cos \alpha$ at the top and an equal one at the bottom

$$= 2f \cos \alpha.$$

(b) due to the two-sided film a surface tension effect

$$= 4Th,$$

i.e.

$$2f \cos \alpha = 4Th. \qquad \ldots(2)$$

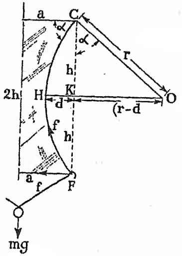


FIG. 65

From (1) and (2) we have

$$\tan \alpha = \frac{mg - 4aT}{4Th}. \qquad \ldots(3)$$

If O is the centre of the circle of which CHF is arc, $\alpha = \angle$ OCK (fig. 65).

So that $\qquad \tan \alpha = \frac{KO}{CK},$

and if $HK = d = a - b$,

$$\tan \alpha = \frac{r - d}{h}.$$

Eliminating r since $\qquad h^2 = (2r - d)\, d,$

$$= \frac{h^2 + d^2}{2d},$$

we have $\qquad \tan \alpha = \frac{h^2 - d^2}{2dh}.$

Equating this to the value given in (3), we get

$$\frac{mg - 4aT}{4Th} = \frac{h^2 - d^2}{2dh},$$

$$2d\,(mg) = 4T\,(h^2 - d^2) + 4aT \,.\, 2d,$$

whence
$$T = \frac{mgd}{2(h^2 - d^2 + 2ad)}.$$

Putting $d = (a - b)$,
$$T = \frac{mg}{2\left(a + b + \dfrac{h^2}{a - b}\right)}. \qquad \ldots(4)$$

Using the very simple apparatus described, a fairly accurate value of T may be obtained simply from a knowledge of a, b, h, and m.

A film is stretched across the string very easily by placing the frame horizontally in a flat dish containing the soap solution.

Care must be taken to avoid excess of soap solution spreading over the part of the apparatus other than the thread. None must be allowed to remain on the lower copper wire.

The measurement of the dimensions of the film can be made by means of ordinary dividers. The method does not justify the use of a travelling microscope. The measurements should be taken fairly rapidly so that they are all obtained before the condition of the film changes appreciably.

In an example $2a = 3{\cdot}5$ cm., $2b = 1{\cdot}2$ cm., $2h = 11{\cdot}7$ cm., $m = 1{\cdot}51$ gm., which gives $T = 23$ dynes per cm.

(4) By Measurement of the Rise of a Liquid in a Capillary Tube

If a clean, fine-bored capillary tube is depressed into a liquid which 'wets' it, and is then clamped vertically, the lower end of the tube being just below the surface of the liquid, it will be found that a column of the liquid remains in the tube, so that the surface in the latter is a height h cm. above the free surface of the liquid in the vessel which contains it.

Suppose r is the radius of the tube and ρ the density of the liquid.

The forces acting on the liquid in the tube are:

(1) *The weight of the liquid.* This is equal to (the volume of the liquid) $\times\, g\rho$.

Now the volume of the liquid is equal to $V = \pi r^2 h +$ (volume of the meniscus) for a uniform tube. If r is small the meniscus is practically hemispherical, hence
$$V = \pi r^2 h + \{(\pi r^2)\, r - \tfrac{2}{3}\pi r^3\}$$
$$= \pi r^2\left(h + \frac{r}{3}\right),$$
i.e. the downward force is
$$\pi r^2\left(h + \frac{r}{3}\right) g\rho.$$

(2) *The upward force due to surface tension.* The line of contact is the intersection of the glass wall and the liquid surface, i.e. a circle of radius r. If α is the angle of contact the upward force, from the definition of surface tension, is equal to

$$2\pi r \,.\, \mathrm{T} \cos \alpha. \qquad \ldots(6)$$

But in the practical cases considered, $\alpha = 0$. Hence for equilibrium, the forces (5) and (6) are equal and opposite,

i.e.
$$2\pi r\mathrm{T} = \pi r^2 \left(h + \frac{r}{3}\right) g\rho,$$

and
$$\mathrm{T} = \frac{g\rho r}{2}\left(h + \frac{r}{3}\right). \qquad \ldots(7)$$

To find T a glass capillary tube is cleaned thoroughly. This may be done by using nitric acid (in which the tube is boiled) and caustic soda; the tube is washed in tap water and dried in alcohol and ether, or the glass tube is allowed to stand for several hours, overnight if possible, in a concentrated solution of sulphuric acid (one part) and potassium bichromate (one part). It is then washed in tap water and dried. It is not advisable to use distilled water.

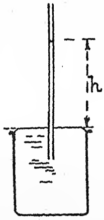


FIG. 66

The tube is then heated and drawn out to a capillary. A length of uniform bore is chosen and clamped vertically in a vessel which is brimful of the liquid whose surface is just above the top of the containing vessel, as shown in fig. 66. Care is taken to avoid touching the tube or the liquid in this adjustment, for even small traces of grease cause a large variation in the value of the surface tension.

The tube is viewed by means of a travelling microscope, provided with a vertical traverse, the lower end of the meniscus is focused and the vernier reading of the microscope noted. The free surface of the liquid in the containing vessel is next focused. If the liquid surface is just above the top of the vessel, this level may be viewed very readily,

and from the vernier reading on the microscope scale in this position the value of h may be obtained.

To find r, the tube is broken at the point at which the meniscus rested, and viewed horizontally by the microscope.

By arranging the cross-hairs in the eyepiece to be tangential in turn to the two ends of a diameter, the internal radius may be measured on the vernier attached to the traverse.

Alternatively, a weighed amount of mercury of density D may be introduced into the tube and its length observed by means of the microscope, when the tube is horizontally on the bed of the microscope.

Hence r, the mean radius of the tube, may be found. If l is the length occupied by the mercury of mass m,

$$\pi r^2 l D = m.$$

The experiment is repeated, using tubes of various diameters, and a mean value of T obtained.

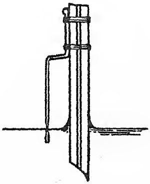


FIG. 67

An alternative method of measuring h, which also overcomes the difficulty of viewing the liquid through the glass beaker, is one which makes use of a pin bent twice at right angles so that the point is displaced about 1 cm. from its original position. The pin is attached to the capillary tube by means of a rubber band (see fig. 67). The point is in this way well removed from the curved surface of the liquid round the tube itself. It is adjusted to coincide with the free surface of the liquid.

The meniscus is first viewed and the reading on the scale of the microscope is noted. The vessel containing the liquid is removed and the microscope is then focused on the pin-point. The difference of the readings obtained gives the value of h.

(b) Another way of carrying out the capillary tube method which has some advantages over the former can be made by using the apparatus illustrated in fig. 77, p. 125. The U-tube and beaker are filled with the liquid ...

by a microscope examination so that its end is circular. It is connected to the rest of the apparatus by means of a rubber tube which fits on to the manometer at A. A bent pin is attached to the tube, as in fig. 67. This serves to mark the surface of the liquid in the beaker and the distance between the pin-point and the end of the tube, denoted by h, measures the depth of the end below the surface. The pressure within the tube is measured by the difference in levels, H, of the liquid in the limbs of the U-tube. By variation of the level of the water in the flask B_1 the liquid in the capillary tube can be forced down until the level of the meniscus is that at the end of the tube. This has the advantage that the radius of the tube can be measured at the section where the surface tension acts, and it is not necessary to rely on a measurement of the average radius as in the last method. The radius should be carefully measured by means of an accurate microscope.

The forces downward are $\rho g H \pi r^2 + \frac{2}{3}\rho g \pi r^3$ and upward $\rho g h \pi r^2 + 2\pi r T$, omitting the effect of atmospheric pressure, which acts equally in both directions.

Thus $$T = \tfrac{1}{2}\rho g r\,(H - h + \tfrac{2}{3}r).$$

The term $\frac{r}{3}$ may be neglected if r is so small with respect to $(H - h)$ that the accuracy is not thereby affected.

It should be noted that a considerable error will arise if H and h do not differ appreciably. This difficulty always occurs when a difference of two measured quantities occurs in a formula.

(5) Ferguson's Method

This is a method especially applicable when only a small quantity of the liquid is available. By means of it an accurate measurement of the surface tension can be made with the use of 1 c.c. or even less of the liquid.

The figure represents, on the left, a piece of capillary tubing into which the liquid is introduced. It should be about 1 mm. in diameter. It is of importance that the effect of gravitation on the liquid should be negligible. Ferguson and Kennedy have shown that this may be safely assumed up to a diameter of 1 mm.

The end, A, of the tube is ground flat to a matt finish. By raising or lowering the pressure flask, F, the liquid in the capillary tube can be moved along it. The difference in pressure on the two sides of the liquid can be measured by means of the manometer, M. The liquid in the manometer should consist of a light oil or aniline. This use of a light liquid is one of the characteristic advantages of this method of measuring surface tension. In the usual capillary method the pressure difference is measured by means of a column of the liquid under

investigation which acts as its own manometer. In the present case an independent liquid of low density can be chosen for the manometer with the advantage that a considerable length of it is measured.

The method is applied in the case of liquids which have an angle of contact with glass equal to zero. The number of such liquids is great, so that the method has a wide application.

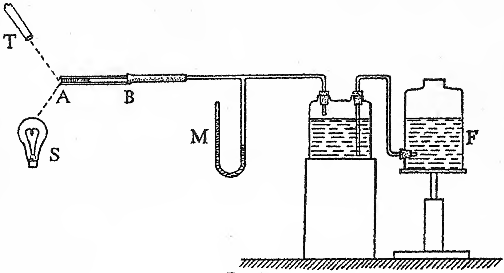


Fig. 68

In fig. 69 the liquid is represented by AB, with the surface at A flat. The pressure at A is equal to that at B, both being at atmospheric pressure. The pressure at C is in excess of that at B by the amount $\frac{2T}{a}$, where T denotes the surface tension and a the radius of the hemispherical end of the liquid. This radius is equal to the radius of the tube.

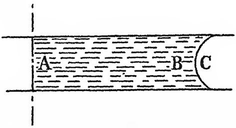


Fig. 69

This excess is measured by the difference in the height of the columns in the manometer.

Thus, if ρ denotes the density of the liquid in the manometer and h the difference of height of the columns,

$$\frac{2T}{a} = g\rho h.$$

The method depends on setting the pressure flask so that the surface at A is flat, which means that the liquid at A and the ground glass surface at the end of the tube form a single plane. This is necessary in order that the above equation shall apply.

By reflecting the light from a small lamp, S, by the end of the tube and observing the surface by means of a short focus telescope, T, this setting may be made accurately. The flask, F, should be raised and lowered about the critical position, when it will be found possible to obtain the condition required with a considerable degree of accuracy.

(6) Surface Tension Determination from Measurements of Bubbles

This method is suitable for measurements of surface tension of soap solutions.

Inside any curved film in equilibrium there is an excess of pressure over the outside by an amount which depends on T, the surface tension, and R, the radius of curvature.

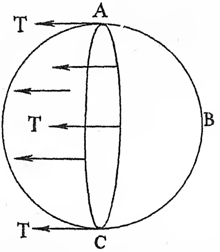


FIG. 70

Consider the equilibrium of the hemisphere ABC of a bubble (fig. 70). The forces are due to the excess pressure, p, inside and to the surface tension T dynes per cm. round the circle AC. Allowing for the two sides of the film the forces from right to left are $4\pi RT$, due to surface tension, and $\pi R^2 p$ left to right, due to the excess pressure inside the bubble.

For equilibrium these are equal and opposite,

i.e.
$$4\pi TR = \pi R^2 p,$$
$$p = \frac{4T}{R}.$$

A suitable form of apparatus with which to obtain a measurement of p and R is seen in fig. 71.

This consists of a fairly wide U-tube, containing water, sealed to a T-piece, T. One arm of the T-piece is bent at right angles and terminates at B, where AB is parallel to the U-tube limbs. To the other arm at C is attached a piece of rubber tube, R, having a glass rod, P, which just fits it, and which can act as a piston.

The piston P is withdrawn and the end B immersed in the soap solution; then P is advanced slightly, causing a bubble to form at B. The excess of soap solution is drained off this bubble by touching the

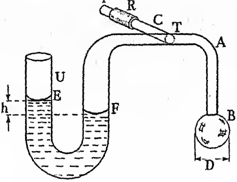


FIG. 71

drop at the bottom with a pencil, or with the side of the dish which contains the solution. The rod is then further advanced, so that a bubble of, say, 1 to 2 cm. diameter is blown. It will be noticed that the water in the U-tube takes up a position as in the diagram indicating a pressure inside the bubble in excess of the atmospheric pressure.

This difference in level may be measured, using a microscope with horizontal and vertical traverse.

Having read this, the microscope is moved until the bubble is focused, first the image of one side, and then the other being brought into coincidence with the cross-hair. The difference in reading of the vernier of the horizontal traverse giving the diameter, D, of the bubble, $p = hg$ dynes per sq. cm., for a *water* manometer.

$$R = \frac{D}{2}.$$

Hence
$$T = \frac{hgD}{8}.$$

(7) Quincke's Method

This method is usually used for the determination of the surface tension of mercury. A large flat drop is formed on a horizontal platform, and from its dimensions, as measured by means of a travelling microscope, the value of T, the surface tension, and θ, the angle of contact of mercury on the ...

We will assume that the radius of the drop, R, fig. 72, is large, so that the drop is flat, i.e. the pressure at a point just above and just below the upper surface is the same.

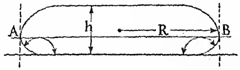


Fig. 72

Consider a section, as in fig. 73, obtained by cutting it by a vertical plane passing through the centre, LMRN, and two parallel vertical planes, AINL and ESRM normal to LMRN. Let ACDE be the horizontal plane of maximum area in the drop, so that a tangent plane at AE is vertical.

By considering the forces on the upper portion, AECDLM, which is in equilibrium, a value of T may be obtained in terms of the dimensions.

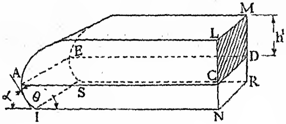


Fig. 73

The horizontal forces acting on this portion of the drop due to the remainder of the drop are

(α) T . LM, left to right,

(β) The hydrostatic pressure over the surface LMDC, from right to left varies from 0 at the upper surface to $g\rho h^1$ at CD, and is on the average $\frac{g\rho h^1}{2}$, so that the total force is $\frac{g\rho h^1}{2}$ × area LMDC, where ρ is the density of mercury.

Equating (α) and (β)

$$T \, . \, LM = \frac{g\rho h^1}{2} \, . \, LMh^1,$$

or

$$T = \frac{g\rho h^{1^2}}{2}. \qquad \ldots(8)$$

When R is not sufficiently large for the surface tension effect of this curvature to be neglected, R. S. Burdon (*Surface Tension and the Spreading of Liquids*) suggests the following amendment to equation (8):

$$T = \frac{1}{2} g\rho h^{1^2} \frac{1{\cdot}641R}{1{\cdot}641R + h^1}.$$

This will be found to make the results more consistent in the smaller drops. Considering now the equilibrium of the whole slab, we have in addition to the corresponding terms, T . LM and $\frac{g\rho h^2}{2}$, where h is now the total height (fig. 72), a term T . IS . $\cos\alpha = -$ T . LM $\cos\theta$ (fig. 73), from left to right,

or
$$T \,.\, LM\,(1 - \cos\theta) = \frac{g\rho h^2}{2} LM,$$

$$2T \sin^2\left(\frac{\theta}{2}\right) = \frac{g\rho h^2}{2},$$

or, substituting the value of T from (8),

$$\sin\frac{\theta}{2} = \frac{h}{\sqrt{2}h^1}. \qquad \ldots(9)$$

The experimental details necessary to carry out such a determination are as follows. A clean glass slab, provided with three levelling screws, is first arranged horizontally, by means of a spirit-level. Mercury is placed on its surface in the form of a circular drop (fig. 74).

FIG. 74

The mercury should be as free as possible from impurities. To ensure this it should, if possible, be submitted to one of the methods of purification described on p. 572.

To obtain the values of h^1 and h, use is made of a travelling microscope, which is provided with cross-hairs. The cross-hairs are arranged so that one is vertical in the eyepiece; this can be done by viewing a thin wire plumb-line. The edge of the drop is then focused. When the vertical cross-wire is tangential to the image of the side of the drop and the intersection of the cross-hairs is at the point of contact, A, of the vertical tangents, figs. 72 and 74, the position on the vernier of the microscope is noted. By making use of the vertical movement the upper and lower surfaces of the drop can be focused and the corresponding vernier reading will enable the values of h^1 and h to be ascertained.

For liquids which wet the surface of the glass a simple modification enables the method to be utilized.

A concave lens of over 1 m. radius of curvature is supported concave side downwards, on three legs inside a glass box, having one plane glass side. The box is filled with the liquid, and an air bubble is blown under the concave

tube is connected to R in place of the U-tube, etc., shown in fig. 77. The slight concave surface enables this air bubble to be blown without much trouble. The section of such a submerged bubble would be similar to that shown in fig. 72, when inverted. Then, if h^1 is the distance between the lower surface of the large air bubble and the plane AB, and if h is the total thickness of the bubble as measured from the lower surface to the plane of contact with the lens, all measured through the plane glass window by means of a microscope, equations (8) and (9) will give T and θ for the liquid in the box.

This method cannot be regarded as accurate for the determination of both the surface tension and the angle of contact. It is difficult to measure h^1 accurately because of the uncertainty in locating the point of contact of the vertical tangent to the drop.

The method may be regarded as suitable for the measurement of the angle of contact when the surface tension is known.

(8) Rayleigh's Method

This method depends on a measurement of the wave-length of ripples formed on the surface of the liquid whose surface tension is to be determined.

The velocity, v, of a harmonic disturbance on the surface of any liquid is given by*

$$v = \sqrt{\frac{\lambda}{2\pi}g + \frac{2\pi T}{\lambda\rho}}, \qquad \ldots(10)$$

when λ is the length of the wave, T the surface tension, ρ the density of the liquid, that is

$$v = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2}{\lambda^2\rho} \cdot T\right)}. \qquad \ldots(11)$$

The surface tension is therefore seen to increase the effective value of g. For long waves the term $\frac{4\pi^2}{\lambda^2\rho} \cdot T$ can be neglected, and the velocity of propagation is $\sqrt{\frac{\lambda g}{2\pi}}$.

For waves of less than 1·5 cm. wave-length, the value of the term involving T becomes more important, and when λ is sufficiently small the velocity becomes more nearly equal to

$$v = \sqrt{\frac{2\pi T}{\lambda\rho}},$$

i.e. the first term in equation (10) becomes negligible in comparison with the second when λ becomes very small.

* See Poynting and Thomson's *Properties of Matter*.

Equation (10) shows that when $\lambda = 0$, $v = \infty$, and when $\lambda = \infty$, $v = \infty$, between these values of λ there is a minimum value of v, corresponding to a value of λ, which is obtained when

$$\frac{\lambda}{2\pi} g = \frac{2\pi T}{\lambda\rho},$$

$$\lambda = 2\pi\sqrt{\frac{T}{g\rho}}; \qquad \ldots(12)$$

this minimum velocity is therefore

$$v_m = \left(\frac{Tg}{\rho}\right)^{\frac{1}{4}}\sqrt{2}.$$

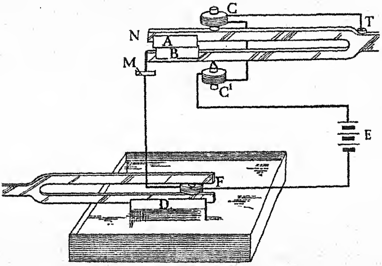


FIG. 75

For water $\lambda_m = 1{\cdot}7$ cm. from equation (11), for $T = 75$, $\rho = 1$, $g = 981$, and hence

$$v_m = 23 \text{ cm. per sec.}$$

Waves having a smaller wave-length than this critical value which corresponds to the minimum velocity are called ripples, and the more important term is the one involving T.

The liquid is placed in a large flat dish, such as a photographic developing dish. As in the case of all experiments for the measurement of surface tension, the dish and everything that comes into contact with the liquid must be thoroughly clean. Ripples are produced in the liquid by means of a dipper attached to a prong of a tuning-fork which is electrically maintained. The dipper may conveniently consist of a thin sheet of aluminium and penetrates the liquid at one end of the dish (fig. 75).

A stroboscopic method is used in order to see the ripples and measure their wave-length. A second tuning-fork of the same frequency as the first is maintained by the

aluminium plates, one attached to each prong, arranged so that when the prongs are wide apart the surface of the liquid can be seen through the slit between the plates. For the remainder of the cycle the plates conceal the surface. In this way, by looking between the prongs of the second fork a short glimpse of the rippled surface is obtained once per cycle. As the ripples have the same frequency as the fork, they appear stationary and their wave-length can be measured.

A long sequence of ripples should be measured and the number of waves in it should be counted. The wave-length, λ, can then be deduced. One way of doing this is to set a pair of dividers so that they contain an integral number of waves.

When performing this experiment with water, the measurement will be found to be somewhat complicated due to shadows cast by the water, and the exact setting of the dividers over the surface for the stretch of waves will not be as easy as in the case of a more opaque liquid.

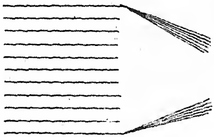


FIG. 76

A method which has been found to give satisfactory results is to suspend an incandescent bulb about 2 m. above the surface of the water. This casts a series of shadows on the bottom of the white porcelain dish. The dividers are arranged also to cast a shadow and adjusted so that the two ends of the dividers' shadow coincides with the corresponding parts of the first and last of the waves.

Now, if n is the frequency of the fork,

$$v = n\lambda = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2}{\lambda^2\rho}\mathrm{T}\right)}$$

or

$$\mathrm{T} = \frac{\lambda^2\rho}{4\pi^2}(2n^2\lambda\pi - g).$$

A suitable experiment to illustrate this method is to find the variation of surface tension of a salt solution with concentration.*

Use first of all pure water. Make sure that the aluminium plate is clean, and that no soft wax, by which it is fastened to the tuning-fork, adheres to it, and so contaminates the water.

Measure the value of λ and calculate T, n being known.

A suitable frequency for the two tuning-forks is about 60.

* Dorsey showed that $\mathrm{T}_n = \mathrm{T} + 1{\cdot}53n$ for sodium chloride solution, where n is the number of gramme equivalents per litre.

Then repeat the experiment with a sodium chloride solution having $\frac{1}{2}$, 1, $1\frac{1}{2}$, to 5 gm. molecules per litre, and plot a graph showing the increase in the value of T with concentration of the solution.

A more convenient way of applying the stroboscopic method is to use a neon lamp controlled by the vibrations of the tuning-fork and to make the measurements in the light of the lamp.

(9) Jaeger's Method

This method is especially to be recommended as a comparative one. It is suitable, for example, for the purpose of studying the variation of surface tension with concentration of a solution of common salt in water.

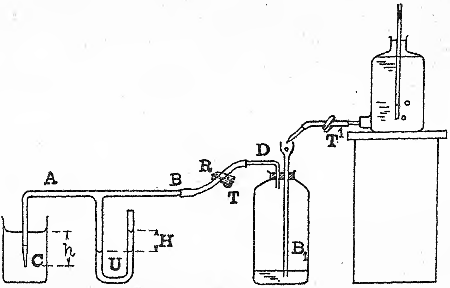


FIG. 77

In order to understand the principle of the method consider fig. 77. This illustrates an apparatus designed to blow bubbles of air through a circular orifice at C, which is placed below the liquid under investigation. If we consider the air bubbles to be blown until they become hemispheres with a diameter equal to that of the exit of the tube and then to break away, the excess pressure within the bubble is p, where

$$p = \frac{2T}{R},$$

where R is the radius of the orifice and the factor 2 occurs here instead of 4 in the case of soap bubbles because there is only one liquid-air surface in the present case.

The actual case is not so simple as this, because before the bubble leaves the tube a neck is formed in it and the formula becomes

$$p \propto \frac{2T}{R}$$

or simply

Some numerical factor must be introduced into the simple formula to take account of the departure from the simple case. For a comparative measurement, all that is necessary is to measure the excess pressure.

The apparatus consists of three parts—the part providing a pressure head, the manometer, and the tube containing the orifice.

In the figure the latter is shown continuous with the manometer, but it is convenient to make this part detachable from the manometer and to make the connexion with rubber tubing at the bend A. The student should take a piece of glass quill tubing and draw it out to form the orifice. He should then examine it to see that it is in the form of a fine circular capillary and should reject the result of his first attempts if they are unsuitable. The diameter of the capillary is not required for the measurements, but it should be carefully measured in

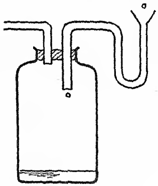


FIG. 78

several places across the orifice in order to verify that the orifice is circular. The travelling microscope may be used for this purpose if the orifice is not too small, but in the case when it is very fine, a microscope carrying a small scale in the eyepiece should be used.

An orifice of 0·2 to 0·5 mm. diameter is satisfactory for the present purpose.

In order to produce the excess pressure two bottles may be used as shown. Drops of water from one are introduced into the other at the rate of one at intervals of a few seconds. The air is trapped in the lower bottle and in this way the pressure gradually rises to force out a bubble at the orifice C. This ensures a regularly recurring process, and with the manometer connected, as in the figure, the level of the liquid gradually rises to a maximum and then falls as the bubble leaves the orifice. Alternatively, the method indicated in fig. 78 may be used. The difference in height, H, of the water levels in the manometer must be carefully measured by means of a travelling microscope, set so that the horizontal part of the cross-wire of the eyepiece is at the extreme level reached by the liquid. The difference of the readings for the two limbs repeated a number of times gives the value of H.

The manometer liquid may be water or a light oil.

The orifice C must be placed at a fixed, measured depth. To ensure this, and to measure the depth accurately, a fine mark should be scratched on the glass tubing around C. Its distance from C must be carefully measured with the microscope and the tube placed so that the mark lies exactly in the surface. If σ denotes the density of the liquid in the manometer and ρ that of the liquid under investigation the excess pressure in the bubble is

$$p = (\sigma \mathrm{H} - \rho h)g,$$

where h is the depth of the orifice.

Thus $$\mathrm{T} = k\,(\sigma \mathrm{H} - \rho h)g,\text{*}$$

where k is a constant which is not required for a comparative determination.

In the experiment for the study of the variation of surface tension with concentration, the first measurement should be made with pure water with the glass surfaces very clean. The value of T in this case can be regarded as known and its value obtained at the temperature of the liquid from a book of tables. In this way k is determined and the value of T for different concentrations can be calculated from the formula.

The density of the solution is required in each case, and should be obtained by means of a specific gravity bottle. Another interesting experiment is to find the variation of surface tension with temperature. This requires the liquid round the orifice to be kept a different constant temperature and the above observations made in each case.

(10) The Method of Sentis

A capillary tube is drawn out to about 0·5 mm. bore, as in Jaeger's method. It is immersed in the liquid under investigation, and then withdrawn and clamped vertically. Some of the liquid will emerge at the lower end and form a drop, as shown in fig. 79 (1), so that the distance from A, the meniscus in the tube, to B, the lowest point of the drop, is h_1 cm., and MN is $2r$.

If now the lower end of the tube is surrounded by a vessel, C, containing the liquid, the column will fall in general, but the meniscus may be brought to the original level by raising C until the free surface of the liquid in the beaker is h_2 cm. below the meniscus, fig. 79 (2).

* If we put $p = \frac{2\mathrm{T}}{\mathrm{R}}$, $k = \frac{\mathrm{R}g}{2}$ the uncertainty introduced by this process is dealt with by Ferguson (*Phil. Mag.*, No. 28, 1914, pp. 128 et seq.), who deduces an expression for T without making this assumption and arrives at the result:

$$\mathrm{T} = g\mathrm{B} + \left(\frac{\mathrm{R}^3\sqrt{3}}{12\sqrt{\mathrm{B}}}\right),$$

where [illegible]

From a knowledge of h_1, h_2, r, and ρ, the density of the liquid, T, may be calculated from the formula

$$T = \frac{g\rho}{2}\left\{r(h_1 - h_2) - \frac{r^2}{3}\right\}.$$

To establish this formula we assume that the portion of the drop, shown in section as MONB, is hemispherical. This approximation is a safe one when the radius of the capillary tube is small, and r small compared with h_1.

Consider the forces acting below the horizontal plane of maximum area shown in section as MN.

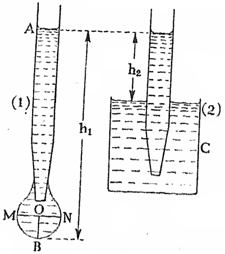


FIG. 79

The length OB $= r$, and hence the column from MN to the meniscus is $(h_1 - r)$; of this a length, h_2, is supported by the upper surface tension as shown in fig. 79 (2), and hence the excess pressure at MN due to the liquid column is $g\rho(h_1 - h_2 - r)$, which contributes a downward force $g\rho(h_1 - h_2 - r)\pi r^2$. The weight of the hemisphere $\frac{2}{3}\pi r^3 g\rho$ also acts downwards.

The surface tension acts vertically in the circle of section of the drop and plane MN, and has a value T . $2\pi r$; hence

$$T2\pi r = g\rho(h_1 - h_2 - r)\pi r^2 + \tfrac{2}{3}\pi r^3 g\rho,$$

or

$$T = \frac{g\rho}{2}\left\{(h_1 - h_2)r - \frac{r^2}{3}\right\}.$$

To make a determination of T for a liquid, the cleaned capillary about 25 cm. long is drawn out to a radius of about 0·5 mm. at one end, and is almost entirely submerged in the liquid, so that the latter fills the greater part of the bore. It is slowly withdrawn, and is clamped vertically. By means of a travelling microscope, the length MN $= 2r$ is measured and the microscope is then focused on the meniscus. A small glass beaker is raised under the tube, until the liquid in the

beaker just touches the drop. The reading of the micrometer screw, which raises the platform carrying the beaker, is noted. The beaker is then further raised until the meniscus, as viewed by the fixed microscope, again acquires its original level. The micrometer screw reading is again noted. The difference between these two readings is equal to $(h_1 - h_2)$.

ρ is determined in the usual manner, whence T may be calculated from the formula deduced above.

If the form of adjustable table with micrometer or vernier attachment is not available, some simple convenient method may be devised for the measurement of $(h_1 - h_2)$. For example, two microscopes may be used. With the first the value of MN is observed, and then the point, B, is viewed and its image brought into coincidence with the cross-hairs; the second microscope is adjusted until, viewing the tube conveniently at right angles to the first, the image of the meniscus is in coincidence with the cross-hairs in the eyepiece.

The beaker, C, is then introduced and adjusted until the meniscus is again as before, producing an image in coincidence with the cross-hairs of the second stationary microscope. A pin is adjusted to coincidence with the free surface of the liquid in the beaker which is then removed. The first microscope is then moved a distance which is measured on the vernier scale attachment, until an image of the point of the pin is in coincidence with the cross-hairs, the vertical distance moved by the microscope is $(h_1 - h_2)$.

Care is, of course, taken that the capillary tube does not move during the experiment.

(11) Anderson and Bowen's Method

A method to determine the value of the surface tension of a liquid, and the angle of contact with glass was described in the *Philosophical Magazine*, April 1916. Another method not so readily adaptable to general laboratory use appears in the same magazine, February 1916.

A small rectangular sheet of thin cover glass is cleaned (by standing it in concentrated sulphuric acid and potassium bichromate, as described previously) and dipped into the liquid whose surface tension is to be measured. It is withdrawn and clamped vertically. The liquid takes up the form shown in the diagram, fig. 80.*

The drop has two curvatures, making the equivalent of a cylindrical lens, concave at the upper half and convex below, the centres being at O and O^1.

The upper limit of the drop may be at O or any point, N, above. If A is the focal point of the concave lens, OA the axis, f_1 the focal

* It was established by the original experiment that for water, glycerine, olive oil, and turpentine, the angle of contact is zero, and hence the form of fig. 80 [illegible]

length, B the focal point of the convex lens, and O^1B the axis of the lens of focal length, f_2,

$OO^1 = h$ cm.,
r_1 = the radius of curvature of the concave surfaces (assumed symmetrical and equal),
r_2 = the radius of curvature of the convex surfaces,
μ = the refractive index of the liquid,
ρ = the density of the liquid,
p_1 = the pressure in the liquid at O,
p_2 = the pressure in the liquid at O^1,
Π = atmospheric pressure,
T = surface tension of the liquid.

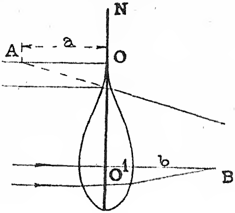


FIG. 80

We have, as the lens is a thin one, using the lens formula

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

$$\frac{1}{f_1} = (\mu - 1)\frac{2}{r_1}, \qquad \ldots(13)$$

$$\frac{1}{f_2} = (\mu - 1)\frac{2}{r_2}. \qquad \ldots(14)$$

Since the pressure inside a cylindrical surface of radius R is greater than the pressure outside by $p = \frac{T}{R}$, we have

$$p_1 = \Pi - \frac{T}{r_1}, \qquad \ldots(15)$$

$$p_2 = \Pi + \frac{T}{r_2}. \qquad \ldots(16)$$

Now $$p_2 - p_1 = g\rho h,$$

also $$p_2 - p_1 = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$ by (15) and (16) above.

Hence $$T\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=g\rho h.$$

But by (13) and (14) above

$$\frac{1}{r_1}=\frac{1}{2f_1(\mu-1)};\quad \frac{1}{r_2}=\frac{1}{2f_2(\mu-1)};$$

i.e. $$\frac{T}{2(\mu-1)}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)=g\rho h$$

or $$T=\frac{2g\rho h(\mu-1)f_1f_2}{(f_1+f_2)}. \qquad ...(17)$$

If one side only of the glass sheet is wet, using the same notation we have

$$T=\frac{g\rho h(\mu-1)f_1f_2}{(f_1+f_2)}. \qquad ...(18)$$

The apparatus used to obtain T is a collimator illuminated by a sodium flame, and adjusted to give a parallel beam. The light passes through the cover glass and the liquid lens, and is viewed by a low-power microscope provided with a vertical traverse, and a *traverse parallel to the axis of the microscope.*

The usual type of travelling microscope will need a little modification to make this latter condition possible.

If a parallel beam of light be sent from the collimator from left to right, and normal to the plate, a virtual image of the horizontal slit will be formed at A by the upper half of the liquid lens. The distance OA may be measured by using the low-power microscope, arranged with its axis parallel to the direction of the incident beam. The microscope is first focused on the image at A, and then moved backwards a measured distance until the glass plate is in focus. The distance moved being $= OA = f_1$. In the same way OB and f_2 may be measured by noting the difference in reading of the microscope when the glass sheet is focused and then when the image of the slit is coincident with the cross-hairs.

It will be found most satisfactory to use one side of the cover glass only, i.e. dry the other side before making the observation, and allow the incident beam to fall on the dry side.

As the incident beam is parallel, it will be found, of course, that the observing microscope must be moved in a vertical direction to enable a focus of first A and then B to be made. This distance, h, is noted.

The refractive index and the density may be obtained from tables, or by one of the many methods available.

Thus, having measured f_1, f_2, and h, knowing μ and ρ, T may be calculated for the liquid used.

Measure in this manner the surf...

The Variation of the Surface Tension of a Liquid with Temperature

The variation of surface tension with temperature may be obtained by Jaeger's method, which enables a good comparison of the relative values of the surface tension at different temperatures to be made.

The details of the experiment are as described on p. 125. The bubbles in this case are formed in the liquid at different steady temperatures, and H is determined for each.

A large beaker is filled with the liquid, say water, and heated to about 90° C., and then allowed to cool, the value of T being obtained every 10° C. The liquid is well stirred before each observation and, if a large volume is taken, will remain sensibly at the same temperature throughout the observation.

A curve is plotted, showing the decrease of T with temperature.

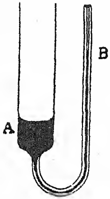


FIG. 81

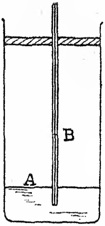


FIG. 82

Capillary Tube Method

The decrease of the value of the surface tension may also be investigated by a capillary tube method. Either form of apparatus shown in fig. 81 or fig. 82 may be employed. One form of apparatus is filled with the liquid and immersed in a water bath whose temperature can be regulated either by a thermostat or by manipulation of a Bunsen burner and a stirrer. The bath is raised to boiling-point and allowed to cool, so that at about 90–95° C. the whole of the apparatus and contents are at the temperature of the bath. The difference in level between the two surfaces, A and B, is measured as quickly as possible with a travelling microscope. This process is repeated at different temperatures, say, every 10° C., and from a knowledge of this difference in level, h, and the radius of the tube as measured by the method given on p. 24, the value of T at each temperature may be calculated and a graph representing this relation plotted.

The value of the density (ρ) is obtained, for each temperature, experimentally, or from tables, and is used in the evaluation of T by the above methods.

It is essential to ensure that all the apparatus, glass tubes etc., are thoroughly cleaned in this as in all experiments on surface tension.

CHAPTER VI

VISCOSITY

When adjacent layers of a fluid move with a relative velocity, forces, known as viscous forces, are brought into play tending to reduce this relative movement.

If we consider a fluid whose upper layer is moving with a velocity v in a fixed direction, the state of affairs shown in fig. 83 will be reached, where intermediate layers between the upper layer AB, which has a velocity v, and the lower layer CD, which is at rest, have a velocity shown by the arrows.

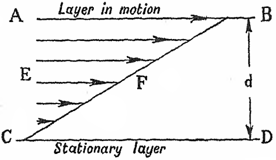


Fig. 83

The force, F, acting on any area in a plane at right angles to the diagram, and parallel to EF, is proportional to the area A, and to the velocity gradient $\frac{v}{d}$.

Thus $\quad$ $F \propto A \times$ (vel. gradient).

Taking the normal to EF, in the plane of the diagram, as the y axis, we have

$$F = \eta A \frac{dv}{dy}, \qquad \ldots(1)$$

where η is a constant for the liquid and is called the coefficient of viscosity.

In the case of a liquid flowing down a tube, the axial stream is moving with a definite velocity and the layer in contact with the wall of the tube is at rest and, provided that the pressure difference which is causing the flow is not too great, the result is the regular type of motion already considered.

If the pressure exceeds a certain limit, the liquid no longer proceeds in this stream-line flow. The result in this case is called *turbulent motion*.

We will assume, in the experiments that follow, that the pressure

applied is below this critical pressure, and that the motion is therefore regular.

The Determination of the Coefficient of Viscosity for a Liquid by Observation of the Flow of the Liquid through a Tube

The value of the coefficient of viscosity of a liquid, such as water, may be obtained by measuring the quantity passing per second through a tube of uniform radius, when a definite pressure difference exists between the ends of the tube.

Consider an axial cylinder of liquid of radius r as shown in fig. 84, within the tube whose radius is R and length l. Let P be the steady pressure difference at the extreme ends of the tube.

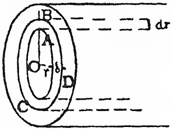


FIG. 84

The force driving the cylinder of liquid is $\pi r^2 \mathrm{P}$ and this is opposed by the viscous forces over the curved surface of the cylinder which have the value, determined by equation (1), of $\eta \,.\, 2\pi r l \frac{dv}{dr}$. In the steady state of flow these forces are equal and opposite or

$$\pi r^2 \mathrm{P} = -\eta 2\pi r l \frac{dv}{dr}$$

or

$$\frac{dv}{dr} = -\frac{\mathrm{P}r}{2l\eta}. \qquad \ldots(2)$$

Integrating, we obtain

$$v = -\frac{\mathrm{P}r^2}{4l\eta} + \text{a constant.}$$

Since $v = 0$ when $r = \mathrm{R}$, the constant is $\frac{\mathrm{PR}^2}{4l\eta}$.

and

$$v = \frac{\mathrm{P}}{4l\eta}(\mathrm{R}^2 - r^2). \qquad \ldots(3)$$

If we consider a thin cylindrical shell of thickness dr (fig. 84) we see that the volume passing through it is

$$2\pi r \,.\, dr \,.\, v \quad \text{or} \quad \frac{\pi \mathrm{P}}{2l\eta}(\mathrm{R}^2 r - r^3)\, dr.$$

Hence, the total volume of liquid passing through the tube per second, Q, is

$$\frac{\pi P}{2l\eta}\int_0^R (R^2r - r^3)\,dr$$

or

$$Q = \frac{\pi R^4 P}{8l\eta}. \qquad \ldots(4)$$

A suitable form of apparatus to use in an experimental determination of η by this method is seen in fig. 85. The liquid, say water, is contained in a large bottle, B, standing a suitable distance above the level of the table. The water flows from this reservoir to the union, X, thence through a capillary tube of known length to the union, Y, and so on, via a length of india-rubber tubing to a graduated jar, J, where the thermometer, C, measures the temperature of the emerging water.

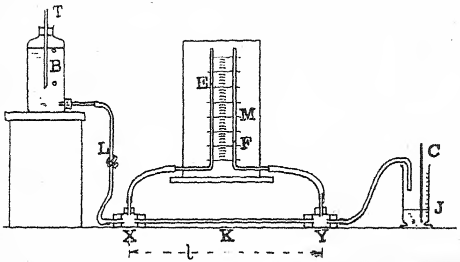


FIG. 85

From the unions, X and Y, two lengths of india-rubber tube make connexion to the manometer M. The difference in the levels between E and F, h, gives the value of the pressure difference between the ends of the experimental tube, K, in cm. of water.

A pinch-cock, L, enables the flow of the liquid to be regulated.

In order to maintain a constant difference of pressure between the two ends, X and Y, whilst the water is flowing, the bottle B is closed by means of a tight-fitting india-rubber cork through which a glass tube passes, to a point well below the surface of the water. This end being open to the atmosphere allows the entrance of air bubbles as water flows through the tube. The lower end of the tube remains at atmospheric pressure, and so, until the whole of the water above this point has passed through the tube, the manometer will register a constant difference of level.

The flow should be so arranged that the emergent water issues as

a slow trickle or succession of drops, so that the kinetic energy of the fluid is small.

When the flow is regulated and a steady flow takes place in the tube the end is inserted into J, as seen in the diagram, and the time is taken in seconds while a measured volume (say 500 c.c.) passes through K. In this way Q is obtained for a measured pressure difference, h, shown by the manometer.

Several values of Q corresponding to different values of h are obtained. This variation in h is brought about either by raising the bottle or by adjusting the level of the end of the tube, B, by sliding it through the cork. The graph of Q against h should be linear. For large values of h the departure from the straight line in the graph indicates turbulent motion in the tube. For intermediate values some departure from the linear relation will be brought about by the cause discussed in the note below.

From the linear part of the graph obtain the value of $\frac{Q}{h}$. Except as in the note below P may be taken as proportional to h or $P = g\rho h$.

The value of the length of the horizontal tube, l, is measured directly to the nearest mm. R is obtained by weighing a measured length of mercury thread.

Care must be taken to maintain the temperature of the liquid at a constant level during the series of experiments, as the value of η, and consequently $\frac{h}{Q}$, decreases with an increase of temperature (e.g. for water η in c.g.s. units is 0·00179 at 0° C., 0·0080 at 30° C., and 0·0055 at 50° C.).

η is calculated from the formula

$$\eta = \frac{\pi}{8} \frac{g\rho R^4}{l} \frac{h}{Q}.$$

A Note on a Correction for the Velocity of Efflux

Suppose that liquid flows through a tube, BC (fig. 86), the velocity being v_1 at B and v_2 at C. Let the potential be V_1 at B and V_2 at C. Suppose that the cross-section of the tube is A_1 at B and A_2 at C. Suppose that in a certain time interval the flow of water carries a short length dx_1 of the column past B so that the mass entering the tube between B and C is $\rho A_1 dx_1$, where ρ denotes the density of the liquid. The liquid is assumed to be incompressible, so that an equal mass flows out past C, and this is denoted by $\rho A_2 dx_2$, where dx_2 is the length of the column flowing past C in the same time.

Thus $$A_1 dx_1 = A_2 dx_2. \qquad \ldots(5)$$

the left exerts a force P_1A_1 on the liquid in the tube and in the movement does work of amount $P_1A_1dx_1$. The energy of the liquid entering at B is

$$\tfrac{1}{2}\rho A_1dx_1v_1^2 + \rho A_1dx_1V_1,$$

the first term denoting the kinetic energy and the second term the potential energy.

Similarly, the liquid in the tube exerts a force P_2A_2 on the liquid outside C and the energy of the liquid leaving C is

$$\tfrac{1}{2}\rho A_2dx_2v_2^2 + \rho A_2dx_2V_2.$$

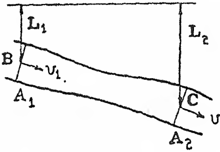


FIG. 86

In the case of steady flow the conditions in the region BC remain unaltered, so that the energy of the liquid there is not changed.

Hence
$$P_1A_1dx_1 + \tfrac{1}{2}\rho A_1dx_1v_1^2 + \rho A_1dx_1V_1$$
$$= P_2A_2dx_2 + \tfrac{1}{2}\rho A_2dx_2v_2^2 + \rho A_2dx_2V_2,$$

and cancelling out the common factor $A_1dx_1 = A_2dx_2$,

$$P_1 + \tfrac{1}{2}\rho v_1^2 + \rho V_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho V_2. \qquad \ldots(6)$$

In other words, the expression $(P + \tfrac{1}{2}\rho v^2 + \rho V)$ remains constant. This theorem of hydrodynamics is known as Bernoulli's theorem.

Suppose that it is applied to the case illustrated in fig. 87. At the wide upper surface, B, it will be supposed that the velocity of the liquid is small enough to be neglected, while at the end, D, the liquid will be supposed to flow out with velocity, v. The potential in this case is due to gravity and the level at B will be supposed to be the zero level of this potential. The potential at D will thus be $-gL$, according to the diagram where D is at a depth L below the zero level.

Thus, if the atmospheric pressure is denoted by π, this will also be the pressure just inside the liquid at the upper surface. Let the pressure in the liquid at C be p. Then, according to Bernoulli's theorem applied to the liquid at B and C,

$$\pi = p + \tfrac{1}{2}\rho v^2 - g\rho L.$$

Thus the pressure at C is

$$p = \pi + g\rho L - \tfrac{1}{2}\rho v^2.$$

The pressure at the end of the tube, D, is atmospheric and is thus π, so that the fall of pressure along the tube, CD, is

$$p - \pi = g\rho L - \tfrac{1}{2}\rho v^2.$$

This is only equal to the pressure head, $g\rho L$, if the velocity of efflux, v, is small.

From equation (3), p. 135, it appears that the velocity is not uniform across the whole surface of the orifice. Let u denote the average velocity over the surface. This means that the total volume flow per sec., Q, is $\pi R^2 u$, where R denotes the radius of the orifice, or $u = \dfrac{Q}{\pi R^2}$.

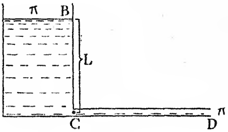


Fig. 87

If we suppose that the velocity, u, is that to be applied in using Bernoulli's theorem, where the velocity is assumed to be uniform all across the orifice, then

$$p - \pi = g\rho L - \frac{1}{2}\frac{Q^2\rho}{\pi^2 R^4}.$$

It is better to leave the precise definition of u open and to write $u = \dfrac{kQ}{\pi R^2}$, where k is a numerical factor which can be determined experimentally.

Thus
$$p - \pi = g\rho L - \frac{1}{2}\frac{k^2 Q^2}{\pi^2 R^4}.$$

If the horizontal capillary tube be of length l, the application of formula (4), p. 136, in which the pressure gradient $\dfrac{P}{l}$ is replaced by $\dfrac{(p-\pi)}{l}$, gives

$$Q = \frac{\pi}{8}\frac{R^4}{\eta l}\left(g\rho L - \frac{k^2 Q^2 \rho}{2\pi^2 R^4}\right) \qquad \ldots(7)$$

or
$$\frac{8\eta l}{\pi R^4 g\rho} = \frac{L}{Q} - \frac{k^2 Q}{2\pi^2 g R^4}.$$

If various values of L are taken and the corresponding values of Q observed, a graph can be drawn of $\frac{L}{Q}$ against Q. The graph obtained should be linear with a slope $\frac{k^2}{2\pi^2 g R^4 \rho^2}$, from which k can be obtained,

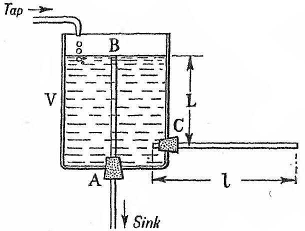


FIG. 88

and with an intercept, $\frac{8\eta l}{\pi R^4 g\rho}$, on the axis of $\frac{L}{Q}$ from which η may be found.

The variation in the value of L may be obtained easily by a device such as is illustrated in fig. 88, or by the method given on p. 136.

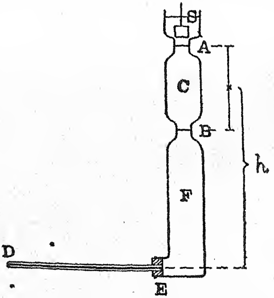


FIG. 89

The vessel, V, is provided with rubber corks, A and C. The capillary tube passes through the latter and a wider quill tube passes through the former. The height of B above the cork can be varied in order to

vary L. Water is slowly run into the vessel from a tap, so that the water may be kept up to the height of the end B of the tube down which it overflows.

With the arrangement of apparatus, illustrated in fig. 85, no correction is required.

In this case the difference of pressure read from the manometer gives the pressure head responsible for the viscous flow.

An alternative apparatus for the determination of η is seen in fig. 89. The horizontal capillary, DE, is fixed in a cork which closes the lower end of the tube, CF, which has constrictions at A and B, on which are scratches on the glass. The whole vessel is filled with the liquid and the ground-glass stopper, S, placed in the neck above A. The volume between the scratches is calibrated, so that its capacity, V, is known. The stopper, S, is removed, and the time, t cm., is taken for this volume to flow through the tube, i.e. when the level reaches A a stop-clock is started, and when the level reaches B the clock is stopped.

p, the pressure, is taken as due to the average height, h, of the liquid above the capillary tube level, i.e. $p=g\rho h$. The other terms are measured as for the first form of apparatus described.

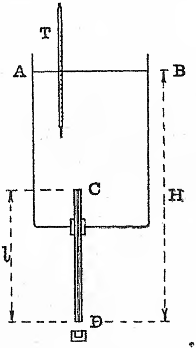


FIG. 90

Flow through a Vertical Tube

The variations of the coefficient of viscosity of water with temperature may be investigated by means of the apparatus shown in fig. 90.

AB is a copper cylindrical vessel into which the capillary tube, CD,

Water is heated in a boiler to a temperature five or six degrees above that at which a determination is to be made, and is transferred to AB, the end, D, being closed by the rubber stopper shown. The water is stirred and the temperature noted by the thermometer, T. It is not very difficult to arrange that the water cools slowly and regularly about the desired temperature. The water is then allowed to flow into a graduated vessel and the time taken for a measured amount to pass. This process may be repeated, using water at various temperatures.

Care must be taken that the temperature throughout the flow is constant, otherwise unreliable results will be obtained.

The radius and the length of the tube may be measured as before.

The theory of this experiment differs somewhat from that in which the capillary tube is maintained in the horizontal position because the force of gravity now affects the flow.

Consider a central column of liquid in the capillary tube of radius, r, the full radius of the tube being R.

This column is under forces of magnitude $\pi r^2 l\rho g$ and $\pi r^2 p$ downwards, the former due to gravity and the latter due to pressure, p being the pressure difference between the ends of the column and l its length.

In the case of steady flow these forces are counterbalanced by the upward viscous force. Since the velocity, v, diminishes with r, the magnitude of the force on the outer cylindrical surface of the column is

$$-\eta\frac{dv}{dr}2\pi rl.$$

Thus the equation for steady flow is

$$-2\eta l\frac{dv}{dr} = l\rho gr + pr;$$

on integrating and remembering that v vanishes when $r = \mathrm{R}$,

$$v = (\mathrm{R}^2 - r^2)\,\frac{l\rho g + p}{4\eta l}.$$

The volume flowing from the tube per sec. is

$$\mathrm{Q} = \int_0^{\mathrm{R}} 2\pi rv dr = \frac{\pi \mathrm{R}^4\,(l\rho g + p)}{8\eta l}. \qquad \ldots(8)$$

The upper surface, AB, is wide in comparison with the surface at the orifice C, so that the velocity there may be regarded as zero. As in the case of the horizontal tube (p. 136), let u denote the average velocity at C and as before write $u = \dfrac{k\mathrm{Q}}{\pi \mathrm{R}^2 \rho}$.

Let h denote the difference in level between AB and C, so that on applying Bernoulli's theorem we have

$$\pi = \pi' + \tfrac{1}{2}\rho u^2 - g\rho h,$$

where π' is the pressure in the liquid at C.

In the formula for Q

$$p = \pi' - \pi,$$

since π is the atmospheric pressure and thus the pressure at D.

Thus $$l\rho g + p = H\rho g - \tfrac{1}{2}\rho u^2,$$

where $H = h + l$ denotes the vertical distance between the upper level of the liquid and the lower end of the tube.

Thus the formula for Q is

$$Q = \frac{\pi R^4}{8\eta l}\left(H\rho g - \frac{k^2 Q^2}{2\pi^2 R^4 \rho}\right). \qquad \ldots(9)$$

When the efflux is small enough the second term in the bracket may be neglected and the simple formula $Q = \dfrac{\pi R^4 H\rho g}{8\eta l}$ results.

This is especially useful for the comparison of viscosities.

Two marks are made on the vessel at the level of AB and at a lower level. One of the liquids to be compared is poured in to a height a little above the higher mark, and the time is taken for the liquid to run out between the two marked levels. The observation is repeated for the second liquid. If the quantities for the two liquids are distinguished by the suffixes 1 and 2, it follows that

$$\frac{Q_1}{Q_2} = \frac{t_2}{t_1}$$

and $$\frac{\eta_1}{\eta_2} = \frac{t_1\rho_1}{t_2\rho_2}. \qquad \ldots(10)$$

The method may also be used for the determination of the variation of η with temperature by maintaining a liquid at various temperatures and observing the times of flow.

Finally, we can determine η by this method, as in the case of the flow through a horizontal tube by determining the efflux per second for various total heights H. These heights may be varied by adding more liquid, thus raising the level AB or by varying the length of the capillary tube emerging from the vessel. By plotting $\dfrac{H}{Q}$ against Q a linear graph should be obtained and both η and k determined as before.

Another way of carrying out this experiment is by means of an apparatus consisting of a siphon of glass tube connected to a length of capillary tube by means of a short length of rubber tubing. A pin, P, bent twice at right angles, is attached to the tube by means of two rubber bands.

The liquid is placed in a beaker and the point of the pin is adjusted to coincide with the surface. The liquid is sucked over and the time taken for a known volume to pass over is obtained.

The volume is found by weighing the liquid which is collected in a measured time. The tube is lowered continuously in such a way that the pin is always in contact with the liquid surface.

The relation developed on p. 136 connects the factors involved, viz.:

$$Q = \frac{\pi}{8} \cdot \frac{P}{l} \cdot \frac{R^4}{\eta},$$

where Q is the volume passing per second. The pressure, P, has the value $g\rho H$, as can be seen from the figure, if it is remembered that the

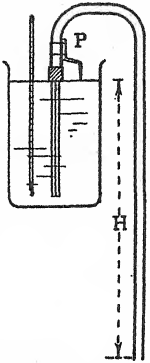


FIG. 91

column of liquid in the capillary tube is supported by the upthrust of the surrounding liquid, leaving the column of $g\rho H$ to provide the pressure necessary to overcome the viscous forces. The correction term depending on Q is neglected, but this can be taken into account, if necessary, by plotting $\frac{H}{Q}$ against Q as before.

To find the variation of the viscosity of water with temperature, it is first boiled for a few minutes in the beaker. The process described above is then repeated at several temperatures as the liquid cools down to room temperature. The beaker may then be surrounded by powdered ice and the values of η may be obtained when the temperature approaches 0° C.

In all cases the mean temperature should not change by more than 0·5° C. during the observations.

During the observations the liquid should pass for two or three minutes; H should be of the order of 50 cm. and l about 15 cm.

This method may also be used to compare viscosities, or to find the variations with the concentration of solution.

Determination of the Viscosity of a Liquid by the Co-axial Cylinder Method

The value of the coefficient of viscosity for a liquid such as glycerine may be obtained, using the apparatus shown diagrammatically in fig. 92. The liquid, e.g. glycerine, is placed in the cylinder, AB, which may be rotated by hand or by a small motor. A belt driven by either means passes over the pulley, P, and the rotation is imparted to the

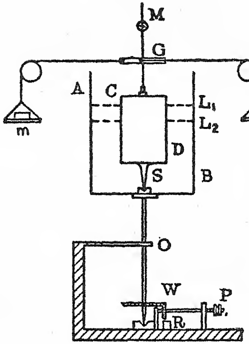


FIG. 92

cylinder by the crown bevel wheels, W. The revolutions may be counted by the revolution counter, R. Hence, if a number of revolutions n, be timed, the angular velocity, ω, may be calculated.

Immersed in the glycerine is a second solid cylinder, CD, which is suspended on a phosphor-bronze suspension, which carries a mirror, M, and which is maintained central by the pivot, S. Due to the viscous forces in the liquid, the inner cylinder will experience a couple, C, which turns it through an angle, θ, such that the torsional restoring couple just balances the turning moment due to the liquid.

If τ is the restoring couple per unit angular twist in the supporting wire, and θ be the constant deflection in radians, we have $\tau\theta =$ the restoring couple due to the torsion of the wire.

To obtain a value for the couple due to the viscous forces, we will first consider the co-axial cylindrical surfaces

Let fig. 93 represent a section normal to the axis of these cylinders, of radius R_1 and R_2, as shown.

Take any thin cylindrical ring in the liquid having points, A and B, on the same radius. If the liquid rotated as a whole, there would be no relative motion. Thus, for no relative motion, when A moves to A′, B moves to C. If ω is the constant angular velocity, $AO = r$, and $BO = r + \delta r$, $AA' = r\omega$; $BC = (r + \delta r)\omega$. Now, in the actual case, the outer cylinder moves with an angular velocity, Ω say, and the layer in contact with the inner cylinder is at rest, i.e. the liquid does not have this constant angular velocity. Actually,

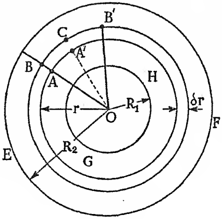


FIG. 93

the particle at B moves with some larger angular velocity $(\omega + \delta\omega)$ and moves to B′ where $BB' = (\omega + \delta\omega)(r + \delta r)$, thus having an excess over the 'no-relative motion velocity' equal to $(\omega + \delta\omega)(r + \delta r)$ $- \omega(r + \delta r)$, i.e. $= \delta\omega(r + \delta r)$, i.e. the velocity gradient is $\dfrac{\delta\omega(r + \delta r)}{\delta r}$, or since δr is negligible in comparison with r, the velocity gradient $\dfrac{dv}{dr} = r \cdot \dfrac{d\omega}{dr}$ in the limit.

Thus, considering the forces on the cylindrical shell, if l is the length of the inner cylinder from the surface of the liquid to the lower extremity, the area of the cylindrical shell is $(2\pi r)l$, and therefore from (1),

$$F = \eta\,(2\pi r l)\,\frac{dv}{dr},$$

where F is force due to the viscosity over the curved surface of such an imaginary shell. The moment of this about the common axis is

$$C = Fr = 2\pi r^2 l \eta r\,\frac{d\omega}{dr},$$

substituting the value for $\dfrac{dv}{dr}$.

Integrating, we have $\int 2\pi l\eta d\omega = \int \frac{Cdr}{r^3}$,

i.e. $$2\pi l\eta\omega = -\frac{C}{2r^2} + B, \qquad \ldots(11)$$

where B is a constant.

Now when $r = R_1, \quad \omega = 0, \quad B = \frac{C}{2R_1^2}$,
$r = R_2, \quad \omega = \Omega,$

i.e. $$2\pi l\eta\Omega = \frac{C}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right],$$

$$C = \frac{4\pi l\eta\Omega R_1^2 R_2^2}{R_2^2 - R_1^2}. \qquad \ldots(12)$$

But $$C = \tau\theta,$$

whence $$\eta = \frac{\tau(R_2^2 - R_1^2)}{4\pi l\Omega R_1^2 R_2^2}\theta. \qquad \ldots(13)$$

In the actual case taken (12) must be modified to include the couple due to the viscous effect on the under surface of the inner cylinder. If this effect were truly due to the parallel circular plates, we might readily deduce an expression for it. But the stream-lines in the liquid will not be composed of two such regular patterns. The effect at the end of the cylinder, CD will be a gradual transition from the one to the other. Some end correction is therefore needed. Let us assume that the couple exerted on the cylinder, CD, due to the under surface and the lower portion of the curved surface, and also the effect due to the support, S, which rests on AB, for an angular velocity, Ω, is $K\Omega$, then the total couple will be

$$C = \frac{4\pi\eta l' R_1^2 R_2^2}{(R_2^2 - R_1^2)}\Omega + K\Omega,$$

where l' is the length of the curved surface of the inner cylinder, measured from the liquid surface to the lower limit of the regular stream-line between the cylinders; the second term deals with the rest of the curved surface and the lower surface, as stated.

If now we have the space between the cylinders filled to a level, such as L_1, we have, if l_1' is the corresponding length and θ_1 the deflection,

$$\tau\theta_1 = \Omega\left[\frac{4\pi l_1'\eta R_1^2 R_2^2}{R_2^2 + R_1^2} + K\right]. \qquad \ldots(14)$$

For a second case, where the length of the cylinder immersed is l_2', corresponding to a level, L_2, in the figure, if θ_2 is the deflection,

$$\tau\theta_2 = \Omega\left[\frac{4\pi l_2'\eta R_1^2 R_2^2}{R_2^2 - R_1^2} + K\right]. \qquad \ldots(15)$$

K may therefore be eliminated from (14) and (15) and, if the angular velocity, Ω, is the same for both determinations, we have by subtraction

$$\tau(\theta_1 - \theta_2) = \Omega\frac{4\pi\eta R_1{}^2R_2{}^2}{R_2{}^2 - R_1{}^2}(l_1' - l_2'). \qquad \ldots(16)$$

The length to be measured is therefore the difference between the two levels, L_1 and L_2.

Thus, by maintaining the speed of rotation constant, and observing the steady deflection corresponding to two levels of the liquid surfaces, knowing $L_1L_2 = l_1' - l_2'$, and the dimensions of the cylinders, η, may be calculated in terms of τ, which itself may be evaluated by observation of the twist of the wire, when the liquid is not in the cylinders, and loads are applied at G, as indicated in the diagram.

For liquids whose viscosity is fairly large, the torsion control will not be sufficient. In this case the restoring couple is increased by adding masses, m, m, thereby increasing the restoring couple by mgD, where D is the diameter of the wheel G.

In many cases it is advantageous to eliminate the torsional restoring couple entirely. This may obviously be done by adding masses, m, m, of such magnitude that the inner cylinder is brought to its rest position, as observed by the reflected beam from the mirror, M, on the scale, which was previously used to measure θ_1 and θ_2.

In that case, if masses of total value, M_1, are supported on the wheel, G, $C_1 = M_1g\frac{D}{2}$, and for the second case with the liquid at level, L_2, $C_2 = M_2\frac{D}{2}g$.

Then equation (16) simplifies to

$$(M_1 - M_2)\frac{Dg}{2} = \Omega\frac{4\pi\eta R_1{}^2R_2{}^2}{R_2{}^2 - R_1{}^2}(l_1' - l_2').$$

This latter method of working will be found most satisfactory for glycerine and similar liquids, whilst the former method, using the torsion of the fibre, will be satisfactory in the case of less viscous substances, such as water.

Determination of the Viscosity of a Transparent Viscous Liquid by Stokes's Method

This method is based on the application of Stokes's law to the fall of spheres through the liquid. A glass jar is filled with the liquid, say castor oil or glycerine, whose temperature may be obtained by means of a thermometer placed in it. A 30° C. thermometer reading to 0·1° C. is advisable, and the liquid should be maintained at constant temperature.

Steel ball-bearings are measured with a screw gauge and placed in a small amount of the liquid in a watch-glass. The ball-bearings are transferred to the glass jar, and after dropping a few centimetres their terminal velocity is obtained by timing a measured fall. To avoid errors due to parallax, paper collars are slipped round the glass jar, and the upper edges of these collars are used as points of reference for the determination. Two or three balls of each size available are used, and are allowed to drop through the liquid in rapid sequence to ensure identical temperature conditions.

According to Stokes's law, the spheres move with a terminal velocity v_0, which is given by

$$F = 6\pi\eta v_0 r,$$

where F is the viscous force acting on the sphere of radius r cm.

In the steady state F is equal to the downward force,

i.e. $$6\pi\eta v_0 r = \frac{4}{3}\pi r^3 (\rho - \delta) g,$$

i.e. $$v_0 = \frac{2}{9} \cdot \frac{r^2}{\eta} (\rho - \delta) g,$$

where ρ is the density of the steel ball (7·72 gm./c.c.), and δ is the density of the liquid (castor oil, 0·96 gm./c.c.; glycerine 1·26 gm./c.c.).

If many sizes of ball-bearings are available, plot v_0 against r^2 and calculate η.

The velocity of fall is influenced by the proximity of the walls of the containing vessel. The spheres should therefore be dropped centrally in the vessel. A relation between v the observed terminal velocity in a vessel of radius R, and v_0, the velocity in a vessel of infinite radius has been given, viz.,

$$v_0 = v\left(1 + 2{\cdot}4\frac{r}{R}\right).$$

This is called the Ladenburg correction, and may be tested by allowing spheres to fall down glass tubes of different radii placed in the containing vessel.

Viscosity of a Liquid (Oscillating Disk Method)

The viscosity of a liquid may be determined by timing the period of oscillation of a flat circular disk in air, and finding the logarithmic decrement in the liquid and in air.

O. E. Meyer has shown that with such a disk, the coefficient of viscosity for a liquid, η, is

$$\eta = \frac{16I^2}{\pi\rho T (r^4 + 2r^3 d)^2}\left\{\left(\frac{\lambda - \lambda_0}{\pi}\right) + \left(\frac{\lambda - \lambda_0}{\pi}\right)^2\right\}^2, \quad \ldots(17)$$

where I is the moment of inertia of the disk and attachments about the axis of suspension,
ρ the density of the liquid,
T the time of a complete swing in air,
r the radius of the disk,
d the thickness of the disk,
λ the logarithmic decrement in the liquid,
λ_0 the logarithmic decrement in air.

The development of the above formula is beyond the scope of this book, and may be found in *Poggendorf Annalen*, No. 113, p. 55.

Without any discussion of the development of the result, we will use it as an empirical formula which agrees with determinations by other methods. It is an excellent method whereby to study the determination of the logarithmic decrement of an oscillating system.

Logarithmic Decrement

Consider a body suspended to oscillate about a vertical axis through its centre of gravity. If I is the moment of inertia of the body about the axis of suspension, F the restoring force per unit angular displacement, we have the equation of motion,

$$\mathrm{I}\ddot{\theta} + \mathrm{F}\theta = 0.$$

If now a frictional resistance acts on the body so that K is the resulting opposing couple per unit angular velocity, the above equation becomes

$$\mathrm{I}\ddot{\theta} + \mathrm{K}\dot{\theta} + \mathrm{F}\theta = 0. \qquad \text{...(18)}$$

This equation can be written in the form

$$\ddot{\theta} + k\dot{\theta} + n^2\theta = 0,$$

where
$$k = \frac{\mathrm{K}}{\mathrm{I}}, \quad n^2 = \frac{\mathrm{F}}{\mathrm{I}}.$$

It should be contrasted with the equation for simple harmonic motion (p. 47).

The solution of this equation can be obtained by substituting $\theta = \theta_0 e^{mt}$. In order that the equation may be satisfied m must take such values that

$$m^2 + km + n^2 = 0,$$

while θ_0 can have any value independent of t.

Thus there are two possible values for m, viz.:

$$m = \frac{-k \pm \sqrt{k^2 - 4n^2}}{2}.$$

Let these be denoted by m_1 and m_2.

Since the value of θ_0 is not determined, it follows that both

$\theta_1 e^{m_1 t}$ and $\theta_2 e^{m_2 t}$ satisfy the equation whatever the values of θ_1 and θ_2, provided they are independent of t.

The solution of the equation must contain two adjustable constants because it is of the second degree, and it thus follows that the solution is

$$\theta = \theta_1 e^{m_1 t} + \theta_2 e^{m_2 t}.$$

Our knowledge of the character of the motion of a body in a viscous medium helps in the choice of the solution applicable in the present case.

It is observed that the motion is like simple harmonic motion in that it has a constant periodicity, but it is unlike it in that the amplitude falls off with time.

The displacement graph is similar to that of fig. 94.

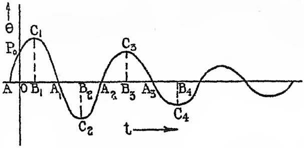


FIG. 94

We notice that the values of m are real if $k^2 > 4n^2$, and the motion cannot in that case be oscillatory for the value of θ then varies exponentially with the time. In the case in which we are interested $k^2 < 4n^2$ and the quantity under the square root is imaginary. The powers of the exponentials thus contain imaginaries, and in that way sine and cosine terms occur in the solution and represent an oscillatory motion.

The expression for θ contains a common factor $e^{-kt/2}$ and terms which introduce $\sin\sqrt{n^2 - \frac{k^2}{4}}\,t$ and $\cos\sqrt{n^2 - \frac{k^2}{4}}\,t$.

Thus the solution can be expressed in the form

$$\theta = e^{-kt/2}\left(\text{A} \sin\sqrt{n^2 - \frac{k^2}{4}}\,t + \text{B} \cos\sqrt{n^2 - \frac{k^2}{4}}\,t\right)$$

which may be written in the form

$$\theta = \text{C}e^{-kt/2} \sin\left(\sqrt{n^2 - \frac{k^2}{4}}\,t + \alpha\right). \qquad \ldots(19)$$

It is to be noted that the solution still contains two constants, C and α, and at this stage the student can assure himself that it satisfies the equation by direct substitution.

This solution describes quantitatively the features of the motion for which experiment has prepared us.

The period is $\quad \text{T} = 2\pi \Big/ \sqrt{n^2 - \frac{k^2}{4}},$

and the amplitude falls off exponentially in accordance with the factor $Ce^{-kt/2}$.

In practice n^2 is usually much greater than $\frac{k^2}{4}$, so that the period is not generally much changed from the undamped case, in which it is $\frac{2\pi}{n}$.

The curve of fig. 94 is the graph of θ against t. It is convenient to choose the origin so that it lies either at B_1 or A, when the solution becomes

$$\theta = Ce^{-kt/2} \cos \sqrt{n^2 - \frac{k^2}{4}}\, t \quad \text{or} \quad \theta = Ce^{-kt/2} \sin \sqrt{n^2 - \frac{k^2}{4}}\, t. \quad \ldots(20)$$

Instead of making use of the constant k directly, another known as the logarithmic decrement (log. dec.) is employed. To illustrate this, take the case with the origin at B_1 and, remembering the value of the period T, write

$$\theta = Ce^{-kt/2} \cos \frac{2\pi t}{T}.$$

The initial amplitude at $t = 0$ is B_1C_1. Half a period later, with $t = \frac{T}{2}$, the displacement is $B_2C_2 = Ce^{-kT/4}$ on the negative side of the axis. A period later, $t = T$, the displacement is $B_3C_3 = Ce^{-kT/2}$, and so on. It is clear that

$$\frac{B_1C_1}{B_2C_2} = \frac{B_2C_2}{B_3C_3} = \ldots = e^{kT/4}.$$

The logarithm of this ratio is denoted by λ. It is called the logarithmic decrement and is equal to $\frac{kT}{4}$.

It is to be noted that the logarithm is taken to the base of Naperian logarithms, so that

$$\lambda = 2{\cdot}303 \log_{10} \frac{B_1C_1}{B_2C_2}.$$

The damping is described by means of λ and not by means of k. It is often inconvenient to measure the extreme displacements at both sides of the zero, but the log. dec. can readily be determined from observations of B_1C_1, B_3C_3, B_5C_5, etc.

It is evident that

$$\log_e \frac{B_1C_1}{B_3C_3} = 2\lambda.$$

In general, if observations are made of a maximum displacement and of another, n periods later on the same side, i.e. of the first and $(2n + 1)$th maximum displacements, the logarithm of the ratio of the displacements is $2n\lambda$.

If the amplitudes of these displacements be denoted by α_1 and α_{2n+1} respectively,

$$\lambda = \frac{1}{2n} \log \frac{\alpha_1}{\alpha_{2n+1}}.$$

λ may be obtained accurately from this formula for the effect of errors in the observations of α_1 and α_{2n+2} is reduced by taking large values of n.

The displacements, α_n, are measured from the zero position of the pointer or spot of light which traces out the oscillations of the vibrating system. It is, however, possible to determine λ without observing the zero position.

Let the successive amplitudes be denoted by α_1, α_2, α_3, etc. α_1, α_3, etc., are on one side of the zero; α_2, α_4, etc., on the other side. The full extent of the successive swings from one side to the other will be denoted by β_1, β_2, etc.

Thus $$\beta_1 = \alpha_1 + \alpha_2,$$
$$\beta_2 = \alpha_2 + \alpha_3, \text{ and so on.}$$

λ can be expressed in terms of β_1, β_2, etc., so that the observations to be made are of the full swings and the zero position is not required.

We have

$$e^{\lambda} = \frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = \frac{\alpha_1 + \alpha_2}{\alpha_2 + \alpha_3} = \frac{\beta_1}{\beta_2}.$$

Similarly, $$e^{\lambda} = \frac{\beta_2}{\beta_3} = \frac{\beta_3}{\beta_4} = \ldots = \frac{\beta_n}{\beta_{n+1}}.$$

Thus $$e^{n\lambda} = \frac{\beta_1}{\beta_{n+1}}$$

or $$\lambda = \frac{1}{n} \log \frac{\beta_1}{\beta_{n+1}}. \qquad \ldots(21)$$

This formula can be used to determine λ accurately.

To illustrate the observations necessary, take the case in which $n = 5$. The first full swing is observed, i.e. the extent of α_1 and α_2. The second full swing is in the opposite direction and has α_2 and α_3 as its limits. The final swing of the series is $(\alpha_5 + \alpha_6)$. Thus, observations are made on the first and fifth swings, the first being in one direction, the second in the opposite direction, and so on. The disadvântage of this method is that in making visual observations it is necessary to take records on two sides of the scale.

In order to make use of the method of determining λ by means of full swings the following procedure may be adopted.

As the first observation is made, start a stop-clock and complete the observation at both ends of the first full swing. Thus β_1 is determined. Allow the oscillations to proceed until the extent of the swing

has been reduced to one-half or one-quarter of the original, and when making the first of the two readings to determine the last swing, β_{n+1}, stop the clock and complete the determination of β_{n+1}. Thus in the time that elapses between the stopping and starting of the clock, n half-periods have elapsed. The period of vibration, T, being known $n = \frac{2t}{T}$, where t is the time recorded on the clock.

Another way of making the determination of λ by the full swing method is to obtain a number of successive values of the full swings, then to wait until the amplitude has died down and to read an equal number of successive values. The time interval must be taken between the first reading of the first sequence and the first reading of the second sequence. In order to illustrate this method it will be supposed that ten values of β are obtained in each sequence.

Thus the first set of values are $\beta_1, \beta_2, \ldots \beta_{10}$. If the number of half-vibrations which occurs between the beginning of each sequence is n, this number can be found from the time interval, t, from the relation $n = \frac{2t}{T}$.

The second set of values are $\beta_{n+1}, \beta_{n+2}, \ldots \beta_{n+10}$.

The formula derived for λ shows that

$$e^{n\lambda} = \frac{\beta_1}{\beta_{n+1}} = \frac{\beta_2}{\beta_{n+2}} = \ldots = \frac{\beta_{10}}{\beta_{n+10}}.$$

Thus
$$e^{n\lambda} = \frac{\beta_1 + \beta_2 + \ldots + \beta_{10}}{\beta_{n+1} + \beta_{n+2} + \ldots + \beta_{n+10}}$$

and
$$\lambda = \frac{1}{n} \log \frac{\beta_1 + \beta_2 + \ldots + \beta_{10}}{\beta_{n+1} + \beta_{n+2} + \ldots + \beta_{n+10}}. \qquad \ldots(22)$$

The method has the advantage that, in effect, the average of a number of swings is used and the use of formula (22) ensures that each reading is used once only.

Experimental Arrangements

A suitable form of apparatus with which to make a determination of η for a liquid is seen in fig. 95.

The flat disk is suspended horizontally by a phosphor-bronze suspension which is attached to a rod rigidly fastened to the centre of the disk. This rod carries a cross-bar whose ends have a screw thread, along which two masses may be screwed to balance the disk horizontally.

A small concave mirror is fastened to the rigid support.

The time of oscillation of this system is first obtained in air. This observation is carried out by the usual lamp and scale arrangement.

A beam of light from a lamp is directed on to the mirror, and is brought to a focus by the latter on a scale about a metre away. As the spot of light passes its rest position on the scale, during its oscillation, a stop-watch is started and stopped after fifty complete swings, hence T, the time for one swing (i.e. the interval between the successive instants when the spot of light passes the zero mark *in the same direction*).

Determination of I. The value of I may be obtained from the method of oscillation described on p. 48. If τ is the restoring couple per unit

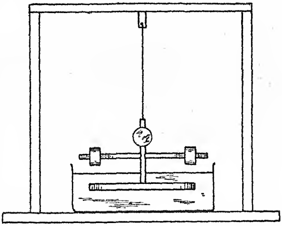


FIG. 95

angular displacement due to the suspension, we have, in air, where the logarithmic decrement is small,

$$T = 2\pi\sqrt{\frac{I}{\tau}}. \qquad \ldots(23)$$

A small ring of thick copper wire is now placed symmetrically on the disk and T′, the time of a complete swing, is obtained from a determination of the time of fifty swings, for the loaded disk.

We have

$$T' = 2\pi\sqrt{\frac{I + I'}{\tau}}, \qquad \ldots(24)$$

where I′ is the moment of inertia of the copper wire ring about the axis of suspension.

If a is the radius of the ring of wire, and if the centre is on the axis of suspension, $I' = ma^2$, where m is the mass of the wire ring.

Squaring (24) and (23) and dividing, we have

$$\frac{T'^2}{T^2} = \frac{I + I'}{I} = 1 + \frac{I'}{I} = 1 + \frac{ma^2}{I}.$$

Hence $$I = \frac{T^2 \cdot ma^2}{T'^2 - T^2}.$$

r, d, and ρ may be readily found by the usual methods

The logarithmic decrement is now found for the disk oscillating in air. Use is made of the method embodied in equation (22) above.

The disk is shielded from draught in an empty glass dish, as in the figure. A straight vertical wire is placed in front of the lamp and its image is focused on the scale by the mirror.

The disk is given a displacement and the oscillations are observed by means of the image. As it passes the position of rest a stop-clock is started. The first turning-point of the image is read off, on the right, say, then one turning-point on the left and so on, tabulating eight or ten readings on each side, as below. The disk is then allowed to perform oscillations until the amplitude has decreased to half or quarter (depending on the time) of the original amplitude. At this stage the reading at the turning of the image on the right is observed, and the stop-clock stopped. The next reading to the left is taken, and so on for eight or ten on each side.

The time being t seconds between the starting of the clock and the first observation of the second set of swings, T being the periodic time of the system in air, there have been $\frac{t}{\mathrm{T}}$ complete swings or $\frac{2t}{\mathrm{T}}$ half periods, hence n is obtained, as shown. The above process is repeated in the liquid whose coefficient of viscosity is to be determined.

Thus, in the table opposite, which gives the observations for the disk in air and paraffin oil, we have, if $\mathrm{T} = 3{\cdot}07$ sec., n, the number of the half-swing at the commencement of the second set of observations in air $= 2\left(\frac{615}{3{\cdot}07}\right) = 400$,

i.e. for air
$$\lambda_0 = \frac{2{\cdot}303}{(400-1)} \log_{10}\left(\frac{411{\cdot}2}{137{\cdot}1}\right)$$
$$= \frac{2{\cdot}3026}{399}(0{\cdot}4770)$$
$$= 0{\cdot}002752.$$

In the same way the value of the logarithmic decrement in paraffin may be calculated,
$$\lambda = \frac{2{\cdot}3026}{(n-1)} \log_{10}\frac{361{\cdot}5}{91{\cdot}9},$$

n being obtained as above from the time interval measured in this case from the beginning of the first set of observations to the beginning of the second set.

Thus, by the methods given, we have a value for I, ρ, T, r, d, λ, and λ_0; hence substituting in equation (17) η may be calculated.

Repeat the experiment, using water and paraffin oil, and calculate the value of η for each.

Air			Paraffin Oil		
Scale Reading		Amplitude β	Scale Reading		Amplitude β
Left	Right		Left	Right	
2·4	45·0	42·6	5·1	47·2	42·1
2·5	45·0	42·5	6·0	46·6	40·6
2·8	44·8	42·0	6·5	46·0	39·5
3·0	44·5	41·5	7·2	45·2	38·0
3·1	44·3	41·2	7·8	44·5	36·7
3·3	44·2	40·9	8·5	43·8	35·3
3·3	44·0	40·7	9·1	43·2	34·1
3·6	43·8	40·2	9·7	42·6	32·9
3·7	43·7	40·0	10·2	42·0	31·8
4·0	43·6	39·6	11·0	41·5	30·5
		Total 411·2			Total 361·5
Time Interval = 615 seconds			Time Interval = 123 seconds		
16·5	30·4	13·9	20·7	31·3	10·6
16·5	30·4	13·9	20·9	31·2	10·3
16·5	30·4	13·9	21·0	31·0	10·0
16·5	30·4	13·9	21·2	30·9	9·7
16·55	30·3	13·7	21·4	30·7	9·3
16·6	30·2	13·6	21·5	30·6	9·1
16·7	30·2	13·5	21·6	30·5	8·9
16·7	30·2	13·5	21·8	30·2	8·4
16·6	30·2	13·6	22·0	30·1	8·1
16·6	30·2	13·6	22·5	30·0	7·5
		Total 137·1			Total 91·9

Viscosity of Air

Consider two parallel circular plates, one suspended by a fibre, and the other rotating at a constant speed. If the space between the plates be filled with any gas a velocity gradient will be set up in the layers of the gas parallel to the rotating plate. The layer in contact with the rotating disk will move with the latter. Due to the viscosity of the gas, the adjacent layer will also acquire a velocity comparable with the former. Thus throughout the space, the air strata will be set in motion, just as in the case of a liquid flowing through a tube. The

layer of the gas in contact with the suspended plate will therefore experience a force tending to rotate it in the same direction as the constantly rotating parallel plate. Due to the force, a couple will act on the plate, which will therefore turn through a definite angle of such a magnitude that the restoring couple due to the torsion in the suspension just balances the displacing couple of the viscous drag.

It can be shown that each stratum of air moves as though it were a solid, i.e. it moves as a whole.

To obtain an expression for the deflection of the suspended plate in terms of η, the coefficient of viscosity of the gas, etc., let us assume that the edge effect is negligible, i.e. the gas between the plates behaves as though the plates were of infinite dimensions, an assumption which is justified by using a guard ring round the suspended plate.

Let d be the distance between the disks,
ω the angular velocity of rotation of the moving plate,
R the radius of the suspended plate.

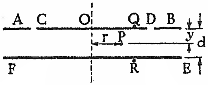


FIG. 96

Consider a stratum shown in fig. 96 by the horizontal broken line, between EF, rotating, and CD, which is suspended. The parts, AB, in the diagram represent a guard ring to eliminate the end effect, CD being of radius R.

In the stratum considered take a point, P, on a circle of radius r cm. about the axis of rotation.

Points Q and R are the projections of P on CD and EF. The velocity of R is $r \cdot \omega$; the velocity of Q is zero. The velocity slope is therefore $\frac{\omega r}{d}$.

We have seen that by the definition of viscosity

$$F = A\eta \frac{dv}{dr} = \eta \frac{\omega r}{d} A.$$

In the stratum considered, let a second circle of radius, $r + \delta r$, be drawn; between the two circles there is an annular ring of width, δr. The area of the ring is $2\pi r \cdot \delta r$. The viscous force acting on such a narrow ring is, from the equation above, equal to F, where

$$F = 2\pi r \delta r \frac{\eta \omega r}{d}.$$

The turning moment about the axis is

$$Fr = 2\pi r^3 \frac{\eta\omega}{d} \cdot \delta r.$$

Such a moment acts on the suspended disk on the projection of this area.

The total couple is the sum of such couples taken over the entire area. Let this couple be C, then

$$C = \int_0^R 2\pi r^3 \frac{\eta\omega}{d} dr = \frac{2\pi\eta\omega}{d} \frac{R^4}{4} = \frac{\pi\eta R^4\omega}{2d}.$$

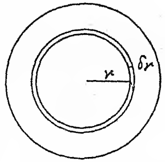


FIG. 97

Let the suspended plate be turned through an angle, θ, due to this couple. The equilibrating couple due to the torsion of the suspension is $\tau\theta$, where τ is the restoring couple per unit angular displacement. This gives

$$\tau\theta = \frac{\pi\eta R^4\omega}{2d}. \qquad \text{...(27)}$$

A suitable form of apparatus is essentially of the form shown diagrammatically in fig. 96. A brass plate is rigidly connected to a shaft which may be rotated by means of a belt drive on a pulley, which is on the shaft. A small counting indicator serves to record the revolutions of the disk, which is steadily turned by hand or by a small motor.

At a distance, which may be adjusted, above the plate is a mica disk, suspended by a phosphor-bronze suspension, and arranged inside a guard ring of brass. The suspension carries a small mirror which serves to measure the deflection by the usual method of lamp and scale; the whole is enclosed in a brass case which serves as a shield. A preliminary experiment will give an indication of the most suitable speed with which to rotate the plate, for any given distance, d, between the plates.

The zero of the spot of light having been determined, the lower plate is rotated until a full-scale deflection is obtained. The speed of rotation is maintained constant. When this is steady, the counting gear, having been read, is thrown into action, and a stop-watch started. Maintaining the spot constant by steady...

(at least several minutes) the value of the deflection is noted. The counter is then thrown out of action and the stop-watch stopped. In this way the number of revolutions n, in a known time t, is obtained,

hence $$\omega = \frac{2\pi n}{t}.$$

As R occurs in the fourth power, several values of the diameter are obtained, as accurately as possible, and the mean value calculated.

d is measured by means of a cathetometer. From the value of the steady deflection and a knowledge of the distance between the mirror and the scale, θ may be calculated in radians. θ is half the value of the angle subtended at the mirror by the length of scale moved over by the spot of light, and $\tan 2\theta$ may be obtained, knowing the linear deflection and the distance between the mirror and the scale.

Thus, from this experiment all the terms in equation (27) are known except τ and η.

To obtain τ the suspended plate is turned from its equilibrium position, and the simple harmonic oscillations set up are timed. If T is the mean value of the periodic time for, say, fifty complete vibrations, we have (p. 49)

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{\tau}}, \qquad \ldots(28)$$

where I is the moment of inertia of the mica plate about the axis of suspension.

If now the mica plate is loaded by placing on it, symmetrically with regard to the suspension, a circle of brass wire of radius a cm. and mass m gm., the moment of inertia has been increased to I', where $\mathrm{I}' = \mathrm{I} + ma^2$.

The time of oscillation, T', of the loaded plate is next determined by timing fifty oscillations, then

$$\mathrm{T}' = 2\pi\sqrt{\frac{\mathrm{I}'}{\tau}}. \qquad \ldots(29)$$

Squaring (28) and (29) and subtracting, we have

$$\mathrm{T}'^2 - \mathrm{T}^2 = \frac{4\pi^2(\mathrm{I}' - \mathrm{I})}{\tau},$$

i.e. $$\tau = \frac{4\pi^2 ma^2}{\mathrm{T}'^2 - \mathrm{T}^2}.$$

Thus, by equation (27)

$$\eta = \frac{2d}{\pi \mathrm{R}^4}\,\frac{4\pi^2 ma^2}{(\mathrm{T}'^2 - \mathrm{T}^2)}\,\frac{t}{2\pi n}\,\theta.$$

$$\eta = \frac{4ma^2td}{\mathrm{R}^4(\mathrm{T}'^2 - \mathrm{T}^2)}\,\frac{\theta}{n}.$$

Having obtained the deflection in degrees, φ° say, we have

$$\theta = \frac{\varphi\pi}{180},$$

so that
$$\eta = \frac{\pi m a^2 t \varphi d}{45 R^4 (T'^2 - T^2)\, n}.$$

The Determination of the Viscosity of a Gas by the Flow through a Capillary Tube

A simple method for determining the viscosity of a gas which may be described as 'The Constant Volume Method', has been described by Prof. A. Anderson (*Phil. Mag.*, Dec. 1921, pp. 1022–3). The apparatus

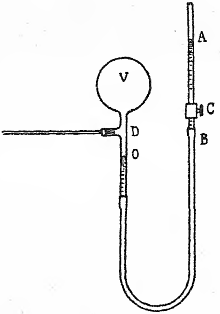


FIG. 98

is illustrated in fig. 98. It consists of a bulb, V, from which a tube, DO, projects downwards and is connected by rubber tubing to a glass tube, AB. These tubes contain mercury up to certain levels marked at A and O.

Just below the bulb a capillary tube leads from the tube, DO, as shown at D. This capillary tube is provided with a piece of rubber tubing which may be closed by a pinch-cock.

The whole apparatus is mounted on a stand and is of about the same size as the constant volume air thermometer.

The arm, AB, is mounted on a carriage which is readily adjusted by means of a rack and pinion regulated by the turn-screw, C.

The volume of the gas within V, down to some mark, O, and extending to the end of the capillary tube is determined. The pinch-cock is then opened to the air and the mercury brought well below O by properly adjusting AB. The pinch-cock is then closed, and the air compressed by raising AB until the mercury stands at O.

A few minutes' interval is allowed to elapse so that the temperature of the gas, which may have been disturbed in this compression, once more attains the temperature of the room. The difference in level of the mercury in the two tubes is observed. The pinch-cock is opened for a definite accurately measured time and then closed. During this interval the level of AB is continually adjusted, so that the mercury always stands at O, and the gas consequently maintains a constant volume. At the end of the time the difference in level of the mercury is again noted. From the observations made, viz. from a knowledge of the volume, V, of the gas, the two pressure differences, p_1 and p_2 respectively, the time interval, t seconds, and the barometric pressure, P, the viscosity is determined from the formula:

$$\eta = \mathrm{P}t\frac{\pi \mathrm{R}^4}{8l\mathrm{V}} \div \log_e \frac{(p_2 + \mathrm{P})(p_1 - \mathrm{P})}{(p_1 + \mathrm{P})(p_2 - \mathrm{P})},$$

where R = radius of capillary tube and l = its length.

The above formula may be obtained from an extension of the result obtained for the flow of a liquid through a capillary tube.

Suppose, in the first case, that the gas enters the capillary at a pressure P_1, and leaves at a pressure P_2. Let V_1 and V_2 be the volume entering and leaving per second; then $\mathrm{P}_1\mathrm{V}_1 = \mathrm{P}_2\mathrm{V}_2$. The volume of gas passing any point in the tube per second depends on the pressure at the point. The velocity is consequently variable along the axis of the tube, and therefore the method of the liquid flow cannot be applied to the full length of the tube.

Consider a small element, δx of the tube. Let p be the mean pressure in the element and δp the difference in pressure at the ends.

Equation (4), p. 136, for the liquid flow, is

$$\mathrm{Q}_1 = \frac{\mathrm{P}}{l} \cdot \frac{\pi}{8} \cdot \frac{\mathrm{R}^4}{\eta}.$$

If q_1 is the volume of gas passing through the element, the above formula may be applied, and since in this case $\frac{\mathrm{P}}{l} = -\frac{\delta p}{\delta x}$ (since the pressure decreases with increase in x), in the limit

$$q_1 = -\frac{dp}{dx} \cdot \frac{\pi}{8} \cdot \frac{\mathrm{R}^4}{\eta}.$$

Now, $\mathrm{P}_1\mathrm{V}_1 = \mathrm{P}_2\mathrm{V}_2 = q_1 p$, since the mass of gas passing any point is constant, $q_1 = \frac{\mathrm{P}_1\mathrm{V}_1}{p}$; therefore

$$\frac{\mathrm{P}_1\mathrm{V}_1}{p} = -\frac{dp}{dx} \cdot \frac{\pi}{8} \cdot \frac{\mathrm{R}^4}{\eta},$$

$$\mathrm{P}_1\mathrm{V}_1 dx = -\frac{\pi}{8} \cdot \frac{\mathrm{R}^4}{\eta} p \,.\, dp.$$

Integrating over the length of the tube

$$P_1V_1 \int_0^l dx = -\frac{\pi}{8}\frac{R^4}{\eta}\int_{P_1}^{P_2} p \, . \, dp,$$

i.e. $$P_1V_1 = \frac{P_1^2 - P_2^2}{l} \cdot \frac{\pi}{16} \cdot \frac{R^4}{\eta}. \qquad \ldots(30)$$

If V is the volume available to enter the tube at a pressure P_1, we may write $$P_1V_1 = P_1 \cdot \frac{dV}{dt}.$$

We may apply Boyle's law to the instantaneous values of P and V at entry,

i.e. $$P_1V = \text{constant},$$

or $$P_1\frac{dV}{dt} + V\frac{dP_1}{dt} = 0, \quad \text{i.e.} \quad P_1\frac{dV}{dt} = -V\frac{dP_1}{dt},$$

whence from (30)

$$\frac{(P_1^2 - P_2^2)}{l}\frac{\pi}{16} \cdot \frac{R^4}{\eta} = -V\frac{dP_1}{dt}, \qquad \ldots(31)$$

where $\frac{dP_1}{dt}$ is the rate of change of pressure at the end of entry, and V the fixed volume.

P_2 in the experiment is the constant atmospheric pressure at the end of the capillary, and P_1 has values, say, p_1 and p_2 at the beginning and ending of the period of observation of t seconds duration. Rewriting (31), we have

$$\frac{\pi R^4}{16 l \eta V} \cdot \int_0^t dt = +\frac{1}{2P_2}\int_{p_1}^{p_2}\left(\frac{1}{P_1 + P_2} - \frac{1}{P_1 - P_2}\right) dP_1,$$

$$\frac{\pi R^4 t}{16 \eta l V} = \frac{1}{2P_2}\left[\log\frac{P_1 + P_2}{P_1 - P_2}\right]_{p_1}^{p_2}$$

$$= \frac{1}{2P_2}\left\{\log\frac{p_2 + P_2}{p_2 - P_2} - \log\left(\frac{p_1 + P_2}{p_1 - P_2}\right)\right\},$$

which becomes, on writing $P_2 = P$ = atmospheric pressure,

$$\eta = \left(\frac{\pi R^4}{8lV}\right)\frac{Pt}{\log\frac{(p_2 + P)}{(p_1 + P)} \cdot \frac{(p_1 - P)}{(p_2 - P)}}.$$

The term $\frac{\pi R^4}{8lV}$ may be calculated and is a constant of the apparatus.

The Constant Pressure Method

The determination of η may be made with constant pressure difference between the ends by maintaining the l...

columns in the two tubes at a fixed difference. The point, O, would be chosen at the commencement near the bottom of the tube, and the time taken for a volume corresponding to a length of tube between the original and final positions of O measured. If p is the total constant pressure inside V, we have

$$V_1 = \frac{(p^2 - P^2)\pi r^4}{16\eta l p} \quad \text{from (30)},$$

in which all terms except η are known.

The difficulty in this modification lies in keeping the mercury levels a fixed distance apart as the gas is driven out.

This method can be easily applied to the determination of the viscosity of a gas, such as oxygen or hydrogen, generated by electrolysis.

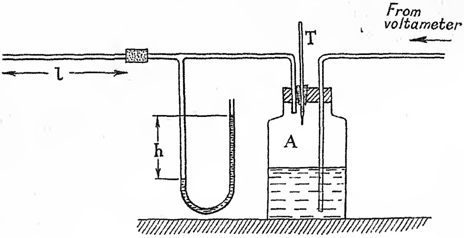


FIG. 99

The gas is led from the voltameter through a drying vessel, A, and along a tube connected to an oil manometer, and finally to the atmosphere through a capillary tube of length l. It is advisable to insert a thermometer at a convenient point to measure the temperature of the gas just before it enters the capillary tube. All joints must be air-tight, and this point requires careful attention. After a short time the manometer will become steady and the gas generated by electrolysis then flows down the tube; the flow per second can be obtained from the current passing through the voltameter and from the electrochemical equivalent of the gas. The conditions are those required for the application of the constant pressure formula in which the atmospheric pressure is P and the pressure p is that deduced from the manometer added to the atmospheric pressure.

A Simple Exercise on the Constant Pressure Method

A simple application of the constant pressure formula can be obtained by joining a piece of quill tubing to a capillary tube and mounting the combination vertically in a stand.

A pellet of mercury a few millimetres long is introduced into the quill tubing. When the end, D, of the capillary is opened to the atmosphere the pellet falls and soon acquires a constant velocity. If the length of the pellet is h, the pressure below it is $(P_0 + g\rho h)$, where P_0 denotes the atmospheric pressure. Thus, the gas below it is forced

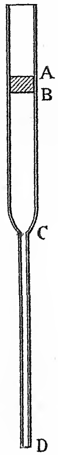


FIG. 100 (*a*)

through the tube with a constant pressure at each end. The above formula thus applies and the efflux of gas per second is

$$V_1 = \frac{(P_0 + g\rho h)^2 - P_0{}^2}{16\eta l\,(P_0 + g\rho h)} \cdot \pi r^4.$$

If terms containing h^2 are neglected this becomes

$$V_1 = \frac{g\rho h}{8\eta l} \cdot \pi r^4.$$

This approximation is justified when h is of the order of magnitude of a few millimetres.

By observing the rate of fall of the pellet and from a knowledge of the cross-section of the tube the rate at which the gas enters the tube and consequently, in the steady state the rate at which

determined. Two marks at a measured distance apart should be made on the tube and the time, t, taken for the pellet to fall between them. If the distance between the marks is l and the radius of the quill tubing is a,

$$V_1 = \frac{\pi a^2 l}{t}.$$

In this experiment no account is taken of the effect of the surface tension of the mercury on the pressure on the gas below it. This is considered in the next experiment.

Rankine's Method for the Determination of Viscosity of Gases

This method was first described by Prof. A. O. Rankine in 1910,* and has since been summarized in the *Journal of Scientific Instruments*, Vol. I (1924), p. 105, to which account the student is referred.

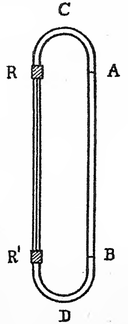


FIG. 100 (*b*)

When the necessary tubes have been carefully cleaned and thoroughly dried, the apparatus may be made by taking a length of quill tubing of from 3 to 3·5 mm. bore. This is bent as in the figure, and joined by rubber tubes R, R′, to a capillary tube of from 0·1 to 0·2 mm. diameter and 50 cm. long. The whole arrangement is then mounted on a board, so that it may be moved about with ease.

Scratches A and B are made on the quill tubing so that the volume above A is the same as that below B.

A pellet of *clean* mercury (about 5 cm. long) is introduced into the tube AB before R is fitted in position. The board is held vertically,

* *Proc. Roy. Soc.* A **83**, 265, 1910.

causing the mercury pellet to descend in AB. This pushes the air in the tube through the capillary. Observation of the mean time of several descents is made, the timing being for the interval from the instant the upper surface of the pellet is at the upper mark (A), to the instant the lower surface is at the lower mark (B). This is repeated by inverting the board several times.

The pellet should move without acceleration as it passes the upper scratch.

As has just been shown, the formula for this case is

$$\frac{v}{t}=\frac{\pi}{8}\frac{p}{l}\frac{\mathrm{R}^4}{\eta}, \qquad \ldots(32)$$

where v is the volume of gas passing through the tube in t seconds,
p is the extra pressure due to the pellet,
l is the length of the capillary tube,
R is the radius of the capillary tube,
η is the viscosity of the gas in the tube.

Using tubes of the dimensions given above, the effect of the circulation within the pellet itself is negligible, but the difference in curvature of the two ends of the pellet introduces a surface tension effect which must be corrected for. The value of p is not $\frac{mg}{\mathrm{A}}$, where m is the mass of the pellet, and A the mean cross-section of the tube, but in (32) above

$$p=\frac{mg}{\mathrm{A}}-\varepsilon,$$

where ε is the amount of the total pressure accounted for by this surface tension effect. The value of ε is constant for a given tube and may be found by altering m. This is readily done by using different lengths of pellet in the fall tube. The value of $\frac{v}{t}$ is plotted against mg; this gives a straight line, indicating the constancy of ε, whose value is found from the intercept on the axis of mg.

Now if v_0 is the volume between the scratches A and B, as determined by experiment,

$$v=v_0-\frac{m}{\rho},$$

where ρ is the density of mercury. Therefore we have

$$\eta=\frac{\pi}{8}\frac{\mathrm{R}^4}{l}\frac{\left(\frac{mg}{\mathrm{A}}-\varepsilon\right)t}{\left(v_0-\frac{m}{\rho}\right)}$$

from which η may be calculated.

When it is desired to find η for gases other than air, another method is used to allow for the surface tension effect.

The mean time obtained for observations on several descents is obtained for a pellet about 2 cm. long. Let this be denoted by t_1.

The pellet is then divided into two parts by gentle tapping of the tube. The parts should be approximately equal and should be separated as little as possible. The effect of this division will be to double the effect of surface tension.

The formula shows that the product pt remains the same, so that, if t_2 denotes the time of descent for the divided pellet, we have

$$\left(\frac{mg}{\mathrm{A}} - \varepsilon\right) t_1 = \left(\frac{mg}{\mathrm{A}} - 2\varepsilon\right) t_2.$$

This enables ε to be found, but another procedure can be adopted. Suppose there were no effect of surface tension and the time of descent were t_0.

The above two quantities would then be equal to $\frac{mg}{\mathrm{A}}\, t_0$.

The formula would then be

$$\eta = \frac{\pi \mathrm{R}^4 \frac{mgt_0}{\mathrm{A}}}{8l\left(v_0 - \frac{m}{\rho}\right)}.$$

This assumes that the volume swept out by the single and divided pellet is the same, viz. $\frac{m}{\rho}$. It is for this reason that the two parts of the pellet should have the least possible separation.

The value of t_0 in terms of t_1 and t_2 is

$$t_0 = \frac{t_1 t_2}{(2t_2 - t_1)}.$$

The value for t is obtained for the gas and then repeated for air in the tube. For the two experiments the conditions are all the same, except the contained gas, and since η is proportional to t, we may calculate η for the gas by assuming the value for air in the comparison.

CHAPTER VII

THERMOMETRY AND THERMAL EXPANSION

The Comparison of a Thermometer by means of a Standard

It is recommended that each student before beginning his experiments on heat should choose a thermometer, test its accuracy, and use it when required throughout his experiments. The method of comparison is to immerse it together with a standard thermometer in a bath and observe temperatures over a suitable range simultaneously by both instruments.

It is convenient for the following experiments to have two thermometers, one reading from 0° C. to 35° C. and the other from 0° C. to 100° C. Both should be calibrated in this way.

A large water or oil bath should be carefully heated over a Bunsen flame and constantly stirred. The thermometers should be placed in the liquid so that the mercury thread shows just above its surface, and with their bulbs close together.

A record of temperatures at intervals of 5° should be taken over the range, and a curve drawn with temperature corrections as ordinates and with readings from the thermometer to be calibrated as abscissae.

By means of this curve the readings of the thermometer in later experiments can be reduced to that of the standard.

A better method of heating the liquid is to place it in a bath standing in a box lined with cotton wool and to supply the heat by passing an electric current through a resistance coil immersed in the liquid. If the current is drawn from storage cells, and a variable resistance included in the circuit, it is possible to adjust the current so that the bath is maintained for a long time at a constant temperature. The most convenient form of stirrer is a small propeller driven by a small electric motor.

The Calibration of a Mercury Thermometer

It is impossible in practice to obtain a perfectly uniform bore in the stem of a mercury thermometer, so that it is not sufficient in an accurate instrument to divide the interval between the fixed points into a number of parts of equal lengths. The makers of thermometers usually attempt to make some correction for this lack of uniformity by adjusting the distance between consecutive divisions to suit the bore of the stem at the various points. But, in spite of this, unless the thermometer is exceptionally carefully constructed, errors remain and a calibration has to be made if accurate observations are required.

In a given thermometer, as a rule, the divisions will be unequally

spaced at different parts of the tube, and the bore will vary from point to point.

The first step is to divide the tube into segments consisting of five to ten degrees each, and over each to find the average distance between each division. We assume for the sake of definiteness that we are considering intervals of ten divisions. Measure each of these, beginning at 0°, by means of a micrometer microscope, and deduce the average length per degree for each of the ten intervals up to 100°. We are assuming that the thermometer is divided into degrees centigrade from 0° to 100°.

When this has been done a thread of mercury is broken off from the main column of length equal to that of about 10° on the thermometer scale.

This thread may be obtained by connecting a small jet, made by drawing a glass tube to a narrow bore, to a gas-pipe, and lighting the gas at the narrow end, adjusting the supply to produce a flame about half a centimetre long. If this flame be applied cautiously at the point of the thread where it is desired to sever it, the thread will divide. During the application of heat the thermometer must be rotated to avoid fracture due to unequal heating.

The thread is moved by gently jerking the thermometer until one end is at 0° and the other near 10°, and its length measured.

The thread is then moved so that one end lies at 10° and the other near 20°, and so on up to 100°.

If it is difficult to get the detached thread down to zero, owing to the projection of mercury from the bulb past the zero mark, the bulb should be cooled by wrapping it in wool and moistening with ether. This will clear the tube and allow the thread to be moved down without its rejoining the main mass of mercury.

Denote the ten lengths of the thread by l_1, l_2, l_3, etc.

Let these be reduced to their equivalents in degrees. This is readily done since the average width of one degree is known in the different parts of the scale. Denote these equivalent lengths by t_1, t_2, t_3, etc., and the mean of these by t.

If the tube had a uniform bore of the same length as that of the actual instrument between 0° and 100°, the readings would be t, $2t$, $3t$, etc., instead of t_1, $t_1 + t_2$, $t_1 + t_2 + t_3$, etc.

Let the values added respectively to t_1, t_2, t_3, etc., to make them equal to t, be δ_1, δ_2, δ_3, etc.

Then
$$t = t_1 + \delta_1,$$
$$t = t_2 + \delta_2, \text{ etc.}$$

The corrections to be applied near 10°, 20°, 30°, etc., are therefore:

$$t - t_1, \quad 2t - t_1 - t_2, \quad 3t - t_1 - t_2 - t_3, \text{ etc.,}$$

or

$$\delta_1, \quad \delta_1 + \delta_2, \quad \delta_1 + \delta_2 + \delta_3.$$

The correct temperature corresponding to t_1 is $(t_1 + \delta_1)$, and corresponding to the point $(t_1 + t_2)$ it is $(t_1 + t_2 + \delta_1 + \delta_2)$, and so on.

We have assumed up to the present that the fixed points at 0° and 100° are correct.

A table is drawn up as shown below. The correction to be applied near 10° is $-0{\cdot}023$, for the thermometer column is too long at this point, near 20° the correction to be applied is $(-0{\cdot}023 + 0{\cdot}032)$ or $+0{\cdot}009$. The correction would have been $+0{\cdot}032$ had the point near 10° been correct, but since that was not the case both errors come in. We correct similarly for other points by adding errors algebraically, and recording in the last column the amount to be added to the thermometer reading to obtain the correct temperature.

Note that in the example given the thread was not very near to the mean length of 10°. It is convenient to arrange this as closely as possible to 10°. Strictly, the error $-0{\cdot}023$ ought to be applied to the recorded temperature of 10·231, but the error will probably not vary very rapidly in the neighbourhood of any given point. Hence we take the corrections in the last column as applied at 10°, 20°, etc.

Mean length of thread, as deduced from column 4, 10·208.

Region of tube	Mean length per scale division	Length of mercury thread	Equivalent of thread in degrees	Difference from mean	Correction to apply to upper reading
0°– 10°	0·2385	2·442	10·231	δ_1 $-0{\cdot}023$	$-$ 0·023
10°– 20°	0·2389	2·431	10·176	δ_2 $+0{\cdot}032$	$+$ 0·009
20°– 30°	0·2391	2·449	10·243	δ_3 $-0{\cdot}035$	$-$ 0·026
30°– 40°	0·2389	2·435	10·193	δ_4 $+0{\cdot}015$	$-$ 0·011
40°– 50°	0·2398	2·438	10·167	δ_5 $+0{\cdot}041$	$+$ 0·030
50°– 60°	0·2350	2·426	10·323	δ_6 $-0{\cdot}115$	$-$ 0·085
60°– 70°	0·2355	2·404	10·208	δ_7 0·000	$-$ 0·085
70°– 80°	0·2353	2·393	10·170	δ_8 $+0{\cdot}038$	$-$ 0·047
80°– 90°	0·2347	2·392	10·192	δ_9 $+0{\cdot}016$	$-$ 0·031
90°–100°	0·2367	2·408	10·173	δ_{10} $+0{\cdot}035$	$+$ 0·004

Draw a curve with thermometer readings as abscissae and the corrections to be applied to obtain the corrected readings as ordinates. The curve should pass through the x-axis at 0° and 100°, since these points have been assumed to be correct.

If, however, the fixed points are incorrectly placed, the errors must be found in the usual manner with ice and steam.

Suppose that the zero correction is δ_0, while that at 100° is δ_{100}.

The upper fixed point must be corrected for pressure, latitude, and height above sea-level.

The barometer must also be corrected, owing to the fact that it is probably not read at the temperature at which the instrument was standardized.

This last correction may be made by the following formula:

If h_0 denotes the height at 0°, and h_t that read at t°,

$$h_0 = h_t\,(1 - 0{\cdot}000162t).$$

For latitude λ and at a height d feet above sea-level, the length of the column which produces the standard pressure at 0° and at sea-level in the standard latitude of 45° is:

$$L = (760 + 1{\cdot}94562 \cos 2\lambda + 0{\cdot}000045466d) \text{ mm.}$$

Under this pressure, L, the boiling-point is 100° at the height and latitude of the place of observation.

Thus, h_0 is equivalent to $\frac{h_0}{L}$ atm. or $\frac{h_0}{L} \times 760$ mm. under normal conditions. From this and the following table the correction to the boiling-point may be made.

Maximum vapour pressure	Corresponding temperatures
720 mm.	98·493° C.
725	98·686
730	98·877
735	99·067
740	99·255
745	99·443
750	99·630
755	99·815
760	100·000
765	100·184
770	100·366
775	100·548
780	100·728

The zero on the scale actually records the reading $-\delta_0$, and the 100° records $(100 - \delta_{100})$.

Thus a correction is required for this.

Plot on the curve the two points (0, 0), $(100, \delta_{100} - \delta_0)$, and join them by a straight line. The ordinate of this line at x° is $(\delta_{100} - \delta_0) \cdot \frac{x}{100}$, since each degree, even if correct as regards bore, would register only $\frac{100 - \delta_{100} + \delta_0}{100}$, we must add to it an amount, $\frac{\delta_0 - \delta_{100}}{100}$, on account of the errors at the fixed points.

To $x°$ we must add $$x \cdot \frac{\delta_0 - \delta_{100}}{100}.$$

Thus, if we draw on the same graph on which the first set of results was plotted this second curve, the difference between the two ordinates taken algebraically will give the true reading corrected for errors due to bore and fixed points.

Redraw a new curve, showing as abscissae temperatures as recorded by the thermometer and as ordinates the difference of the ordinates of the two curves, and the resulting curve will give the amount to be added to any recorded temperature to give the true temperature.

Throughout the taking of measurements the temperature of the detached thread should remain constant, and in order to be sure that this condition holds, place another thermometer close by and observe whether it varies or not. Do not handle the thermometer under examination more than is necessary, and only do so by holding it at the tip away from the bulb.

Newton's Law of Cooling

The object of this experiment is to verify Newton's Law of Cooling, which states that the rate at which a body cools is proportional to the difference of temperature between itself and the enclosure in which it is placed. The constant of proportionality depends on the surface exposed and the thermal capacity of the exposed body, and the law is true for small differences of temperature only.

The conditions under which this law applies* approximately must be considered. The body must be supported in the air in a place in which there are no draughts. The heat of the body produces local convection which is chiefly responsible for the loss of heat. In addition, there is a small loss due to radiation and conduction. The apparatus to be described is designed to comply with the conditions and to ensure a definite temperature of the surroundings.

The apparatus required is a small metal thimble, into which water at about 80° C. can be placed, provided with a cork through which a thermometer passes for noting the temperature of the liquid.

The enclosure consists of two calorimeters, one inside the other, and containing water in the space between, to provide an enclosure at nearly constant temperature. A thermometer placed in this water, which should be stirred occasionally, gives the temperature during the experiment.

Observe the temperature recorded by T_1 and T_2 (fig. 101), at intervals of half a minute during the initial stage of the fall, and, as the rate decreases, the interval between observations may be increased. The record of T_2 should not vary very much.

* The conditions described are not exactly those originally stated in association with this law.

Tabulate the results thus:

Time (mins.)	Record of T_1	Record of T_2
0·0	72·5° C.	12·2° C.
0·5	71·3° C.	12·2° C.
1·0	70·1° C.	12·2° C.
1·5	69·0° C.	12·2° C.
2·0	67·8° C.	12·2° C.
2·5	66·7° C.	12·2° C.
3·0	65·6° C.	12·2° C.
3·5	64·5° C.	12·2° C.
4·0	63·5° C.	12·3° C.
4·5	62·5° C.	12·3° C.
—	—	—
—	—	—
—	—	—

Draw a curve showing the relation between the temperature T_1 and the corresponding time.

Make the temperatures the ordinates and times the abscissae.

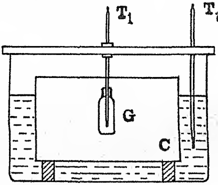


FIG. 101

The rate of fall of temperature may be obtained from this curve by measuring the tangent of the angle of inclination of the tangent to the curve to the axis of t. This measures the value of $\frac{dT_1}{dt}$ at the various points of the curve.

According to Newton's law these values should be proportional to the differences between T_1 and T_2. If T_2 does not vary very much its mean value may be regarded as the mean temperature of the enclosure.

If T_2 varies too much to permit this approximation, draw a curve showing the relation between T_2 and the time on the diagram which shows the relation between T_1 and the time.

Then at each time we can determine the value of $(T_1 - T_2)$ and the corresponding value of $\frac{dT_1}{dt}$ from the same graph.

Make another table containing two columns, one for values of $(T_1 - T_2)$ and the other for the corresponding values of $\frac{dT_1}{dt}$.

Draw a curve with the values of $(T_1 - T_2)$ as ordinates, and those of $\frac{dT_1}{dt}$ as abscissae, when, if these two quantities are proportional, the result should be a straight line.

The Use of the Weight Thermometer

The weight thermometer consists of a glass bulb, B, drawn out at the upper end into a capillary stem, A.

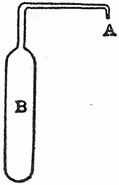


FIG. 102

A convenient size is obtained by making B about one and a half to two inches long, and between a quarter and half an inch wide. It forms a good exercise to make the apparatus from a piece of glass tubing. If this is done, care must be taken to get rid of any blob of glass likely to accumulate at the bottom of B, otherwise in the subsequent heating B will be very likely to crack.

The apparatus measures the expansion of any liquid placed in B relative to glass. In order to deduce the real expansion it is necessary to know that of B.

The coefficient of expansion of mercury has been found with great accuracy, and may be taken as 0·0001818.

We may therefore use mercury to find the expansion of B, and then use B to determine the expansion of other liquids. We shall consider the determination of the expansion of water.

(1) *Determination of the Expansion of the Weight Thermometer*

Carefully weigh the apparatus.

Surround B by wire gauze and warm carefully in order to drive out

air. Place the end A under clean mercury contained in a dish and allow B to cool so that a little mercury enters. Then, when sufficient has entered, boil the mercury so that the space above it may become full of mercury vapour. Once more place A below the mercury, when B will fill as the vapour condenses on cooling.

It is best to carry out the process gradually, heating and cooling B several times, and allowing a little mercury to enter at a time.

Warm up the mercury first to prevent the bulb from cracking when it enters.

Allow B to cool gradually, and finally surround it with ice, keeping A immersed all the time.

Bring up a small, weighed, empty dish, and remove the mercury at A, replacing it by the empty dish. Remove the ice and allow B to acquire the temperature of the room. Mercury will, of course, flow over into the dish. As soon as the flow ceases, weigh the bulb, B, and the dish. Let the total weight of mercury within B at 0° be denoted by W_0.

Immerse B and as much as possible of the stem in boiling water, and catch the mercury that flows out in the little dish.

After the bulb has remained in the water for a quarter of an hour, to allow it to assume the temperature of the boiling water, remove it, dry and carefully weigh it. Suppose that w is the amount that has flowed out. As a check reweigh the dish and again determine w.

Let ρ_0 denote the density of mercury at 0°C. and ρ_1 that at temperature T.

If V_0 denote the volume of the apparatus at 0° and V_1 that at T, while β denotes the coefficient of expansion of glass,

$$\frac{V_1}{V_0} = 1 + \beta T.$$

But

$$V_0 = \frac{W_0}{\rho_0}, \quad V_1 = \frac{W_0 - w}{\rho_1}.$$

$$\therefore \quad 1 + \beta T = \frac{W_0 - w}{W_0} \cdot \frac{\rho_0}{\rho_1} = \frac{W_0 - w}{W_0} \cdot (1 + \alpha T).$$

$$\therefore \quad \beta = \frac{W_0 - w}{W_0} \cdot \frac{1 + \alpha T}{T} - \frac{1}{T}$$

$$= \frac{W_0 - w}{W_0} \alpha - \frac{w}{W_0 T},$$

or

$$\alpha = \frac{W_0}{W_0 - w} \beta + \frac{w}{W_0 - w} \cdot \frac{1}{T},$$

i.e.

$$\alpha - \beta = \frac{w}{W_0 - w} \beta + \frac{w}{W_0 - w} \cdot \frac{1}{T}.$$

The ratio $\frac{w}{W_0 - w}$ is small, and since β is also small we may often neglect the first term on the right and use simply

$$\alpha - \beta = \frac{w}{W_0 - w} \cdot \frac{1}{T}.$$

(2) *The Coefficient of Expansion of Water*

The value of this coefficient varies considerably throughout the range 0° C. to 100° C. In this experiment the average value between two temperatures, say, 20° C. and 60° C. will be determined. If both temperatures are above that of the room the experimental difficulties are not so great. We assume this to be the case though the method is quite general, and exactly the same precautions have to be taken as in (1) if water flows out on moving B from the lower temperature enclosure to the balance. Carefully fill the weight thermometer with water that has been recently boiled to get rid of contained air, and place it completely immersed in water at temperature t_1. Remove it, dry, and weigh.

Place the thermometer in water at t_2 and repeat.

Use the formula given above to deduce α for water, making use of the value of β previously determined.

Determination of the Density of Water at Various Temperatures by means of a Glass Sinker

In this experiment a solid is weighed while totally immersed in water at different temperatures, so that by the principle of Archimedes the weights of the fluid displaced by the solid corresponding to the various temperatures are known.

Let V_0 denote the volume of the solid at 0° C. and α the coefficient of expansion of the solid, so that at temperature $t°$, the volume is $V_0(1 + \alpha t)$. If ρ_t is the density of the water at temperature, t, the loss of weight due to immersion is $V_0(1 + \alpha t)\rho_t$. This value is observed by the balance; let it be W_t.

Then $$\rho_t = \frac{W_t}{V_0(1 + \alpha t)}.$$

In order to carry out the experiment, a long wire is attached to the scale-pan of a balance and passed through a hole in the base of the balance to support the solid, which hangs in the water. Since during some part of the experiment the water will be at a temperature considerably above that of the immediate surroundings of the balance, it is necessary to use a wire about 40 cm. long, so that convection currents may not disturb the equilibrium of the balance.

The wire should have a diameter not greater than $\frac{1}{10}$ mm., so that

surface tension may not cause any appreciable effect where it enters the water. In practice thin copper wire is often employed, though it is preferable to use a short length of platinum wire, specially treated to diminish surface tension effect for immersion in the water.

A stirrer is necessary to keep the temperature of the liquid uniform, and as soon as the whirls, due to stirring, have died away, the balance is made and the temperature taken by a thermometer immersed in the water with its bulb as near as possible to the solid.

It is preferable to heat up the water to the highest desirable temperature and allow it to cool down. In this case the weights in the scale-pan will require to be continually diminished. The weights should be adjusted before an observation so that the solid appears a little too heavy. After a short interval the scale pointer will cross the zero position, and at this instant the temperature of the water should be observed.

In this way a series can be obtained very conveniently for temperatures above that of the atmosphere.

The solid is first weighed in air so that the values, W_t, corresponding to different temperatures may be obtained by subtraction.

As a sinker it is usual to employ a glass bulb containing lead shot, and α may then be taken as 0·0000232.

When it is not possible to reduce the temperature right down to zero it may be taken down to some convenient low temperature, t_0. By finding the weight, W_{t_0}, of the submerged vessel at this temperature, the volume, V_{t_0}, may be deduced by the help of the density table appearing on p. 179, taken from Kohlrausch's *Physical Measurements*,

since
$$V_{t_0} = \frac{W_{t_0}}{\rho_{t_0}}.$$

The densities at the other temperatures may then be deduced by the formula

$$\rho_t = \frac{W_t}{V_{t_0}\{1 + \alpha(t - t_0)\}}.$$

In the laboratory it is convenient to begin at a temperature of about 80°, and make observations about every 10° C.

A curve should be drawn exhibiting the relation between temperature and density.

Alternatively, the experiment may be made in order to determine the coefficient of expansion of water for varying intervals of temperature from the formula: $\rho_0 = \rho_t(1 + \beta t)$.

The Constant Pressure Air Thermometer

In this form of thermometer temperature is defined by means of the equation
$$V_t = V_0(1 + \alpha t);$$

DENSITY OF WATER BETWEEN 0° AND 20° C. PER C.C.

Temp.	Density	Temp.	Density
0°	0·99988	11°	0·99965
1°	0·99993	12°	0·99955
2°	0·99997	13°	0·99943
3°	0·99999	14°	0·99930
4°	1·00000	15°	0·99915
5°	0·99999	16°	0·99900
6°	0·99997	17°	0·99884
7°	0·99993	18°	0·99866
8°	0·99988	19°	0·99847
9°	0·99982	20°	0·99827
10°	0·99974		

t denotes the temperature, V_t and V_0 the volumes occupied of a certain mass of gas at two temperatures, the former at $t°$ and the latter at a convenient fixed point: the zero of the scale.

By choosing the melting-point of ice as $t = 0°$, and the boiling-point of water as $t = 100°$, under standard conditions we can find the value of α.

We may therefore say that the equation is assumed, and that t is defined by it,

or $$t = \frac{1}{\alpha}\left(\frac{V_t}{V_0} - 1\right).$$

It is assumed that the pressure remains constant throughout.

The diagram shows a simple form of constant pressure thermometer.

The mercury reservoir is adjusted so that the mercury stands at the same level in the two tubes, CD and EF. When this is the case the pressure is equal to that of the atmosphere in both tubes.

Suppose it is desired to determine a certain temperature with this instrument, say the melting-point of wax. First surround the bulb completely with powdered ice and turn the three-way tap, T, so that B and CD are open to the room, and adjust G so that the mercury stands at the zero division on the graduated scale of CD.

Wait for about ten minutes to allow the bulb to cool exactly to 0° C., and turn the tap so that B and CD are connected to each other but cut off from the atmosphere. In this way the zero on the scale is made to correspond to 0° C.

Now immerse the bulb in boiling water, and lower G until the mercury stands at the same level in the two tubes, and observe the scale reading.

Read the barometric height, and deduce the boiling-point of the water.

Surround the bulb with warm water and adjust its temperature to the melting-point of the wax. To do this, put a small piece of the wax in a small test tube and immerse it in the water. Heat the water until the wax melts, and then let it cool a few degrees, and then warm up

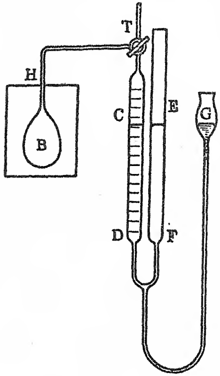


Fig. 103

very slowly, keeping the water stirred until the wax begins to melt again and then take the reading of the thermometer. Of course G must be adjusted so that the level of the mercury is the same in both tubes.

From the observations made we can deduce the melting-point of the wax. Note the temperature also by means of an ordinary mercurial thermometer.

Theoretical Considerations

The fundamental equation of gas thermometry, whatever may be the form of thermometer, is simply:

Total mass of gas in the instrument = constant.

In practice it is not possible to maintain all the gas at the same temperature; some of it is necessarily remote from the point of application of the body examined. These remote regions are described by the term 'dead space'.

In our apparatus the dead space extends from H to the level of the mercury in C.

We shall suppose that the scale readings are in c.c.s, beginning at the zero and extending downwards.

Let the reading corresponding to the case when boiling water surrounds B be denoted by x_b, and let x_w be the reading when the wax is melting.

Let the volume from the top of the bulb at the point where it is immersed to the zero of the scale be denoted by v, with a suffix to indicate the temperature at which it is measured.

The temperature of the dead space, which has a total volume $(v + x)$, will vary from one end to the other; but we shall make our calculations by assuming that this temperature is uniform throughout and equal to that measured, by placing a thermometer in a position approximately midway between the two ends of this space.

We shall denote this by the letter τ, and when the bulb temperature is t we shall write τ_t for the corresponding temperature of the dead space.

Let the volume of the bulb together with that part of the tube which is immersed be V_0 at the temperature zero, let β denote the coefficient of cubical expansion for glass, and ρ_0 the density of air at zero.

The mass of gas in terms of the quantities measured when the bulb is at 0° C. is

$$V_0\rho_0 + v_{\tau_0}\,\rho_{\tau_0}.$$

The temperature of the dead space is, of course, not necessarily at 0°, it has some value τ_0.

This may be expressed in the form:

$$\rho_0\left\{V_0 + v_0\frac{(1+\beta\tau_0)}{(1+\alpha\tau_0)}\right\}. \qquad \ldots(1)$$

When the bulb is at a temperature, t, let x_t denote the reading on the scale. This denotes the volume between the zero of the scale and the mark x_t at the temperature at which the apparatus was graduated. This is often done at about 15° C., and the laboratory temperature is usually in this neighbourhood. We shall not introduce any great error into our calculations if we regard this as measuring the true volume of this part of the apparatus under the conditions of the experiment.

The temperature of the dead space is now τ_t, and the total volume is therefore

$$v_0(1+\beta\tau_t) + x_t.$$

The mass of gas in the dead space is

$$\frac{v_0(1+\beta\tau_t)\,\rho_0}{1+\alpha\tau_t} + \frac{x_t\rho_0}{1+\alpha\tau}.$$

Hence the total mass is measured by

$$\frac{V_0(1+\beta t)\rho_0}{1+\alpha t}+\frac{v_0(1+\beta\tau_t)\rho_0}{1+\alpha\tau_t}+\frac{x_t\rho_0}{1+\alpha\tau_t}. \qquad \ldots(2)$$

If the expressions (1) and (2) be equated, since they denote the same quantity, it will be found that:

$$x_t = V_0(1+\alpha\tau_t)\left\{\frac{(\alpha-\beta)t}{1+\alpha t}+\frac{v_0}{V_0}\left(\frac{1+\beta\tau_0}{1+\alpha\tau_0}-\frac{1+\beta\tau_t}{1+\alpha\tau_t}\right)\right\}. \qquad \ldots(3)$$

The second term in the square bracket may be written

$$\frac{v_0(\alpha-\beta)(\tau_t-\tau_0)}{V_0(1+\alpha\tau_0)(1+\alpha\tau_t)},$$

where we have neglected the product $\alpha\beta$ in the numerator.

In practice the difference between τ_t and τ_0 is not very large, and the construction of the apparatus provides that $\frac{v_0}{V_0}$ is small.

We may thus neglect this term in comparison with the first term in the bracket without introducing any great error. The student is recommended to find an approximate value of the two terms, so that he may better appreciate the effect of this neglect of the second term.

We then have $$x_t = V_0(\alpha-\beta)t\cdot\frac{1+\alpha\tau_t}{1+\alpha t}. \qquad \ldots(4)$$

If we apply this to the case when the bulb is surrounded by boiling water, of which the temperature is b, corrected, of course, for any variation of the barometric height from normal, we have from (4)

$$x_b = V_0(\alpha-\beta)b\cdot\frac{1+\alpha\tau_b}{1+\alpha b}. \qquad \ldots(5)$$

Hence from (4) and (5)

$$\frac{x_t}{x_b}=\frac{1+\alpha\tau_t}{1+\alpha\tau_0}\cdot\frac{1+\alpha b}{1+\alpha t}\cdot\frac{t}{b}$$

$$=\frac{\frac{1}{\alpha}+\tau_t}{\frac{1}{\alpha}+\tau_b}\cdot\frac{\frac{1}{\alpha}+b}{\frac{1}{\alpha}+t}\cdot\frac{t}{b}. \qquad \ldots(6)$$

The value of $\frac{1}{\alpha}$ may be taken as 273·1.

τ_t and τ_b may be observed on a mercury thermometer, although this is introducing into the experiment the mercury scale.

Equation (6) is a linear equation in t, which is thus determined from the reading on the scale of the air thermometer.

This enables us to deduce the temperature of the melting wax.

The Constant Volume Air Thermometer

The diagram illustrates a common form of the apparatus.

The bulb, B, is connected by a capillary tube to rubber tubing, DE, and to the glass tube, EF.

EF slides against a scale, SS, by means of which the height between the mercury levels in the tubes, CD and EF, can be read.

EF can be clamped in any desired position, so that the level of the mercury in the tube CD stands always at a definite mark, C.

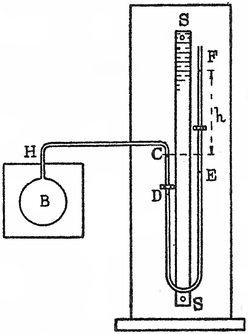


FIG. 104

In this case the temperature is defined by assuming the relation:

$$p_t = p_0 (1 + \alpha t), \quad \text{or} \quad t = \frac{1}{\alpha} \cdot \frac{p_t - p_0}{p_0}.$$

p_t denotes the pressure within B at a temperature t, and p_0 that at a standard zero position—the temperature of melting ice, while the volume of gas remains constant.

The pressures are measured by adding the atmospheric pressure to that due to the mercury column of length, h.

In practice, the part of the apparatus containing the air expands with rise of temperature, and there is the dead space, HC, to be allowed for as in the last experiment.

Let V_0 denote the volume of the bulb and immersed portion of the apparatus, v_0 that of the dead space at 0°.

Suppose that the dead space remains at temperature τ_0 during the experiment when B is surrounded with ice, and τ_b when it is surrounded by steam.

We shall show how α may be determined experimentally.

If ρ_0 denotes the density of air under normal conditions, and ρ that at pressure, p, and temperature, t,

$$\rho = \frac{\rho_0}{1 + \alpha t} \cdot \frac{p}{76}.$$

The mass of gas contained in the thermometer is

$$\frac{p_0 \rho_0}{76} \cdot V_0 + \frac{p_0 \rho_0}{76} \cdot v_0 \frac{(1 + \beta \tau_0)}{(1 + \alpha \tau_0)}. \qquad \ldots(7)$$

expressed in terms of the conditions prevailing when B is at the temperature of melting ice.

Similarly, the mass of gas expressed in terms of the conditions prevailing when B is at temperature, t, is

$$\frac{V_0 (1 + \beta t) \cdot p_t \rho_0}{76 (1 + \alpha t)} + \frac{V_0 (1 + \beta \tau_t) p_t \rho_0}{76 (1 + \alpha \tau_t)}. \qquad \ldots(8)$$

Hence, on equating (7) and (8):

$$p_0 \left\{ V_0 + \frac{v_0 (1 + \beta \tau_0)}{(1 + \alpha \tau_0)} \right\} = p_t \left\{ V_0 \frac{1 + \beta t}{1 + \alpha t} + v_0 \frac{1 + \beta \tau_t}{1 + \alpha \tau_t} \right\}. \qquad \ldots(9)$$

Again, on account of the smallness of the ratio $\frac{v_0}{V_0}$ we have as an approximation

$$p_0 = p_t \frac{1 + \beta t}{1 + \alpha t}. \qquad \ldots(10)$$

When the bulb is at the temperature, b, of boiling water,

$$p_0 = p_b \frac{1 + \beta b}{1 + \alpha b}. \qquad \ldots(11)$$

We may assume the value 0·0000232 for β and thus calculate α from (11) by observing the pressures when the bulb is surrounded by melting ice and by steam respectively.

Equation (10) then enables us to deduce the temperature corresponding to any pressure, p_t.

Take a mercury thermometer and immerse it close to the bulb, observing its readings and the corresponding pressures. From the latter deduce the temperatures from (10) and draw up a table recording these in one column opposite to the records of the mercury thermometer in a second column.

Draw a graph with air temperatures as ordinates and mercury temperatures as abscissae, exhibiting the deviations between the two temperature scales.

In order to calculate the value of $\frac{v_0}{V_0}$ if this is necessary, first adjust the mercury to the mark C, immersing B in water that has come to the room temperature.

Read off the pressure, P, to which the air is now subjected.

Carefully raise EF so that the mercury approaches the bend at H, and so fills nearly all the dead space. Let the pressure within the bulb be now P_2. Then, since the conditions are isothermal and the whole volume, v_0, has been very nearly filled with mercury in the second case,

$$P_2V_0 = P_1 (V_0 + v_0).$$

$$\therefore \quad \frac{v_0}{V_0} = \frac{P_2 - P_1}{P_1}.$$

We may read the temperatures τ_0 and τ_t by means of a mercury thermometer placed close to the dead space and obtain a closer approximation to the value of α.

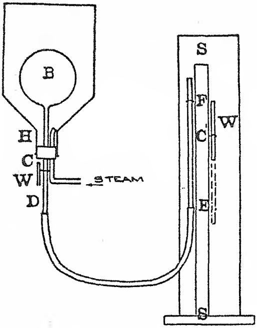


FIG. 105

In the second form of apparatus (fig. 105), into which the first may be readily converted, the volume of the dead space is made negligible.

In order to measure the difference of level between C and F, two tubes, WW, connected by a rubber tube containing water are adjusted so that the level on the left is the same as that at C, and consequently this is the same on the right at C'.

The distance, C'F, is readily observed on SS.

CHAPTER VIII

CALORIMETRY

The Specific Heat of a Solid by the Method of Mixture

THE student will be familiar with the principle of the method of mixture. The main object in this description is to give an account of the method of making a correction for the error arising from exchanges of heat with the surroundings.

If W is the water equivalent of the calorimeter and contents, m the mass of the solid, and s its specific heat, and if t_1 is the initial temperature of the calorimeter, t_2 the final temperature, and T that of the solid initially, then if there has been no loss of heat we have

$$ms\,(\mathrm{T} - t_2) = \mathrm{W}\,(t_2 - t_1).$$

In practice there is a loss or gain of heat from or to the calorimeter, which should be added on the right-hand side of this equation, since all the heat from the solid has not been retained in the calorimeter.

We may make the correction by the consideration that the final temperature, t_2, would have been $t_2 + \Delta t$, where Δt is an interval of temperature which must be small if the experiment is to be successful.

Hence the corrected equation is

$$ms\,(\mathrm{T} - t_2) = \mathrm{W}\,(t_2 + \Delta t - t_1).$$

Δt will be small if during half the experiment the calorimeter gains heat, and in the other half loses heat. This can be arranged by adjusting the initial temperature so that the room temperature is approximately a mean between it and the final temperature.

A preliminary experiment is made to find out roughly the temperatures that will be attained during the experiment.

The same amounts of the materials are used in a second case, but the calorimeter is cooled down by adding small pieces of ice or warmed up, as may be necessary, so that t_1 and t_2 may lie at nearly equal temperature intervals below and above the temperature of the room.

The temperature of the calorimeter is noted immediately before immersing the hot body, and then at quarter- or half-minute intervals until the maximum temperature is attained, the observations being continued beyond this point at definite intervals.

These results should be plotted on a graph (fig. 107).

The curve obtained will be similar to ABC. Had there been no losses or gains, the curve would have been similar to ADE, the final temperature remaining constant at the level DE.

Regnault's method of making the correction is actually to make the correction of the ordinates and draw the curve ADE.

For this purpose we require to know the rate of cooling at any particular temperature.

Find the rate at one temperature, e.g. at the mean temperature over the range BC, and plot $\frac{d\theta}{dt}$ against θ on a curve. In doing this it is assumed that the relation is linear in accordance with Newton's law.

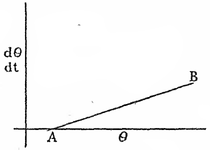


FIG. 106

At the temperature of the room $\frac{d\theta}{dt}$ vanishes; for if the calorimeter and surroundings were at the same temperature then there would be no loss or gain.

We therefore have two points on the graph, and by joining these by a straight line we can determine $\frac{d\theta}{dt}$ at any value of θ (fig. 106).

Divide ABC into sections very nearly straight, as AR, RQ, etc. Note the mean temperature over AR and from the graph for $\frac{d\theta}{dt}$ note the rate $\frac{d\theta}{dt}$ for this temperature. Multiply this by the time, ON, and add the result to the ordinate, NR, thus obtaining NR^1. Let this correction be $\delta\theta_1$.

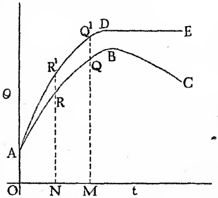


FIG. 107

In the same way find the amount $\delta\theta_2$, lost during the interval, NM. Add the sum $(\delta\theta_1 + \delta\theta_2)$ to MQ and so obtain MQ^1. Continue this

The curve obtained in the experiment should attain this horizontal branch very nearly, and the ordinate is the quantity $t_2 + \Delta t_2$.

Another way of making the correction for the interchange of heat with the surroundings without actually drawing the corrected graph is as follows.

Suppose that a calorimeter containing a liquid has been mounted in an enclosure in conditions in which Newton's law holds and that a hot body is ready to be immersed.

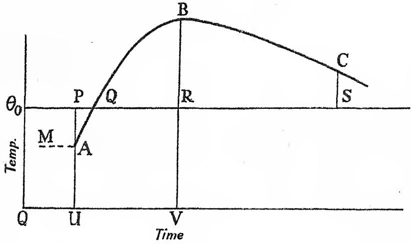


FIG. 108

The temperature of the calorimeter and contents should be recorded for a few minutes before immersion and the graph of temperature against time drawn. This will resemble MA in fig. 108. The time of immersion of the body should be noted at frequent intervals during the rise along AB to the maximum. This change may occur rapidly and it is advisable to record the temperature at quarter- or half-minute intervals. Beyond B the fall of temperature is slower and a reading every minute will suffice.

The line PQRS at the level of the temperature of the surroundings should be drawn.

The point A corresponding to the point of immersion of the solid can be found by noting the time at which the body entered the liquid. Draw perpendiculars from A, from the point of maximum temperature, B, and from some convenient point, C, on the descending part of the curve on to PQRS.

According to Newton's law the rate of loss of heat from the calorimeter is given by the formula

$$\frac{dQ}{dt} \propto (\theta - \theta_0).$$

If W is the water equivalent of the calorimeter and its contents

$$\frac{dQ}{dt} = W\frac{d\theta}{dt}.$$

Thus
$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

or
$$\frac{d\theta}{dt} = k(\theta - \theta_0).$$

The constant k depends on the water equivalent, on the area of the surface of the calorimeter and on the nature of the surface, all of which remain unaltered during the experiment.

Thus the temperature change due to the operation of Newton's law is $k(\theta - \theta_0)\,dt$.

In an extent such as AB, the total change due to this cause is $k\int(\theta - \theta_0)\,dt$. This will be described as $\Delta\theta$ to denote that it is only part of the change of temperature in this extent, the other change being due to the hot body. $\Delta\theta$ is conveniently represented on the graph for $\int\theta dt$ taken between the limits AB is equal to the area AQBVU, while $\int\theta_0 dt$ between the same limits is PRVU. Thus the difference of the two quantities is (QBR $-$ APQ). This is the area between the curve AQB and the line of the surrounding temperature, the part APQ being subtracted if it lies below this line.

If the graph be drawn on squared paper this area can be measured in terms of unit squares with sides 1 second and 1 degree. We cannot determine $\Delta\theta$ until k is found.

This can be found from the extent BC, for in this case the loss is $k \times$ BCSR. In this region the only temperature loss, $\Delta\theta'$, occurs according to Newton's law, so that $\Delta\theta'$ is equal to the temperature difference between B and C in this extent. We can write $\Delta\theta' = \theta_B - \theta_C$.

Thus
$$\frac{\Delta\theta}{\theta_B - \theta_C} = \frac{\text{Area between AB and line of } \theta_0}{\text{Area between BC and line of } \theta_0}.$$

There is some difficulty in locating the exact position of the maximum, B, but this will lead only to a small error if care is taken. It must also be remembered that the correction $\Delta\theta$ must be small if the conditions are to be favourable to accurate work in calorimetry. It should not exceed $\frac{1}{2}$° C., and if the calculation leads to a value of 1° C. or more the result is not likely to be reliable, and the experiment should be repeated paying attention to the conditions described at the beginning of this chapter.

Specific Heat of a Liquid by the Method of Cooling

The rate of loss of heat of a body depends only on the temperature of the body and that of its surroundings, on the area, and on the nature of the surface exposed.

If the difference of temperature between the body and its surroundings is not large, the rate of emission of heat is proportional to the temperature difference. This is Newton's Law of Cooling, and it is assumed that the conditions in this experiment are such that this law is applicable. It can be shown, however, that the same method and the same final formula can be used in the more general case in which the rate of emission of heat is some regular function of the temperature of the body.

Suppose a mass of liquid, M_1, is enclosed within a calorimeter of mass m_1, and let S_1 and s_1 denote the specific heats respectively. The thermal capacity of the system is $(M_1S_1 + m_1s_1)$. If the temperature falls from t_1 to t_2 in n_1 seconds, the average rate of loss of heat is $(M_1S_1 + m_1s_1)\dfrac{t_1 - t_2}{n_1}$.

In the case of the second liquid under the same conditions, let n_2 denote the number of seconds required for a fall of temperature from t_1 to t_2, and the loss of heat per second is

$$\left(M_2S_2 + m_1s_1\right)\left(\frac{t_1 - t_2}{n_2}\right).$$

We have by Newton's law:

$$(M_1S_1 + m_1s_1)\frac{t_1 - t_2}{n_1} = (M_2S_2 + m_1s_1)\frac{t_1 - t_2}{n_2};$$

$$S_1 = \frac{n_1(M_2S_2 + m_1s_1)}{n_2M_1} - \frac{m_1s_1}{M_1}. \qquad ...(1)$$

The apparatus for carrying out the determination consists of a small calorimeter fitted with a rubber stopper through which a thermometer may pass (fig. 109). A calorimeter of aluminium of about an inch diameter and three inches high serves the purpose very well. The calorimeter should be supported by threads, or should stand on a non-conductor within a double-walled enclosure, the thermometer passing through a cork in the lid of the enclosure.

In order to secure a uniform temperature, the space between the walls of the enclosure may be partly filled with water. In this case care must be taken that the inner box does not float, or it may happen that it will touch the calorimeter and there will be loss of heat by conduction.

The enclosure may consist of two calorimeters—an outer large one fitted with a lid, and an inner smaller one standing on blocks.

First fill the aluminium calorimeter about two-thirds full of water, and warm it to a temperature of about 70° C. by immersing it in hot water. Place the apparatus in the position shown in the diagram, and take readings of the thermometer at intervals of half or whole minutes down to a temperature below 30° C.

Note from time to time the temperature of the enclosure, which should hardly vary during the experiment.

Some time must of necessity elapse between the observations on one liquid and those on another, and although it is not difficult to maintain a constant enclosure temperature throughout each set of observations, it often happens that the mean temperatures recorded by the thermometer, T_2, are appreciably different in the two cases.

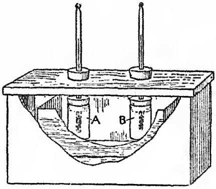


FIG. 109

This difficulty may be avoided by using a second aluminium container similar to G, and suspending it by the side of G inside C. The records of the temperatures of the two liquids are then made almost together, and the enclosure temperature is the same for each.

Make up a table containing the liquid temperatures opposite the times of observation, and in a third column record the enclosure temperatures.

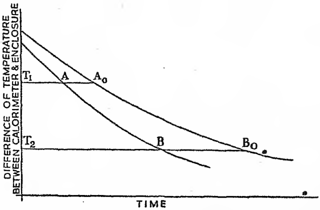


FIG. 110

Draw on the same graph as illustrated in fig. 110 the curves, one for each liquid, with the differences of temperature between liquid and enclosure as ordinates, and with the times for abscissae.

Let AB denote the curve for paraffin (say) and A_0B_0 that for water.

Draw the horizontal lines, T_1AA_0 and T_2BB_0 to cut the curves at A, A_0 and B, B_0 as shown.

Let t_1 denote the temperature of the liquids above that of the enclosure in the first case, and t_2 the corresponding temperature in the second.

Then in the time that elapses between the instant measured by T_1A until that measured by T_2B the paraffin cools down the interval $(t_1 - t_2)$, and the water cools down the same amount during the interval between T_1A_0 and T_2B_0. Denote the two periods of cooling by n_1 and

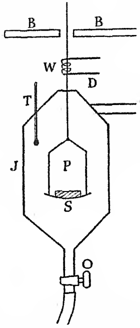


FIG. 111

n_2. In this case the value of S_2 is unity, and for aluminium the value of s_1 is 0·219.

By weighing the liquids and calorimeters we obtain sufficient data to give the value of s_2 for paraffin by means of the formula (1).

Determination of the Specific Heat of a Solid by means of Joly's Steam Calorimeter

A metal jacket, J, enclosed in a casing of felt surrounds a platinum pan, P, suspended by means of a fine wire attached to one arm of a balance, whose base is shown at BB.

The upper end of J is closed by a light metal disk, D, through a hole in which the wire passes. This disk is free to move, and when oscillations occur in P it finally settles down so that the wire passes through the hole without contact with the disk. Just above D, a small coil of wire carrying a heating current round the suspension prevents condensation on it and also on D.

In the first place, J is allowed to attain the temperature of the room, and the inlet and outlet pipes are then closed.

The pan is balanced in the usual way, and in the meantime water is boiled in a container ready to supply J with steam by means of I.

When a good supply is obtained O is opened and steam passed through I. When the steady state is attained it will be found that additional weights are required to counterpoise P on account of the condensation of steam on it. Suppose w gm. are condensed and let the initial temperature of P, which has been observed by a thermometer placed in T, be t_1, and that of steam t_2.

Once more allow the apparatus to dry and weigh the solid S. When steam is again passed into J with S in the pan, a greater amount of steam will be condensed on account of S. Let this be W and suppose the initial temperature now is T_1.

Previously the scale-pan condensed w gm. of steam and rose in temperature through the interval $(t_2 - t_1)$. In the second case it rises from T_1 to t_2. The amount of condensation per degree rise of temperature is $\frac{w}{(t_2 - t_1)}$, hence the weight of steam condensed in the second case is $\frac{w}{(t_2 - t_1)} \times (t_2 - T_1)$.

Denote the mass of the solid by m and the specific heat by s.

The mass of steam condensed by the solid is

$$W - \frac{w\,(t_2 - T_1)}{(t_2 - t_1)}.$$

Hence we have the equation

$$ms\,(t_2 - T_1) = \left[W - \left\{\frac{w\,(t_2 - T_1)}{(t_2 - t_1)}\right\}\right] L,$$

by means of which the value of s may be calculated from a knowledge of the value of L or, conversely, L may be determined if the specific heat of the solid is known.

L denotes, as usual, the latent heat of steam.

A correction to account for the differences in apparent weights of the solid in air and steam has been neglected. The temperature of the solid is supposed equal to that of the apparatus and surrounding air. An interval of certainly not less than twenty minutes is required to allow the solid to acquire this temperature.

Bunsen's Ice Calorimeter

Description and Preparation for Use

A diagram of the apparatus is shown in fig. 112. The calorimeter is represented by ABCFED. It consists of a test tube, B, fitted into the glass jacket, A, which is drawn out at its base into the tube, CFE.

This tube ends in a cup, E, closed by a cork, through which passes the narrow glass tube, D.

A is filled partly with clean mercury and partly with distilled water containing no air.

The surrounding jacket, J, is a calorimeter closed with the cork or wooden stopper, S, which supports the apparatus.

In order to keep J and its contents at the freezing-point, it is placed in a larger vessel, standing on non-conducting blocks and packed round with a mixture of ice and snow or with flaked ice.

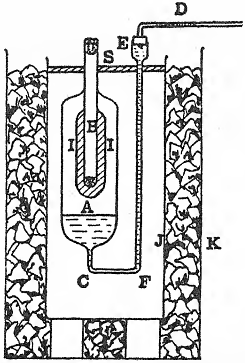


FIG. 112

By cooling the inner surface of B sufficiently, a layer of ice may be formed round the outside, as indicated at I.

On melting 1 gm. of ice the volume diminishes by 0·0907 c.c., so that if heat be added at B the amount may be determined by noting the change of volume as a result of the partial melting of I. The change of volume is observed by noting the movement of the end of the mercury column at D along the capillary tube. If this has been previously calibrated the change can be observed directly.

In order to fill the apparatus, remove the capillary tube, D, and the stopper in E, and introduce into A sufficient distilled water to fill it to about half. Invert the apparatus with the open end of the test-tube downwards and carefully boil the water, continuing until A is about one-third full.

While this is proceeding, boil some distilled water in a large beaker,

and towards the end of the evaporation of the water in A, place the end, E, well under the surface in the beaker.

Cease boiling the water in the calorimeter, and allow more to flow over from the beaker. In this way the inside of the calorimeter and the tube, CFE, become filled.

Clean mercury must now be passed in to lie below the water in A.

Introduce it gradually from a pipette held under the surface of the water in E, allowing displaced water to overflow. Take care that no air bubbles are introduced with the mercury, particularly when it becomes necessary to tilt the apparatus to allow water to pass over the mercury in A towards the tube. Fill up with mercury to E, place the stopper in position, and by carefully adjusting it make the end of the thread coincide with any desired position along D.

The apparatus should then be placed in a calorimeter containing water and ice to reduce the temperature as nearly as possible to zero.

This will probably take an hour at least, and the progress of the fall may be tested by placing a thermometer in B.

When the temperature is about 2° C. introduce cooled ether into B. The ether may be cooled by placing it in a cooled test-tube, and standing it in the calorimeter with the apparatus.

Draw air through the ether and cause it to evaporate, continuing until a cap of ice surrounds B.

The solidification will cause D to move farther along the capillary, and enough ice should be formed to cause more expansion than is likely to be required in succeeding experiments with the apparatus.

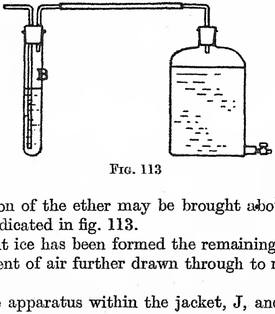


Fig. 113

The evaporation of the ether may be brought about by some such device as that indicated in fig. 113.

When sufficient ice has been formed the remaining ether is evaporated, and a current of air further drawn through to remove all traces of it.

Now place the apparatus within the jacket, J, and stand it in the vessel, K, packing it round as described above with ice and snow.

Leave this standing for an hour or two until a steady state has

been reached, and the movement in the capillary tube is only slight and steady.

It is not possible to maintain the end of the mercury thread quite steady with this arrangement, so that the slight motion must be accounted for in determining results from observations.

The capillary tube may be calibrated by the method described on p. 24.

Calibration of the Apparatus

By placing warm water within B of known mass and temperature, we may note the movement of D at various parts of the capillary tube for a known absorption of heat.

In performing the experiment it will be sufficient to calibrate the tube for one particular region, and in using the instrument again a slight pressure on the stopper or a slight easing of it will drive the end of the thread into the calibrated strip.

Heat up pure water to about 25° C., and transfer carefully to B. Allow the apparatus sufficient time to become steady, and note the displacement of the end of the mercury thread. Let this be l, and let the time be noted between the insertion of the water and the return to steady conditions. This will be denoted by t.

In order to correct for the small creep of the thread, observe the rate of motion just before adding the warm water, and also just after the absorption of heat.

All these measurements are to be made with a travelling microscope mounted conveniently opposite the capillary tube.

If the rates of creep are respectively p and p', then the average rate may be taken as $\frac{1}{2}(p + p')$ during the time t.

Thus the displacement of the thread due to absorption of heat from the water is $l - \frac{1}{2}(p + p')\,t$. We shall call this quantity L.

If m is the mass of water added and θ its initial temperature, then a motion of L units corresponds to the absorption of $m\theta$ calories.

Determination of the Specific Heat of a Substance

Let M gm. of the substance be heated to a temperature θ', and let s denote the specific heat.

The substance when placed in B will transfer to the ice $\mathrm{M}\theta' s$ calories.

If the mercury thread moves a distance, L′, in the calibrated region, the heat absorption is $\frac{m\theta}{\mathrm{L}} \cdot \mathrm{L}'$ calories.

Thus
$$\mathrm{M}\theta' s = \frac{m\theta \mathrm{L}'}{\mathrm{L}},$$

or
$$s = \frac{m\theta}{\mathrm{M}\theta'} \cdot \frac{\mathrm{L}'}{\mathrm{L}}.$$

The correction for creep must again be applied by observing the motion just before and just after the insertion of the mass. L′ is the corrected length.

When a solid is put into B, a pad of cotton wool should be placed at the bottom of the tube to prevent breakage when it falls. In order to facilitate removal it is a good plan to tie a light thread round it. This will introduce only a slight error. During the absorption of heat, and generally while the apparatus is in use, the end of the tube, B, should be stopped with a plug of cotton wool.

In the case of determining the specific heat of a liquid the experiment is almost exactly a repetition of the calibration.

In order to dry the tube after liquid has been put in, a roll of clean blotting paper may be used.

The Determination of the Density of Ice

In this experiment we require to know the volume per unit length of the capillary tube. We have assumed that this has been previously calibrated. It thus remains to determine the shrinkage due to absorption of a definite quantity of heat. The experiment may be performed in conjunction with the calibration just described.

Let L denote the latent heat of fusion of ice, and suppose warm water added to B imparts k calories.

The amount of ice melted is $\frac{k}{\mathrm{L}}$ gm.

Let δv denote the shrinkage as measured by the movement of the mercury thread.

Then $\frac{k}{\mathrm{L}}$ gm. of ice have become $\frac{k}{\mathrm{L}}$ gm. of water at 0° C.

Let d_0 denote the density of water at this temperature.

The volume of water is $\frac{k}{d_0 \mathrm{L}}$ c.c., and the volume of the ice is thus

$$\left(\frac{k}{d_0 \mathrm{L}} + \delta v\right) \text{ c.c.}$$

Hence the density of ice is

$$\frac{\frac{k}{\mathrm{L}}}{\frac{k}{d_0 \mathrm{L}} + \delta v} \text{ gm. per c.c.}$$

The Latent Heat of Fusion of Ice

It is assumed that the student is familiar with the principles of the determination of the latent heat of fusion and has carried out the

experiment without making calorimetric corrections. We are concerned in this description chiefly with an account of how this correction may be made. Care is taken, as in the determination of specific heat, to adjust the initial and final temperatures so that the room temperature is the mean of the two. In addition, care must be taken that the calorimeter is not cooled down so low that the dew-point is reached, otherwise there will be a deposit of dew on the apparatus and a liberation of latent heat in consequence.

We may make the correction for radiation as in the experiment on specific heat, but an alternative method will be described.

Note the temperature when the ice is placed in the calorimeter, and at intervals of half a minute until it is melted, and finally at intervals of one or two minutes, during which the calorimeter is absorbing heat from the surroundings.

Let the temperatures observed in this second period be

$$T_1, T_2, T_3, \ldots, T_{n+1}.$$

Then the change of temperature during these intervals will be

$$\delta\theta_1 = A\,\{\tfrac{1}{2}(T_1 + T_2) - t_0\},$$
$$\delta\theta_2 = A\,\{\tfrac{1}{2}(T_2 + T_3) - t_0\},$$
$$\vdots$$
$$\delta\theta_n = A\,\{\tfrac{1}{2}(T_n + T_{n+1}) - t_0\}$$

by Newton's Law of Cooling, where A is a constant depending on the calorimeter but not on its temperature, and t_0 is the temperature of the surroundings.

Thus the total change of temperature $\Delta\theta$ is given by

$$\Delta\theta = A\left\{\frac{T_1 + T_{n+1}}{2} + (T_2 + T_3 + \ldots + T_n) - nt_0\right\}.$$

But $\Delta\theta = T_{n+1} - T_1$, so that A can be calculated.

In the first part of the experiment during the melting of the ice, let the observed temperatures be $t_1, t_2, \ldots t_{n+1}$.

Then we have

$$\Delta t = A\left\{\frac{t_1 + t_{n+1}}{2} + (t_2 + t_3 + \ldots + t_{n+1}) - nt_0\right\}.$$

Thus, since the observed minimum temperature is t_{n+1}, the corrected minimum is $(t_{n+1} + \Delta t)$.

In the above the $\delta\theta$s and δts are to be treated algebraically, for some will be negative and some positive if the initial temperature is adjusted in the manner described.

Thus, if W is the water equivalent of the calorimeter and contents, and m the mass of ice melted, we have

$$mL + mt_{n+1} = W\,\{t_1 - (t_{n+1} + \Delta t)\}.$$

The Latent Heat of Vaporization. (Berthelot's Apparatus)

One form of this apparatus is depicted in fig. 114. Its essential feature is the condenser spiral, C, with the receptacle below. This is immersed in water in a calorimeter and the calorimeter shielded by packing it round with a non-conductor and placing it in a convenient vessel, or, better still, by standing it on non-conducting blocks in an empty larger calorimeter and enclosing within the larger vessel (see fig. 114).

The condenser is dried and weighed. It is placed in the water and allowed to stand until the temperature becomes steady as recorded

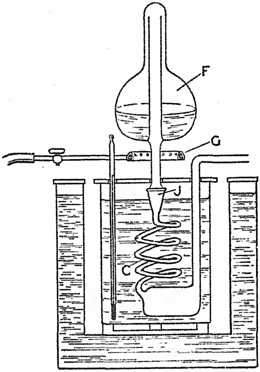


FIG. 114

by the thermometer, the uniformity of temperature throughout the calorimeter being procured by means of a stirrer.

A quantity of vapour is introduced into the condenser, and the spiral provides a large area of contact with a cold surface, so that liquefaction takes place and the liquid collects in the receptacle. The water is constantly stirred, and its temperature observed by means of the thermometer.

When a suitable rise is obtained the supply of vapour is cut off and the thermometer watched until the maximum temperature is reached. In this interval, immediately after cutting off the supply, the end of the condenser outlet tube must be closed by a cork to avoid convection effects.

The condenser is removed, dried, and weighed, and the mass condensed thus determined. Denote this by m, so that if L denotes the latent heat the supply of heat on condensation is mL calories.

Let T_1 denote the temperature of vaporization, and T_2 the final temperature of the calorimeter and contents, while T_0 denotes the initial temperature of the calorimeter.

Let s denote the specific heat of the liquid, and W the water equivalent of the calorimeter, condenser, and remaining calorimeter contents. Then $W(T_2 - T_0) = mL + m(T_1 - T_2)s$; so that L may be determined.

One of the weak points of this form of apparatus is the mode of introduction of the vapour.

As the diagram shows, the liquid is vaporized over a small gas ring in the reservoir. It thus easily happens that the vapour gets super-

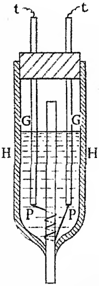


Fig. 115

heated, and, in addition, it is difficult to shield the calorimeter effectively from the heat of the flame. The screen, S, is introduced to reduce this effect.

It is preferable to use an electrical method of heating, and this is done in the more recent forms of the apparatus.

Kahlenberg's heater is illustrated in fig. 115. The liquid is contained in the vacuum flask, H, and is heated by passing an electric current through the platinum wire, PP.

For ordinary laboratory practice, a test-tube may take the place of H without introducing much difficulty in shielding the calorimeter.

Correction for loss due to radiation in the calorimeter, C, must be made by one of the usual methods (p. 187).

The Heat of Solution of a Salt

When a salt is dissolved by a liquid the solution is accompanied by an absorption or liberation of heat. The amount varies with the proportion of the salt to the solvent, i.e. with the resulting concentration, and with the amount of salt dissolved.

The number of calories absorbed or liberated when 1 gm. of salt is dissolved in a certain amount of solvent is said to be the heat of solution for the particular concentration.

The experiment described below is designed to measure this quantity.

The apparatus necessary is illustrated in fig. 116. It consists of an outer protecting calorimeter of metal, C, which carries a cork through which the inner vessel, A, is supported.

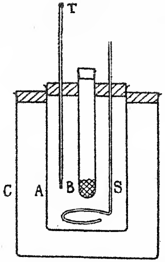


FIG. 116

This vessel also carries a cork through which pass a thermometer, stirrer, and thin test tube, B, into which the salt may be placed.

The vessel, A, and stirrer, S, are usually of glass, since many solutions attack copper. There is uncertainty about the specific heat of the glass, so that the water equivalent of the vessel, A, and its contents should be determined in a separate experiment. For many purposes, however, we may assume the specific heat of the glass to be 0·16. The pure solvent is placed inside A, and it should surround the lower part of the tube, B, which contains the salt.

The apparatus is allowed to stand so that the salt may acquire the temperature of the solvent. Half an hour should be allowed for this, and during this interval the salt may be occasionally stirred by a clean, dry, glass rod.

The weight of solvent is obtained in the usual way before the insertion of B. Let this be denoted by W.

Suppose the water equivalent of A and its contents is w, and the

specific heat of the solution s. This quantity must be determined later by the method of cooling (p. 189), or by any other convenient experiment.

It will be necessary, also, to make a correction by one of the methods previously described (p. 186).

Let the initial temperature of the salt and solvent be denoted by t_0 and the final corrected temperature by t_1, and suppose that the weight of the salt dissolved is q. The heat of solution being Q, we have

$$qQ = \{(W + q)s + w\} (t - t_0),$$

for $(W + q)$ is the weight of the solution.

The value of Q will be positive for the liberation of heat, and negative for absorption.

In order to mix the salt and solvent, the glass rod used above for stirring should remain in the salt until just before mixing. At this instant observe the temperature, t_0, recorded on the thermometer, and push the rod through the bottom of the thin tube, B.

This tube must, of course, be clean and dry, and the salt will then fall into the solvent and be dissolved. The rod should then be removed, and care taken not to carry with it any of the solvent. The solution is assisted by the stirrer, S, and observations of T taken every quarter- or half-minute in the initial stages, and later at longer intervals.

A graph is drawn showing the variation of temperature with time from the instant of mixing, and this curve corrected, as explained above, or we may use the non-graphical method.

To find the Ratio of Specific Heats at Constant Pressure and Constant Volume for Air. (Clement and Désorme's Experiment)

Apparatus and Experimental Details

A glass reservoir, provided with a tap, T, giving a wide opening to the air is connected to an oil manometer, G, and to a pump; an ordinary bicycle pump is convenient. A small excess pressure is applied to A, the difference between it and the atmospheric pressure being measured by the manometer (fig. 117).

In the first stage of the experiment the temperature under these conditions is allowed to become steady.

In the second the tap, T, is opened and closed suddenly by giving one half-turn.

By this means the pressure falls to that of the atmosphere in so short an interval that we may suppose there is no passage of heat to A during this expansion.

The condition of the expansion is therefore adiabatic.

Finally, the temperature is allowed to return to that at the beginning of the experiment, during which process the pressure in A increases, though it does not recover its original value.

Theory

Suppose that gas occupying the volume below the dotted line remains in the flask all the time.

Denote its volume by V_0 while that of the flask is V.

Let the initial pressure be P_0, and let that immediately after the adiabatic expansion be B.

Then $$BV^\gamma = P_0V_0^\gamma,$$

where γ = ratio of specific heats, the value of which we require.

The flask is open to the air so that B is the atmospheric pressure.

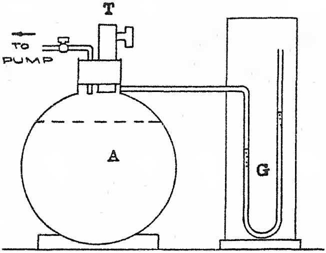


FIG. 117

It is in order that we may ensure a fall of pressure from P_0 to B during a short interval that the tap is wide.

In the final stage let the pressure become P when the temperature has attained a steady value.

We have now passed from volume, V_0, and pressure, P_0, by an isothermal process to volume, V, and pressure, P.

$$\therefore \quad PV = P_0V_0.$$
$$\therefore \quad P^\gamma V^\gamma = P_0^\gamma V_0^\gamma.$$

Hence, by dividing this by BV^γ and $P_0V_0^\gamma$, we have

$$\frac{P^\gamma}{B} = P_0^{\gamma-1},$$

or $$\left(\frac{P}{P_0}\right)^\gamma = \frac{B}{P_0};$$

$$\gamma = \frac{\log \dfrac{B}{P_0}}{\log \dfrac{P}{P_0}}.$$

Let the difference in heights of the manometer be h_0 initially, and h finally.

Then if pressures be measured in terms of heights of the liquid columns

$$P_0 = B + h_0, \quad P = B + h.$$

$$\therefore \quad \gamma = \frac{\log\left(1 - \frac{h_0}{P_0}\right)}{\log\left(1 - \frac{h_0 - h}{P_0}\right)} = \frac{\frac{h}{P_0}}{\frac{h_0 - h}{P_0}} \text{ approx.,}$$

by expansion in logarithmic series and neglecting higher powers of $\frac{h_0}{P_0}$ and $\frac{h_0 - h}{P_0}$ than the first.

This is permissible since the values of h used are only a few centimetres. A convenient value for h_0 is 5 cm.

Hence $$\gamma = \frac{h_0}{h_0 - h}.$$

The result may also be obtained by another method.

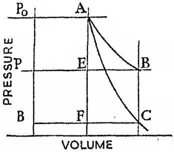


FIG. 118

In fig. 118 let AC denote any curve relating the pressure and volume of a gas. The elasticity is defined to be the ratio

$$\frac{\text{stress}}{\text{strain}}.$$

The stress will be measured by a slight change in pressure, and the strain by the corresponding slight change in volume per unit volume. Let us consider the volume, v, represented by BC. Let a change of pressure, δp, denoted by FA, bring about the change in volume denoted by CF. We shall record this by δv, but $\delta v = -$ CF on account of the diminution of volume on the addition of pressure.

Thus, the elasticity, E, is measured by AF $\div \frac{\text{CF}}{v}$, for $\frac{\text{CF}}{v}$ denotes the change of volume per unit volume.

$$\therefore \ \mathrm{E} = -v\frac{\delta p}{\delta v}.$$

When we proceed to the limit and make the changes very small we have

$$\mathrm{E} = -v\frac{dp}{dv}.$$

The value of E depends on how the change is made, and we shall consider two cases, first the case of an isothermal, and then that of an adiabatic change. Let AC denote the adiabatic curve and AB the isothermal; the former is steeper than the latter. Let E_ϕ denote the adiabatic and E_θ the isothermal elasticity.

From the formula for E we have

$$\mathrm{E} = -v \times \text{slope of curve}.$$

Thus $$\mathrm{E}_\phi = -v \times \text{slope of adiabatic},$$

and $$\mathrm{E}_\theta = -v \times \text{slope of isothermal}.$$

If we start at A with a particular volume, v, measured for each curve by $\mathrm{P}_0\mathrm{A}$, we have

$$\frac{\mathrm{E}_\phi}{\mathrm{E}_\theta} = \frac{\text{slope of adiabatic}}{\text{slope of isothermal}}.$$

The changes of pressure and volume in the experiment are small, so that the curves, AB and AC, are approximately straight, and the slopes can be measured by

$$\frac{\mathrm{AF}}{\mathrm{FC}} \quad \text{and} \quad \frac{\mathrm{AE}}{\mathrm{EB}} \text{ respectively.}$$

$$\therefore \ \frac{\mathrm{E}_\phi}{\mathrm{E}_\theta} = \frac{\mathrm{AF}}{\mathrm{AE}}.$$

Now, AF is the change in pressure during the adiabatic part of the expansion, viz. h_0, and AE is the change during the isothermal part. In our case the atmospheric pressure is that at the end of the adiabatic expansion, i.e. that at C, and since the point, B, on the curve represents the final state, CB denotes the pressure, h.

Thus $$\mathrm{AE} = h_0 - h.$$

Thus $$\frac{\mathrm{E}_\phi}{\mathrm{E}_\theta} = \frac{h_0}{h_0 - h}.$$

For the adiabatic expansion we have

$$pv^\gamma = \text{constant},$$

$$\therefore \ \log p + \gamma \log v = 0.$$

Differentiating, we have

$$\frac{1}{p}\frac{dp}{dv} + \frac{\gamma}{v} = 0,$$

i.e. $$\frac{dp}{dv} = -\frac{p}{v}\gamma.$$

In our case p and v denote the values of the co-ordinates at A, and $\frac{dp}{dv}$ is the tangent of inclination of the curve AC to the axis of x.

Similarly, for the isothermal case we have

$$pv = \text{constant},$$

and
$$\frac{dp}{dv} = -\frac{p}{v}.$$

p and v are the co-ordinates of A, but the slope is now for AB.

Thus the ratio of the slopes is γ.

$$\therefore \quad \frac{E_\phi}{E_\theta} = \gamma = \frac{h_0}{h_0 - h}.$$

In carrying out the experiment make about six independent determinations, and increase the pressure cautiously so as not to expel oil from the manometer.

CHAPTER IX

VAPOUR DENSITY AND THERMAL CONDUCTIVITY

Vapour Density. (Victor Meyer's Method)

THE vapour density of a substance can be found by measuring the volume of the vapour produced from a small quantity of the solid or liquid whose weight is known. In Victor Meyer's method this volume is found from the volume of air displaced by the vapour.

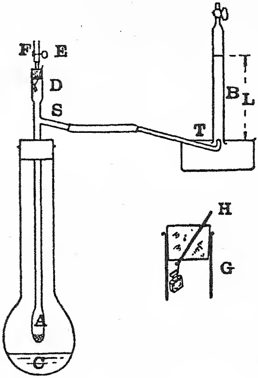


FIG. 119

The apparatus (fig. 119) consists of a vertical glass tube provided with a bulb at the lower end, A, and a side tube, ST. The side tube dips under water in a beaker and a rubber cork closes the upper end of the tube.

As it is necessary, on introducing the vapour, to allow a small bottle, D, to fall the length of the tube, it is advisable to place a little asbestos at the bottom of A to prevent breakage.

The tube, A, is surrounded by a larger tube containing a liquid which boils at a higher temperature than the substance to be experimented on.

In the case of the determination of the vapour density of ether, water may be used in the bath.

The various parts of the inner tube should be kept at a constant temperature during the experiment, and in order to maintain this condition the outer tube is screened from draughts by surrounding it with a cylinder of asbestos or cardboard which fits it above the bulb.

Before beginning the experiment the inner tube must be quite dry. If necessary, it should be warmed over a Bunsen flame while a current of air is blown through it.

The apparatus is set up as shown in the diagram, and heating is kept up until no more bubbles come out from the side tube.

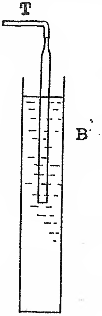


FIG. 120

The substance is weighed and enclosed in the small bottle, D, and suspended close to the upper end of the inner tube. When everything is quite steady the bottle is allowed to fall; vaporization takes place, and the cork of the bottle is blown out. Air passes over into the side tube and may be collected in the burette, B. It is better to collect it by the method illustrated in fig. 120, in which case the gas collected can be brought to atmospheric pressure by raising or lowering the burette. In the other case it is necessary to correct for the height of the water column, L.

In order to cause no disturbance on introducing the substance to be vaporized, a piece of thread or thin wire should be passed through the cork and be held by a stop-cock, E, which pinches a piece of tubing, F. The bottle is allowed to fall by opening E, and then closing it immediately.

A better method is to use the apparatus shown at G. By turning the wire, H, through 180° the bottle will be caused to slip off the hook.

The air collected at B is over water at a temperature, T, and will be saturated with water vapour at this temperature. Let B be the saturation pressure at this temperature. If v is the measured volume of air, and H the total pressure, then v_0 the volume under normal conditions is given by

$$v_0 = v \cdot \frac{\mathrm{H} - \mathrm{B}}{76} \cdot \frac{273}{273 + \mathrm{T}}.$$

If w = weight of substance enclosed in D, the density of the vapour is

$$\frac{w}{v_0} = \frac{w}{v} \cdot \frac{76}{\mathrm{H} - \mathrm{B}} \cdot \frac{273 + \mathrm{T}}{273}.$$

1 c.c. of hydrogen at 0° and 76 cm. pressure weighs 0·0000900 gm., and its molecular weight is 2.

Hence the molecular weight of the substance examined is

$$\frac{2}{0{\cdot}00009} \cdot \frac{w}{v} \cdot \frac{76}{\mathrm{H} - \mathrm{B}} \cdot \frac{273 + \mathrm{T}}{273}.$$

At the conclusion of the experiment remove the stopper from the end of the inner tube to prevent any sucking back of water from B into the bulb, A, as the apparatus cools. It is important to cause the bulb, A, to be heated by the steam from the bath, and it should be adjusted to prevent actual contact with the water in the bath, and to be out of reach of splashes when boiling takes place.

Vapour Density. (Dumas's Method)

A large flask is cleaned and dried, and fitted with a cork provided with a bent piece of glass tubing drawn to a fine point so that it can be easily sealed by the application of a Bunsen flame (fig. 121).

The flask is suspended from the arm of a balance and weighed. Let the observed weight be W_1, and let w_1 denote the weight of air displaced by the wall of the flask, while w_2 denotes the weight of air displaced by the closed flask, so that $w_2 - w_1$ denotes the weight of air within it. This will be denoted by w_a. Hence if W denote the real weight of the flask

$$\mathrm{W}_1 = \mathrm{W} - w_1. \qquad \ldots(1)$$

Now introduce a small quantity (5–10 c.c.) of the liquid to be vaporized into the flask. The actual quantity will depend on the size of the flask and will be discovered by trial. The amount given is of the right order when a flask of capacity about a half litre is employed to find the vapour density of chloroform. This liquid is very suitable in the case of a laboratory exercise, since it has a high vapour density. The flask is placed within an enclosure, J, over a sand bath, with the tube, T, projecting through the lid.

Heating is continued until the liquid is vaporized and no more issues

from T. This may be tested by placing a polished surface near the end of T. It will become dimmed if vapour is still coming out.

When the steady state is reached, T is sealed off.

Let this happen at a temperature, $t°$, measured by means of a thermometer hanging close to A, within the enclosure, J, and let the volume of the vapour be V_t, and the density ρ_t. Allow the flask to cool, and weigh it. Break off the end of T under water. The flask will fill with water, the space occupied by the condensed vapour becoming negligible under the new conditions.

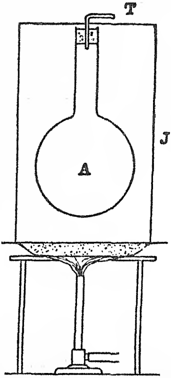


Fig. 121

Preserve the broken pieces from T, and after drying the outside of the tube reweigh the flask and water, noting the temperature of the water, $t_0°$.

Let W_2 denote the weights in the scale-pan when the flask and vapour are weighed, and let w_v denote the weight of the vapour.

Then
$$W_2 = W + w_v - w_2;$$
$$\therefore\ W_2 - W_1 = w_v - (w_2 - w_1) \text{ by equation (1)}$$
$$= w_v - w_a.$$

Let V_{t_0} denote the volume of the flask obtained from the weight of water it contains at $t_0°$. Then if β denote the coefficient of cubical expansion of the glass, which may be taken as 0·0000232,

$$V_t = V_{t_0}\{1 + \beta(t - t_0)\},$$
$$w_v = \rho_t V_t.$$

If the barometric pressure be P, we can reduce ρ_t to normal conditions by the formula

$$\rho_0 = \frac{(273 + t)}{273} \cdot \frac{76}{P} \cdot \rho_t$$

$$= \frac{(273 + t)}{273} \cdot \frac{76}{P} \cdot \frac{w_v}{V_{t_0}\{1 + \beta(t - t_0)\}}.$$

w_a denotes the weight of air filling the flask at the temperature of the air within the balance. Let this be t'° C.

Then if d_a denote the density of air under these circumstances,

$$w_a = d_a V_{t'} = d_a V_{t_0}\{1 + \beta(t' - t_0)\}$$

$$= d_0 \cdot \frac{P}{76} \cdot \frac{273}{273 + t'} \cdot V_{t_0}\{1 + \beta(t' - t_0)\},$$

$d_0 = 0{\cdot}001293$ gm. per c.c.

From these two formulae, since $w_v = w_a + W_2 - W_1$, we may now calculate ρ_0.

One of the difficulties of the experiment is to drive out all the air from the flask and replace it by vapour. It frequently happens that on attempting to fill the flask with water some air is left behind. More of the liquid is required for driving off the air in this case.

If the volume of air left over is small, we may apply a correction. At the temperature t_0, of the water, let the volume of air be v. This may be determined by filling up the flask with water from a measuring flask.

The total pressure to which the mixture of vapour and air is subjected is the sum of the partial pressures, P' and p', of the vapour and air respectively. The weight of the vapour is now w_v', obtained by subtracting the weight of air of volume, v, from w_v, determined as above

$$w_v' = \rho^t V_t$$

$$= \rho_0 \cdot \frac{P'}{76} \cdot \frac{273}{273 + t} \cdot V_t\ \{1 + \beta(t - t_0)\}.$$

V_{t_0} denotes as before the total volume of the flask, i.e. the volume of the water after the bubble of air has been replaced.

We can find P' by remembering that the air of volume, v, and at atmospheric pressure occupied a volume, V_t, under the partial pressure, p', the temperatures being respectively t_0° and t° C.

Thus

$$\frac{vP}{273 + t_0} = \frac{V_t p'}{273 + t},$$

$$p' = P - P';$$

$$\therefore\ P'V_t = PV_t - \frac{273 + t}{273 + t_0} \cdot Pv,$$

$$w'_v = \rho_0 \cdot \frac{273}{273+t} \cdot \frac{1}{76}\left(\mathrm{PV}_t - \frac{273+t}{273+t_0} \cdot v\right)$$

$$= \rho_0 \cdot \frac{273}{273+t} \cdot \frac{\mathrm{P}}{76} \cdot \mathrm{V}_{t_0}\{1 + \beta(t - t_0)\} - \rho_0 \cdot \frac{\mathrm{P}}{76} \cdot v \cdot \frac{273}{273+t_0},$$

$$\text{or } \rho_0 = \frac{w'_v}{\dfrac{273 \, . \, \mathrm{P}}{76} \cdot \left[\dfrac{\mathrm{V}_{t_0}\{1 + \beta(t - t_0)\}}{273+t} - \dfrac{v}{273+t_0}\right]}.$$

The term, $\dfrac{v}{273 + t_0}$, will be small if the experiment is to be successful at all, so that it will only be necessary to change the value, w_v, to the actual weight of vapour, w_v'.

It should also be remarked that the volume, v, is a mixture of air and water vapour, whereas it has been assumed to consist of air only. We shall not, however, further consider this correction, which is small and affects a term which must be already small. The discussion shows how the error affects our formula, and it would be sufficient to measure the volume of the bubble, multiply by the density, 0·001293 gm. per c.c., and subtract from w_v.

After a few attempts the bubble will usually be sufficiently small to be neglected altogether.

Conductivity of a Copper Bar

The apparatus consists of a bar of copper, CC (fig. 122), with two holes bored well into it to carry thermometers, T_1 and T_2, mercury being placed in the holes to ensure good thermal contact. At the ends of the bar are two metal boxes, through one of which, A, is passed steam, and through the other, B, a steady stream of water from a constant pressure head.

The shelves, LL, within B, serve to prevent any flow of water straight from inlet to outlet. The temperature of the water is taken just before it enters and leaves B by the thermometers, T_3 and T_4. The apparatus is allowed to attain a steady state, when all the thermometers will record steady temperatures.

It is usual to pack loosely round CC and the boxes some cotton wool, the whole being enclosed in a felt-lined wooden box, through which T_1 and T_2 project. T_3 and T_4 are kept as close as possible to the box, and it is a good plan to wrap the T-pieces loosely with wool.

In this way the heat from A is transmitted by conduction to B, and the heat passing across in any time, t, is noted by collecting water as it leaves B and weighing it. If the mass collected is m the rate of transmission of heat is

$$\frac{m\,(\mathrm{T}_4 - \mathrm{T}_3)}{t}.$$

T_3 and T_4 denote the initial and final temperatures of the water. Since this quantity of heat is transmitted from T_1 to T_2—a distance, d, say, the amount is $k \cdot (T_1 - T_2) \cdot \frac{A}{d}$, where A is the area of section of the bar, and k the conductivity. A is measured by finding the diameter of the bar if it is cylindrical, or by measuring its breadth and height if rectangular.

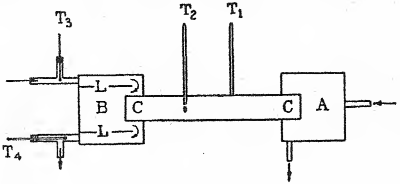


FIG. 122

We have, therefore:

$$k (T_1 - T_2) \cdot \frac{A}{d} = m \frac{(T_4 - T_3)}{t}.$$

In another form of apparatus the cold water is passed through a metal spiral wound round the end instead of through the metal box. Good thermal contact is made between the spiral and bar, and the temperature of the water measured at entrance and exit as before.

In the apparatus usually provided for this experiment two thermometers, T_1 and T_2, are provided. These are not sufficient to make it certain that the condition of linear flow prevails, and it is essential that this should be the case. At least four thermometers should be placed along the bar and the conditions must be adjusted until linear flow is taking place. It will be necessary to provide more efficient lagging along the bar if this is not the case. A graph should be drawn showing the temperature against the distance along the bar. This gives a ready means of seeing whether the conditions are linear or not, and the temperature gradient should be calculated from the graph.

Thermal Conductivity of Rubber Tubing

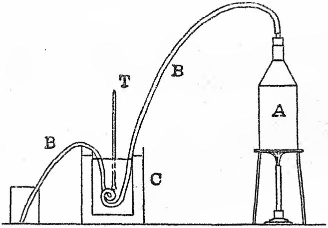
The apparatus required consists of a length of rubber tubing, B, a copper heater, A, for producing steam, a calorimeter, C, thermometer, T, and a measuring glass, as illustrated in fig. 123.

The method of procedure is as follows:

A quantity of water is introduced into the calorimeter, C, and weighed.

Steam is passed through B until a rise of temperature of the calorimeter and water of from 10° to 20° C. has occurred. The initial and final temperatures are noted and also the time of passage of the steam. The tube is removed and the length that has been immersed is noted—l (say). Two pieces of cotton should be tied round the rubber at the points where it enters and leaves the water.

Let the initial temperature be T and the final T′. If there had been no loss of heat the final temperature would have been some other value, $T' + \Delta T$, and it is necessary to make the correction ΔT.



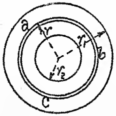


FIG. 123

In order to do this, observe temperatures, at intervals of half a minute or some convenient period, beginning at T and ending at T′, during the passage of the steam.

These will be denoted by $t_1, t_2, \ldots t_{n-1}$.

If the temperature of the room be t_0 we have for the change of temperature due to heat losses or gains, as the case may be, according to Newton's Law of Cooling

$$\delta\theta_1 = C\left(\frac{T + t_1}{2} - t_0\right),$$

$$\delta\theta_2 = C\left(\frac{t_1 + t_2}{2} - t_0\right).$$

$$\delta\theta_3 = C\left(\frac{t_2 + t_3}{2} - t_0\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\delta\theta_n = C\left(\frac{t_{n-1} + T'}{2} - t_0\right),$$

where $\delta\theta$ denotes the change in any interval, the suffix denoting which interval.

Thus

$$\Delta T = \Sigma\delta\theta = C\left\{\frac{T + T'}{2} - nt_0 + (t_1 + t_2 + \ldots + t_{n-1})\right\}.$$

C is a constant depending on the calorimeter and contents, and must be determined if Δt is to be calculated.

When the final temperature, T′, has been attained, cut off the supply of steam and allow the calorimeter to cool, observing temperatures, T_1, T_2, . . . T_n, at equal intervals.

Then as before if δt denotes the loss of temperature in each interval, we have:

$$\Sigma\delta t = C\left\{\frac{T_1 + T_n}{2} - (n - 1)\, t_0 + (T_2 + T_3 + \ldots + T_{n-1})\right\}.$$

But $\Sigma\delta t = T_1 - T_n$, so that C can be calculated and may be employed in the above case.

The ranges of temperature in the two cases should be as nearly as possible the same.

If M denotes the water equivalent of the calorimeter and contents the heat transmitted through the tubing is

$$M\,(T' + \Delta T - T).$$

We can connect this with the conductivity, k, in the following way.

Let the outer and inner radii of the tubing be r_1 and r_2, and consider a portion of unit length of the tube between radii, r and $r + \delta r$, at which the temperature is t.

The rate of change of temperature at the distance, r, is $-\dfrac{dt}{dr}$. The negative sign expresses the fact that the temperature diminishes as r increases.

Thus, if Q denotes the quantity of heat transmitted per second per unit length, i.e. across an area, $2\pi r$,

$$Q = -k\,.\,\frac{dt}{dr}\,.\,2\pi r.$$

$$\therefore \quad \int dt = -\frac{Q}{2\pi k}\,.\int\frac{dr}{r},$$

where the integration is to be taken between the limits, r_1 and r_2 for r, and between the inner and outer temperatures of the tubing for t.

Let the steam temperature be τ, the outer temperature is taken as the mean of T and T′,

$$\therefore \tau - \tfrac{1}{2}(\mathrm{T} + \mathrm{T}') = \frac{\mathrm{Q}}{2\pi k} \log_e \frac{r_1}{r_2},$$

or

$$k = \frac{\mathrm{Q}}{2\pi\left(\tau - \dfrac{\mathrm{T} + \mathrm{T}'}{2}\right)} \log_e \frac{r_1}{r_2}.$$

But

$$\mathrm{Q} = \frac{\mathrm{M}}{lp}(\mathrm{T}' + \Delta\mathrm{T} - \mathrm{T}),$$

where p = time of flow in seconds.

$$\therefore k = \frac{\mathrm{M}}{2\pi l} \times 2{\cdot}303 \times \frac{\mathrm{T}' + \Delta\mathrm{T} - \mathrm{T}}{\tau - \dfrac{\mathrm{T} + \mathrm{T}'}{2}} \log_{10} \frac{r_1}{r_2}.$$

(Change being made to logarithms to base 10.)

The value of r_1 may be determined by means of a screw gauge, and in order to find r_2, place a length of the tube of 5 to 10 cm. in water in a measuring glass and note the volume, v, displaced.

Then if L denotes the length of tubing:

$$v = \pi \mathrm{L}\,(r_1^2 - r_2^2).$$

All the quantities except r_2 are known, so that this value can be determined.

Another way of determining the radii is to cut the tube clean, normal to its length and use it as a rubber stamp, pressing it lightly on a clean sheet of paper.

The impress of the outer and inner circumferences will be distinct and the diameters may be measured by means of a travelling microscope.

The Conductivity of Glass

The conductivity of glass in the form of a tube may be found by the method described in the last experiment. A different arrangement of apparatus is required, but the theory is identical in both cases.

Steam is passed through a jacket, J, round the tube, B. Within B a stream of water is caused to flow from a supply which provides a constant head of pressure.

Within B is a spiral made of cord or rubber so that as it progresses up the tube the water is caused to traverse it spirally. This is important as the temperature at any cross-section of B must be the same throughout the section. The rate of flow is adjusted to cause a difference of temperature of about 20° C. between the initial and final temperatures. The thermometers are enclosed in T-pieces as near as possible to the

ends of B, the T-pieces being covered with felt or cotton wool to prevent loss of heat by radiation before the temperature is taken.

In order to measure Q, water is collected on exit for a measured time.

It is usual, as stated in the previous experiment, to take the average temperature of the water which receives heat to be the arithmetic mean of its initial and final temperatures. This point is worthy of consideration.

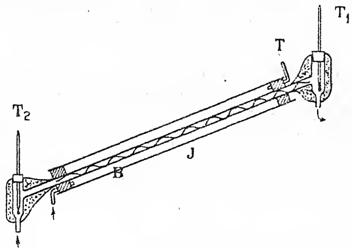


FIG. 124

Let the temperature of the water at an element of length dx at a distance x from the colder end of the tube be T. The heat transmitted across this element per second is

$$dQ = \frac{2\pi k\,(\tau - T)\,dx}{\log_e \dfrac{r_1}{r_2}}$$

in the notation of the previous experiment.

Thus the total transfer of heat all along the tube is

$$Q = \frac{2\pi k \int_0^l (\tau - T)\,dx}{\log_e \dfrac{r_1}{r_2}}.$$

The integral has the value ($\tau l - \int_0^l T dx$), and in order to evaluate it, it is necessary to know the distribution of temperature in the water all along the tube. By plotting a graph of T against x the integral within the bracket can be obtained from the area under the graph between $x = 0$ and $x = l$. If the distribution of temperature is linear in x, the value of the integral is $\frac{1}{2}(T_1 + T_2)\,l$, where T_2 and T_1 are the initial and final temperatures, as indicated in fig. 122. The experiment is made complicated if the temperature has to be measured at several points along the inside of the glass tube, but the distribution has been

carefully examined under the conditions usually prevailing in the laboratory and it has been found that it is linear to a degree of accuracy which makes further complication of the apparatus unnecessary. It may therefore be assumed for the purpose of illustration of this method of determining conductivity that the distribution is linear.

Conductivity of Cardboard by the Method of Lees and Chorlton

The apparatus consists of a retort stand provided with a clamp (fig. 125) and metal ring, AB, from which hangs a cylindrical slab of copper or brass, DE, of diameter about 12·5 cm. On this rests a hollow cylinder, C, of the same diameter, provided with inlet and outlet tubes, G and H, through which steam may be passed.

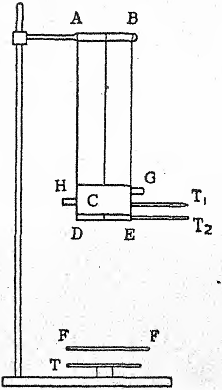


Fig. 125

Towards the base of C and into DE holes are bored so that the thermometers, T_1 and T_2, may be inserted and the temperatures read.

The two cylinders are nickel-plated in order to produce a surface of uniform emissive power.

Suppose a thin slab of material of the same diameter as the cylinders is placed between them, and let the loss of heat from the edge of the slab be small enough to be neglected in the calculation. Then all the heat transmitted across the slab is lost from DE during the steady state.

Let A denote the area of cross-section of the slab and d its thickness. Let the thermal capacity of DE be denoted by W, and let the thermometers, T_1 and T_2, record temperatures, T_1 and T_2 in the steady state, while the temperature of the surrounding air is T_0. The heat transmitted through the slab per second is

$$k \cdot \frac{T_1 - T_2}{d} \cdot A,$$

where k is its thermal conductivity.

The heat lost per second from DE is

$$C \cdot (T_2 - T_0),$$

where C is a constant.

Thus $$k \cdot \frac{T_1 - T_2}{d} \cdot A = C\,(T_2 - T_0).$$

We may readily use the apparatus to give comparative results.

If two sheets be cut of the same diameter as DE, one of glass and the other of cardboard, about 1 mm. thick, and if they be inserted between the cylinders, we may observe two sets of temperatures, one for each. Let the letters without dashes be used to describe the experiment with glass while those with dashes correspond to cardboard.

In the latter we have

$$k' \cdot \frac{T_1' - T_2'}{d'} \cdot A = C\,(T_2' - T_0').$$

On dividing we have

$$\frac{k'}{k} = \frac{T_2' - T_0'}{T_2 - T_0} \cdot \frac{d'}{d} \cdot \frac{T_1 - T_2}{T_1' - T_2'}.$$

d and d' are measured by means of a screw gauge.

In order to find T_0 place the thermometer below a sheet of cardboard, FF, to protect it from direct heat from DE, but it should be placed directly below DE, so as to give the temperature of the air which rises upwards to DE.

An absolute value may be found for k by determining the rate of fall of temperature of DE at the temperature, T_2.

This may be found by removing C and allowing a Bunsen flame to play on DE until T_2 registers a temperature about 10° above that recorded during the steady state.

Observe the temperature recorded by T_2 as the slab cools to about 10° below that recorded during the steady state. The observations should be made every half-minute, or more frequently if the change is rapid, and a graph drawn relating the time and temperature.

From the graph determine the value of the rate of change at various temperatures.

Plot a second graph with the rates, $\frac{dT}{dt}$, as ordinates and the corresponding temperatures, T, as abscissae (fig. 126). From this determine the particular rate for the temperature, T_2. Denote this by $\left(\frac{dT}{dt}\right)_2$.

Weigh the slab and determine its water equivalent, using for the specific heat of copper the value 0·094, or of brass the value 0·09. Denote this by W.

Hence the loss of heat is expressed by $W\left(\frac{dT}{dt}\right)_2$, and we have the equation

$$k \,.\, \frac{T_1 - T_2}{d} A = W \,.\, \left(\frac{dT}{dt}\right)_2,$$

by means of which k is determined.

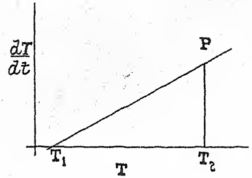


FIG. 126

Make a table of the observations during cooling as follows:

Time (min.)	Temp. (T_2)	Average temp.	Rate $\frac{dT}{dt}$
0			
$\frac{1}{2}$			
1			
$1\frac{1}{2}$			
2			
$2\frac{1}{2}$			
3			
.			
.			
.			
.			
.			
.			
.			

Comparison of Conductivity of a Good and Bad Conductor

The conductivity of substances such as glass may be compared with that of copper by a method in which the condition of linear flow prevails. Since the conductivity of copper is a well-known quantity the method provides an accurate and direct determination of the conductivities of these substances.

The method consists in taking two equal solid cylinders of copper, half to one inch in diameter and about eight inches long. The material to be investigated is of the same diameter and, in the case of glass, may be one-tenth to one-fifth inch thick. It is placed between the two cylinders so that the three parts form a cylinder and good thermal contact can be obtained by a thin film of glycerine on the glass surfaces.

One end of one of the copper rods is set into a chamber, through which steam is passed in order to maintain this end at a constant temperature.

It is convenient to set the complete cylinder vertically and to measure the temperature at various points along the copper bars by thermo-electric means. All that is necessary is to solder constantan wires at six equally spaced points along the rods. The junctions will form the hot junctions of copper-constantan couples and the constantan wires should be brought to a key through which they can be connected to a single constantan wire leading to the cold junction. The return copper wire should be soldered to the cold end of the cylinder, but in order to provide an electrical connexion across the non-conducting specimen between the two copper cylinders, these must be joined by a thin copper wire. The key may consist conveniently of a slab of paraffin wax provided by a ring of holes containing mercury into which the constantan wires dip. The common connexion to the cold junction may be made through a hole at the centre of the ring, also containing mercury, which may be connected in turn to the other holes by a stirrup of wire. The galvanometer should be inserted in the copper lead between the copper cylinder and the cold junction. The cylinder may be clamped in a vertical position and protected from lateral loss of heat by cotton wool or felt. A graph is plotted of the temperature against distance from one end. Actual temperatures are not required, but it is important to know the relation between the temperature and the galvanometer deflection. It will often be found that a linear relation exists, but this should be examined in a subsidiary experiment.

If this is the case, all that is required is to plot galvanometer readings instead of temperatures.

When the condition of linear flow is obtained the graph takes the form shown in fig. 127.

The parts AB and CD of the graph result from observations of the galvanometer deflections. If they are linear and parallel no heat has been lost laterally, and the flow is linear in the metal and the material

between the metal cylinders. The fall of temperature through the material is required, and is the difference between the readings of temperature at B and C on the graph. It would, however, give an inaccurate result if this difference were measured directly from the graph.

Measure carefully the thickness of the disk of the material and draw two lines vertically on the graph at this distance apart. It does not matter where they lie on the graph, they will cut the lines AB and CD produced, if necessary, to form a parallelogram with a diagonal equal to BC and the temperature difference may be read off. Very careful drawing on a graph drawn on a large scale is necessary to ensure even a fair degree of accuracy.

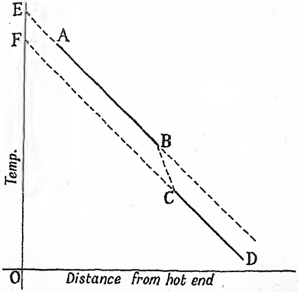


FIG. 127

The difference of temperature may also be obtained from the formula

$$\Delta T = EF + md,$$

where EF is denoted in fig. 127, m is the slope of the graph, and d is the thickness of the material.

It is assumed that the conductivity, k, of the copper is known. Since the rate of conduction of heat is the same throughout, it follows that

$$kg = k'g',$$

where g is the temperature gradient in copper and g' that in the material, it being assumed that the area of cross-section is the same throughout.

Thus
$$k' = \frac{g}{g'}\,k,$$

and since $\frac{g}{g'}$ contains the ratio of temperatures, it is sufficient to obtain the ratio of corresponding galvanometer deflections.

$g' = \left(\frac{\text{EF}}{d} + m\right)$ and g is obtained from the straight portions AB or CD of the graph.

Forbes's Method of Determining the Conductivity of a Metal Bar

This experiment consists of two parts, in the first of which a metal bar is heated at one end until the steady state is reached.

In the second part a bar of the same material and cross-section, but shorter, is allowed to cool under similar external conditions.

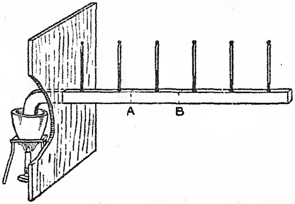


FIG. 128

In order to maintain constant external conditions the experiment should be performed in a part of the laboratory sheltered from draughts.

The bar (fig. 128) usually has one end curved and dipping into a convenient molten metal contained in a vessel on the other side of a screen, which protects the bar from direct radiation from the source of heat.

The metal may conveniently be molten lead or solder. The bar is provided with a series of holes which lie regularly along its length into which thermometers fit.

If these are small and contain mercury or the molten liquid, it is found that the process of conduction is not disturbed by their presence. The holes near the hot end should contain the molten liquid and the remainder mercury.

By observation of the temperatures at various distances along the bar measured from the screen, the temperature slope may be found. This is best done by plotting a curve (fig. 129) and calculating the slope, $\frac{d\theta}{dx}$, from the inclination of the tangents at various points.

A thermo-junction may be used alternatively to find the temperatures by dipping one junction successively into the holes. For the calibration of the junction the reader is referred to p. 582.

The importance of maintaining steady external conditions will be appreciated in this part of the experiment, and great care will be required to maintain the whole length of the bar simultaneously steady.

The bar is assumed to be sufficiently long that its end is at the external temperature, θ_0. A convenient length is one metre and its section may be about 2 cm. square.

In the steady state all the heat passing a section, B, escapes from the surface between B and the end.

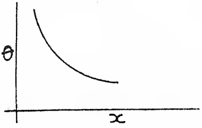


FIG. 129

The rate of flow at B is $$kA\frac{d\theta}{dx},$$

where k is the conductivity and A the area of section.

This rate of flow is calculated from the second part of the experiment by determining the rate of cooling of the portion of the bar beyond B. The equation thus obtained serves to find k.

The second bar may be conveniently 10 cm. long. It is heated to the temperature of the molten metal, but so that its surface is not damaged and remains similar to that of the first bar. To do this and at the same time to prevent sudden cooling of the molten mass, the rod is wrapped in several layers of paper and completely immersed.

It is provided also with a hole to carry a thermometer and is suspended under the same conditions as prevailed round AB. Its temperature is observed at successive times so as to include in the range those values which prevailed along the bar.

A curve is drawn showing the relation between the temperature, θ, and the time, t, for the short bar.

This bar is similar to the long bar and cools under the same conditions, so that the rate of cooling for both is the same. From the curve we can deduce the rate of cooling, $\frac{d\theta}{dt}$, in the usual manner.

We require for the purpose of the calculation the rate of cooling for points along the bar in the first part of the experiment. We must

correlate the values of x and $\frac{d\theta}{dt}$, x denoting distances measured from the screen.

This may be done from the curves.

Take a series of values of x from the curve illustrated in fig. 129 and observe the corresponding value of θ. From the curve relating θ and t take the values $\frac{d\theta}{dt}$ for these particular values of θ.

We then have the corresponding values x and $\frac{d\theta}{dt}$.

Plot these on a curve as illustrated in fig. 130.

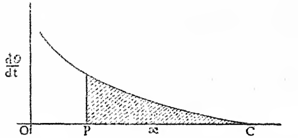


FIG. 130

This curve will cut the axis at a point, C, where $\frac{d\theta}{dt}$ vanishes, or where the temperature of the bar is equal to that of the surrounding air.

If s is the specific heat and ρ the density, the rate of loss of heat between two points separated by distance δx is

$$A\delta x \rho s \frac{d\theta}{dt}.$$

The total loss per second between B and the end of the bar is thus

$$\int_{B}^{\text{end}} A\rho s \frac{d\theta}{dt}\, dx$$

$$= A\rho s \int_{B}^{\text{end}} \frac{d\theta}{dt}\, dx$$

$$= A\rho s \times \text{shaded area of fig. 130,}$$

where OP on the graph measures the distance from the hot end up to B.

This area may be found by the planimeter or calculated if carefully drawn on squared paper.

Thus $$kA\left(\frac{d\theta}{dx}\right)_B = A\rho s \times S.$$

S = shaded area and $\left(\frac{d\theta}{dx}\right)_B$ denotes the temperature slope at the point B.

Thus $$k \cdot \left(\frac{d\theta}{dx}\right)_B = \rho s S.$$

We can thus determine k, the values of ρ and s being given as constants of the apparatus or determined in the usual way.

These values may be taken to be 8·93 gm. per c.c. and 0·094 respectively when the bar is of copper.

The Determination of the Conductivity of a Bar of Metal by Ångström's Method

In this method heat is supplied to a long bar by alternately heating and cooling it at one region in regular periods.

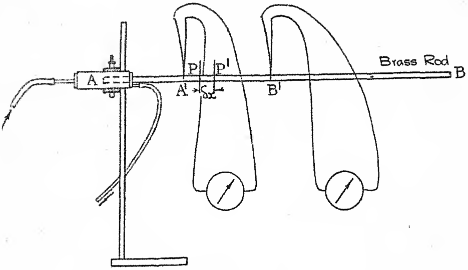


FIG. 131

In this way the temperatures at points along the bar fluctuate periodically, and on account of surface radiation the temperature amplitudes diminish as the distance from the region of supply increases, while the maximum and minimum values occur at later times with the increasing distance.

The bar must be chosen so that it is sufficiently long to allow us to neglect the effect of the terminal faces. The fluctuations should die away at a short distance from the cooler end, which thus has the same temperature as the air surrounding it.

When the heating is continued long enough the periods develop themselves completely, in which case the mean temperature at any point of the bar preserves some constant value.

We first consider the theory of the experiment.

Let us consider a bar, AB (fig. 131), with one end, A, exposed for a definite time, T_1, to a current of steam and for a succeeding time, T_2, exposed to a current of cold water. Suppose that this process is continued regularly until the steady fluctuations throughout the bar are

developed. These fluctuations will have a complete period of $(T_1 + T_2)$ which we may denote by T, and for the sake of convenience we shall write

$$p = \frac{2\pi}{T}.$$

Let the conductivity of the bar be k, ρ its density, s its specific heat, A its cross-section, P its perimeter, and θ its temperature at a point, P, distant x from a convenient origin. We shall choose this to be at A.

Consider a point, P′, distant δx from P, as in the diagram, and let its temperature be $(\theta + \delta\theta)$. The heat flowing into the element, PP′, of the bar across a plane at P drawn normally to its length is equal to

$$k \cdot \frac{\theta_P - \theta_{P'}}{\delta x} \cdot A \text{ per sec.},$$

where the subscripts denote the points at which θ is measured, i.e. the heat flow amounts to

$$-k \,.\, \frac{\delta\theta}{\delta x} \,.\, A$$

$$= -k \,.\, \frac{d\theta}{dx} \,.\, A = F \text{ (say)},$$

in the limit when δx is made infinitesimal.

Thus this expression denotes the flow from left to right at the point, P, of the bar. Since θ depends on x, in the general case this flow will also depend on x, i.e. F depends on x.

Again consider the element PP′. We have calculated the flow, F, into it at P, and the flow out at P′ will be

$$F + \frac{dF}{dx} \,.\, \delta x.$$

Hence the total flow of heat *into* PP′

$$= -\frac{dF}{dx} \,.\, \delta x$$

$$= k \,.\, \frac{d^2\theta}{dx^2} \,.\, A \,.\, \delta x \text{ per sec.}$$

This heat is used up, partly in warming up the part of the bar concerned, and partly in losses from the surface.

If the temperature is changing at the rate, $\frac{d\theta}{dt}$, t denoting the time, the first part amounts to

$$A \,.\, \delta x \,.\, \rho \,.\, s \,.\, \frac{d\theta}{dt},$$

and the second to

$$h \,.\, P \,.\, \delta x \,(\theta - \theta_0).$$

h denotes a constant for the surface and θ_0 is the external temperature.

We shall, however, suppose that θ is measured in degrees above the surrounding temperature, so that we may write the last expression simply

$$h \,.\, \mathrm{P} \,.\, \delta x \,.\, \theta.$$

We, therefore, arrive at the equation

$$k\mathrm{A}\,\delta x\,\frac{d^2\theta}{dx^2} = h\mathrm{P}\delta x\theta + \mathrm{A}\rho s\delta x\,\frac{d\theta}{dt}.$$

or

$$\frac{d\theta}{dt} = \mathrm{K}\frac{d^2\theta}{dx^2} - \mathrm{H}\theta, \qquad \ldots(2)$$

where

$$\mathrm{K} = \frac{k}{\rho s}, \quad \mathrm{H} = \frac{h\mathrm{P}}{\mathrm{A}\rho s}. \qquad \ldots(3)$$

Any function which satisfies certain conditions can be expanded as a Fourier series.

Students of physics should make themselves acquainted with the Fourier analysis and they may be referred to Carslaw's work on this subject. For the present purpose we need nothing more than the statement that the expansion is possible.

Fourier's series is the following:

$$\left.\begin{aligned} f(t) = \mathrm{A}_0 &+ \mathrm{A}_1 \cos pt + \mathrm{A}_2 \cos 2pt + \mathrm{A}_3 \cos 3pt + \ldots \\ &+ \mathrm{B}_1 \sin pt + \mathrm{B}_2 \sin 2pt + \mathrm{B}_3 \sin 3pt + \ldots \end{aligned}\right\}. \quad \ldots(4)$$

Here $f(t)$ is any function existing over some interval from $t = a$ to $t = b$ and satisfying certain conditions with regard to continuity, etc., which we need not enter into here. We merely mention that the series is applicable to the function we shall use in our experiment.

With regard to the values of the A's and B's the rule is that:

$$\mathrm{A}_0 = \frac{1}{b-a}\int_a^b f(u)\,du;$$

$$\mathrm{A}_n = \frac{2}{b-a}\int_a^b f(u)\cos\frac{2n\pi u}{b-a}\,du;$$

$$\mathrm{B}_n = \frac{2}{b-a}\int_b^a f(u)\sin\frac{2n\pi u}{b-a}\,du;$$

n has the integral values, 1, 2, 3, etc.

The value of p in the above series is in this general case $\dfrac{2\pi}{b-a}$.

When $f(t)$ is periodic and has the complete period, T, we take the limits, a and b, at the ends of this period and write $a = 0$, $b = \mathrm{T}$, so that the series represents $f(t)$ from $t = 0$ to $t = \mathrm{T}$, and on account of its periodic character represents it also from T to 2T, etc. In this case:

$$p = \frac{2\pi}{T};$$

$$A_1 = \frac{1}{T}\int_0^T f(u)\,du,$$

$$A_n = \frac{2}{T}\int_0^T f(u).\cos npu\,du,$$

$$B_n = \frac{2}{T}\int_0^T f(u)\sin npu\,.\,du.$$

The periodic function with which we are concerned is that which expresses the fact that from $t = 0$ to $t = T_1$ the temperature has some value, θ_1, and from $t = T_1$ to $t = T_2$ it has the value θ_2. Such a function can be represented by a Fourier series and we shall assume that the particular expression is (4), with coefficients calculated according to the rule. These, however, are not required for the experiment.

It is shorter and more convenient for our purpose to write (4) in the form

$$a_0 + a_1 \sin (pt + \gamma_1) + a_2 \sin (2pt + \gamma_2) + \ldots \qquad \ldots(5)$$

Let us suppose that this is the expansion which expresses the temperature at A.

Now the temperature, θ_1, at the point whose distance is x_1 from A is a fluctuating function, and so is that for any other point, x_2.

We may therefore write for θ_1 some such value as (5), viz.

$$\theta_1 = C_0' + C_1' \sin (pt + \delta_1') + C_2' \sin (2pt + \delta_2') + \ldots \quad (6)$$

and for the point, x_2,

$$\theta_2 = C_0'' + C_1'' \sin (pt + \delta_1'') + C_2'' \sin (2pt + \delta_2'') + \ldots \quad (7)$$

The quantities C_0' C_1' . . ., δ_1' δ_2' . . ., will differ from C_0'' C_1'' . . ., δ_1'' δ_2'' . . ., since they correspond to different values of x.

We have to solve the equation (2), and we have a clue from the experimental observations of diminishing amplitudes and lagging maxima and minima. Moreover, the value of θ is known at the point A where x is zero. This value is given by (5).

The solution is, in fact

$$\theta = a_0 e^{-\alpha_0 x} + a_1 e^{-\alpha_1 x} \sin (pt + \beta_1 x + \gamma_1) + a_2 e^{-\alpha_2 x} \sin (2pt + \beta_2 x + \gamma_2) + \text{etc.} \qquad \ldots(8)$$

Where
$$\left.\begin{aligned} \alpha_0{}^2 = \frac{H}{K},\quad K(\alpha_n{}^2 - \beta_n{}^2) = H, \\ 2K\alpha_n\beta_n = -np \end{aligned}\right\} \qquad \ldots(9)$$

When $x = 0$ this expression reduces to that of equation (5), so that the solution satisfies the end condition.

The reader may verify that the solution satisfies equation (2) by substitution. He will observe that the coefficients of the separate

sine and cosine terms and the terms independent of trigonometrical functions all vanish if (9) holds.

Now if we consider the two points, x_1 and x_2, at a distance, l, apart, by equations (6), (7), and (9), we have

$$C_1' = a_1 e^{-\alpha_1 x_1}, \qquad C_1'' = a_1 e^{-\alpha_1 x_2}, \qquad \delta_1'' - \delta_1' = \beta_1 l,$$

for $\qquad \delta_1' = \beta_1 x_1 + \gamma_1$ and $\delta_1'' = \beta_1 x_2 + \gamma_1$.

Hence $\qquad \alpha_1 l = \log \dfrac{C_1'}{C_1''}$ and $\alpha_1 \beta_1 = \dfrac{\delta_1'' - \delta_1'}{l^2} \log \dfrac{C_1'}{C_1''}$.

But by the last of the conditions (9)

$$K = -\frac{p}{2\alpha_1\beta_1} = \frac{\pi l^2}{T(\delta_1' - \delta_1'') \log \dfrac{C_1''}{C_1''}} \qquad \ldots(10)$$

and in the same way

$$K = \frac{n\pi l^2}{T(\delta_n' - \delta_n'') \log \dfrac{C_n'}{C_n''}}. \qquad \ldots(11)$$

In this formula we note that the constant, h, does not appear and we are not troubled with the difficulties always associated with it. The conductivity, k, is given by $K\rho s$, so that if our experiment is performed carefully and as a consequence the quantities, C_n and δ_n, accurately known, we have to rely on the accuracy of the knowledge of the density and specific heat. Both these are accurately known. This is the reason of the importance of Ångström's method. The student is recommended to refer to a translation of the original paper in the *Philosophical Magazine*, series 4, Vol. XXV, p. 130, 1863.

From (11), we observe that each coefficient, C_n, gives rise to a value of K. As the carrying out of the experiment will show, the coefficient, C_1, gives the most reliable result, but the value corresponding to C_2 should also be worked out. It remains to describe the experiment and to show how to find the Cs and δs.

The bar, which may be of copper, iron, or brass, has one end, A, inserted in a chamber so arranged that steam and cold water may be passed in alternately as described above. The total period is to be measured and care exercised to reproduce the conditions exactly in each successive period.

Steam from a conveniently large tin flask and water from the tap will set up the right conditions.

The steady temperature fluctuations will be the more easily attained if the bar is sheltered to avoid draughts in its neighbourhood and consequent troublesome convection effects. The temperature at a point on the bar should be observed by means of a thermo-junction placed in a small cavity in the bar. Two such cavities will be required and

should contain mercury. The thermo-junction must be calibrated in the usual way (see p. 237).

When this is done read the temperature at convenient intervals by means of the galvanometer in the circuit and obtain as many readings as possible throughout two complete periods or more. Plot a curve for θ against the time very carefully on squared paper, as illustrated in fig. 132. Suppose that this refers to the point, x_1. Repeat the process for x_2.

We shall determine the necessary quantities from the graphs.

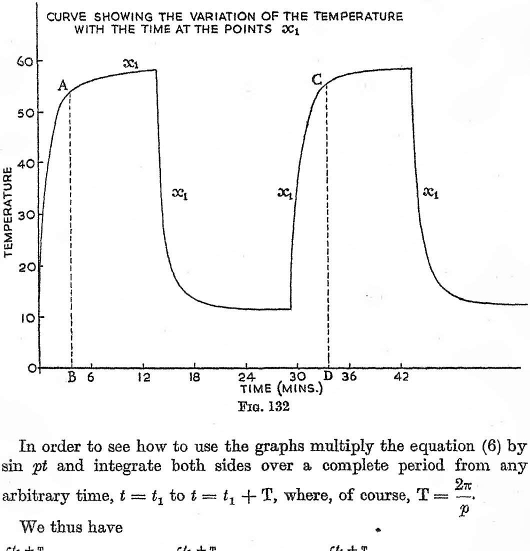


FIG. 132

In order to see how to use the graphs multiply the equation (6) by $\sin pt$ and integrate both sides over a complete period from any arbitrary time, $t = t_1$ to $t = t_1 + T$, where, of course, $T = \frac{2\pi}{p}$.

We thus have

$$\int_{t_1}^{t_1+T} \theta_1 \sin pt\,dt = C_0' \int_{t_1}^{t_1+T} \sin pt\,dt + C_1' \int_{t_1}^{t_1+T} \sin(pt + \delta_1') \sin pt\,dt$$
$$+ C_2' \int_{t_1}^{t_1+T} \sin(2pt + \delta_2') \sin pt\,dt + \text{etc.} \quad \ldots(12)$$

In doing this it is well to remark that we are integrating the series on the right-hand side term by term, adding all the integrated terms

and equating to the integrated function on the left. This process is not always legitimate, but on account of the properties of the particular series we may apply it in the present case.

On performing the integration the only term on the right-hand side which does not vanish is the second, and this has the value

$$\tfrac{1}{2}TC_1' \cos \delta_1'.$$

Denote the integral on the left by S_s.

Then $$S_s = \tfrac{1}{2}TC_1' \cos \delta_1'.$$

In the same way, if S_c denote the value of

$$\int_{t_1}^{t_1+T} \theta_1 \cos pt\, dt,$$

we find

$$S_c = \tfrac{1}{2}TC_1' \sin \delta_1'.$$

We can determine C_1' and δ_1' from these two equations, for

$$\tan \delta_1' = \frac{S_c}{S_s} \qquad \ldots(13)$$

and $$C_1'^2 = \frac{4}{T^2}(S_s^2 + S_c^2). \qquad \ldots(14)$$

In the same way, if we multiply the series by cos $2pt$ and sin $2pt$ and integrate, we find

$$S_{2c} = \int_{t_1}^{t_1+T} \theta_1 \cos 2pt\, dt = \tfrac{1}{2}TC_2' \sin \delta_2',$$

and as before we can express C_2' and δ_2' in terms of the quantities S_{2c} and S_{2s}, the meaning of the latter being, of course,

$$\int_{t_1}^{t_1+T} \theta_1 \sin 2pt\, dt = \tfrac{1}{2}TC_2' \cos \delta_2'.$$

This suggests a graphical method of determining the necessary quantities.

Measure as many ordinates of the curve (fig. 132) as is conveniently possible and multiply each by sin pt or sin $\frac{2\pi t}{T}$.

T is the complete period of the periodic heating and t is the value of the time appropriate to each ordinate.

Plot a new curve with these new quantities as ordinates and times as abscissae and thus obtain fig. 133.

Draw any ordinate A^1B^1 for some arbitrary time, t_1, and construct C^1D^1 the ordinate at one complete period later.

Measure the area between these ordinates, the curve and time axis carefully with a planimeter.

This gives S_s.

Go through the similar process to find S_c.

Multiply θ_1 by sin $2pt$ and cos $2pt$ to find S_{2s} and S_{2c}. All this has to be repeated for the second point, x_2, and we then have all the data necessary to deduce k from formulae (10) and (11).

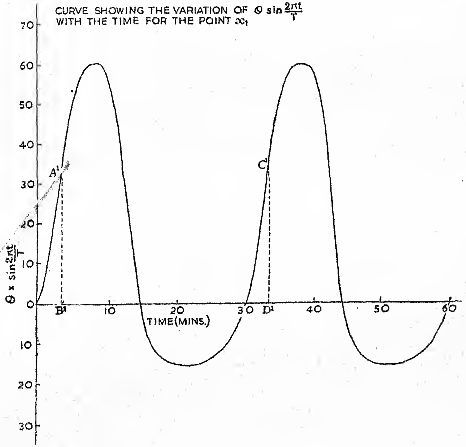


FIG. 133

Another way of making the calculation is to make two experiments with different periods, T′ and T″. If α_n' and α_n'' are the values of α_n in these cases respectively we find from (9) the result:

$$K = \frac{n}{2\alpha_n\alpha_n'}\sqrt{\frac{p'^2\,\alpha_n'^2 - p''^2\,\alpha_n''^2}{\alpha_n'^2 - \alpha_n''^2}}.$$

α_n' and α_n'' are derived as before, and

$$p' = \frac{2\pi}{T'}, \quad p'' = \frac{2\pi}{T''}.$$

It is to be noted that we do not in this case require the values δ.

The density and specific heat may be taken from tables or be measured by any of the usual methods.

In drawing the curves it is a good plan to have two complete periods shown to give a means of testing the accuracy of the areas, S_s and S_c.

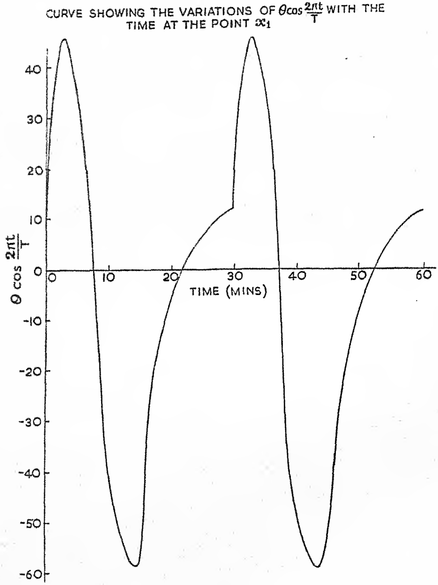


FIG. 134

Always test the end of the bar remote from the heat supply to determine if it remains at the air temperature. It will suffice to place a thermometer close to this end. A longer bar must be used if the fluctuations continue right to the end.

Ångström used a square bar of side 2·375 cm., and the length, l, between the two points, x_1 and x_2, was 5 cm.

It has been found, however, that a cylindrical bar of diameter from 1 to 2 cm., and of length about 60 cm., will suffice, with such a range of temperature as described above. In the original experiment Ångström heated his bar at the central region and, of course, had similar conditions on both sides.

CHAPTER X

MISCELLANEOUS EXPERIMENTS IN HEAT

Determination of the Radiation Constant. (**'Phil. Mag.', ser. 6, 1905, p. 210**)

By Stefan's Law the total radiation from a black body is proportional to the fourth power of the absolute temperature, or

$$R = \sigma T^4 \text{ per sq. cm. per sec.}$$

It is the object of this experiment to determine σ.

Apparatus

The diagram (fig. 135) illustrates the arrangement of apparatus which consists of a blackened hollow metal hemisphere, B, about ten

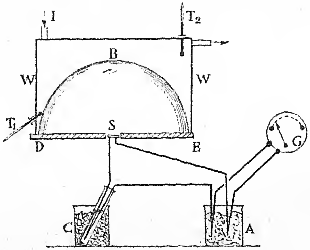


FIG. 135

inches in diameter, fitted into a wooden box, W, lined with tin. This fits on to a table, of which the top, DE, is shown, containing a small hole at S, which lies at the centre of the hemisphere. B is heated to a uniform temperature measured by the thermometers, T_1 and T_2, by passing steam through the box above the hemisphere. The black surface of B is the radiator, and the heat is received by a small disk of silver placed at S, and blacked on the upper surface to prevent reflection. It is better to fit the disk in a vulcanite frame rather than to allow it to touch the table directly.

From the disk are led away two wires, one of constantan, and the other of silver, to a galvanometer and second junction.

S is thus one of the junctions of a thermo-electric couple, the other

is placed in a tube containing oil standing in a calorimeter, C, containing water or ice.

The junctions to the galvanometer are kept in a canister, A, packed with cotton wool to prevent any electrical effect due to difference of temperature at these junctions.

It may be further necessary to include a resistance in the circuit to keep the deflection on the scale if the galvanometer is too sensitive.

A rise of temperature due to absorption by the disk is thus recorded on the galvanometer.

Theory

Let R_1 = radiation absorbed by the silver disk per unit area per second, and R that emitted. Let the temperature of the radiator be T_1, and of the disk T.

If the whole enclosure, the disk included, had temperature T_1, there would be equilibrium, and the disk would both emit and absorb R_1 in unit time. The energy absorbed would arise, of course, from B. This same energy falls on the disk when at the lower temperature and is absorbed, but the energy emitted is now R. Thus the gain of energy per second = $(R_1 - R)$ A, where A denotes the area of the disk.

Let m denote the mass of the disk, s its specific heat, and $\frac{dT}{dt}$ its rate of change of temperature.

Then we have

$$ms \frac{dT}{dt} = \frac{R_1 - R}{J} . A = \frac{\sigma A}{J} . (T_1^4 - T^4),$$

or

$$\sigma = \frac{Jms}{A (T_1^4 - T^4)} . \frac{dT}{dt},$$

where J = Joule's equivalent ($4{\cdot}2 \times 10^7$ ergs per calorie).

All the quantities on the right are measured, and hence σ is calculated.

Experimental Details

It is first necessary to ascertain the relation between the readings of the galvanometer scale and the difference in temperature between the two junctions.

In order to make this comparison the disk is surrounded with cotton wool, and the cold radiator placed above it.

The calorimeter, C, is then heated and the difference in temperature between it and the disk recorded on a graph against the readings of the deflection.

We thus obtain: difference in temperature per scale division = $\frac{AB}{BC}$.

By measuring the temperature of C, we can then deduce from the galvanometer deflection the temperature of S from this graph.

Secondly, we require the rate of rise of the temperature of the disk when the radiator is put on.

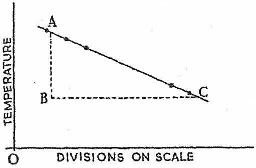


FIG. 136

As soon as the temperature in the enclosure, B, has become steady, the box is placed over S, keeping the latter at the centre, and the galvanometer is read at equal intervals of time; these may conveniently be every five or ten seconds.

The table shows a record of observations made every ten seconds.

Time	Scale div.	Time	Scale div.
0	227	40	190
10	218	50	181
20	208	60	173
30	199	70	165

These results are plotted on a graph.

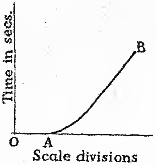


FIG. 137

Draw a tangent to the curve and measure the value of as $\frac{dT}{dt}$ close to A as possible, since errors soon arise by conduction from the silver disk.

m and *s* are found in the usual way. It will be convenient to regard

these as constants of the apparatus, and to record them on the occasion of making the apparatus. It is inconvenient to make determinations at each experiment.

Otherwise s may be measured by the method of mixture using a piece of silver identical with that in the apparatus.

The Rise of Boiling-point of Solutions

The object of this experiment is to determine how the temperature at which a solution boils depends on the concentration, to determine the rise of the boiling-point above that of the solvent, and to compare experimental results with those given by the thermodynamic formula.

If P denotes the osmotic pressure of a solution, M the molecular weight of a salt, and s, the number of grammes dissolved in 1 c.c., we have, in the case of dilute solutions,

$$\frac{P}{s} = \frac{cR\theta}{M},$$

where c depends on the degree of dissociation, R is the value of the gas constant per gm. molecule, and θ is the temperature of the solution. Thus for any particular salt and solvent and for a temperature θ,

$$P \propto s.$$

A calculation based on the second law of thermodynamics shows that the rise of temperature of the boiling-point is given by the formula

$$T = \frac{P\theta}{L\rho},$$

where P = osmotic pressure,
L = latent heat of vaporization of the solvent,
ρ = its density, and
θ = temperature of boiling of the solvent.

For the purpose of the calculation it may be assumed in the case of water, that $\rho = \frac{1}{1{\cdot}003}$ at boiling-point.

The standard osmotic pressure P_0 is $22{\cdot}4 \times 10^6$ dynes per gm. molecule per litre at 0° C.

If θ = B.P., the approximate value for P at temperature θ is

$$\frac{\theta}{273} \times 22{\cdot}4 \times 10^6 \text{ dynes.}$$

θ is in absolute units, and the value for any particular concentration may be deduced by simple proportion.

L, for water at ordinary pressures, may be taken as $537 \times 4{\cdot}2 \times 10^7$ ergs, in which units it must be expressed in the equation.

Suppose w gm. of solute are dissolved in W gm. of solvent.

The volume of the solvent is $\frac{W}{\rho}$ c.c. or $\frac{W}{1000\rho}$ litres.

If M denotes the molecular weight of the solute, we have $\frac{w}{M}$ gm. molecules in $\frac{W}{1000\rho}$ litres, i.e. $\frac{1000\rho w}{MW}$ gm. molecules per litre.

Thus the depression according to the theory should be

$$\frac{1000\rho w}{MW} \cdot \frac{P_0\theta}{L\rho} = \frac{1000wP_0\theta}{MWL}.$$

Verify this expression by taking several values of w.

A modern form of apparatus for carrying out this experiment will be described. It is represented diagrammatically in the fig. 138. A is a glass tube which holds the solvent from which proceed two tubes, B and C, provided with corks.

The thermometer passes down B, and the bulb is immersed in the solution or pure solvent in A.

Weighed quantities of solute may be added through C.

Through the cork, H, passes the metal tube of the condenser, F, which is cooled by passing in water by one of the tubes, T, and allowing it to flow out by the other.

The small tube, G, carries garnets, which are placed at the bottom of A, and which together with a short wire through the base of A, assist the beginning of boiling and tend to prevent superheating. A stands on a cylinder of asbestos, D, which rests on a plate, E, of pipeclay, and the liquid is heated by two burners placed below E.

The thermometer is set and its reading taken when the pure solvent is boiling, and the difference noted when the solute is added and the solution boils.

We have first to weigh the vessel, A, when detached from F, and not carrying the thermometer. Add a weighed amount of solute, w, to a convenient quantity of solvent. After solution, and after taking the boiling-point of the solution, F is again removed and also the thermometer, and A is again weighed.

Since w is known, we can obtain W, the weight of solvent. Care must be taken not to remove any of the liquid with removal of the thermometer. The thermometer bulb must be placed in the solution. If it is above, it attains the temperature of the steam coming from the solution, and this is not at the temperature of the solution, but at the temperature of steam appropriate to the prevailing pressure.

The Beckmann Thermometer

The magnitude of the change of temperature is not very great, and great care is necessary to measure it accurately. The most convenient form of thermometer for this purpose is one of the Beckmann type.

A common form of this thermometer is made of Jena glass, having an apparent coefficient of expansion for mercury of magnitude $\frac{1}{6370}$. It is illustrated in fig. 139.

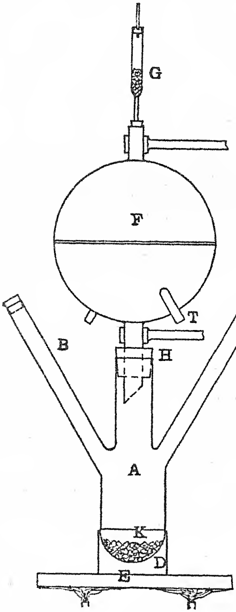


FIG. 138

A special feature is the large bulb, B, containing a comparatively large quantity of mercury from which a fine capillary tube extends, bent as shown at A, where the capillary is widened.

The mercury in the bulb being of considerable volume gives a large additional volume on expansion, and this causes a large movement in the fine capillary tube when the change of temperature round the bulb is only slight.

If, however, the quantity of mercury in the bulb were fixed, the range of temperature for which the instrument could be used would be very small unless the capillary were inconveniently long.

The thermometer is used to measure a small change in temperature above or below some particular temperature. Suppose that we require to measure a small change of temperature above T_0.

FIG. 139

It would be best to arrange so that the mercury extended from the bulb up to the zero of the scale when at the temperature, T_0. A slight increase in temperature would then drive the mercury along the capillary, which is marked in hundredths of degrees at intervals of sufficient width to enable an experimenter to estimate to thousandths.

Suppose now that it is required to read temperatures slightly above another temperature, T_1, which is greater than T_0. If we could withdraw sufficient mercury from the bulb it would be possible to arrange that the mercury extended up to the zero mark at the temperature, T_1, and slight increases would drive the thread along the fine tube as before.

Of course, the expansion per degree rise of temperature in the second case is not strictly the same as in the first, since the initial volumes are not equal in the two cases.

In order to see how this affects the observations, let us suppose that the graduations on the stem are correct at t_0° C., i.e. with mercury filling the bulb and extending to the zero mark, when the surrounding temperature is t_0°, a rise of temperature of 1° C. would cause an expansion to the first degree mark above the zero.

Let the volume of mercury at t_0° under these conditions be denoted by V_0, and let the coefficient of apparent expansion of the mercury in glass be α.

Then the volume of the capillary per 1° is $V_0\alpha$.

Now suppose that mercury is drawn off until at temperature, t, the mercury fills the bulb and extends to zero. The volume drawn off is $V_0\alpha(t - t_0)$.

The remaining mercury had a volume at t_0° of

$$V_0\{1 - \alpha(t - t_0)\},$$

and on further heating of one degree would expand by an amount

$$V_0\alpha\{1 - \alpha(t - t_0)\}.$$

Since the volume of the capillary per 1° is $V_0\alpha$ this expansion will be registered as $\{1 - \alpha(t - t_0)\}$ of a degree.

The degrees on the scale are too large for the temperature t.

A correction could be made by dividing the scale readings by $\{1 - \alpha(t - t_0)\}$, and it would be sufficient to add to the observations

an amount $\alpha(t - t_0)$ per degree registered on the scale to obtain the corrected rise.

As has been stated, α has the value $\frac{1}{6370}$, so that if we suppose the thermometer correct at 90°C. and we record a rise just above 100°C. we should add to each degree an amount

$$\frac{10}{6370}° = 0{\cdot}0016°.$$

This is an appreciable amount since we can record up to thousandths of a degree.

The correction can be neglected when we are recording to hundredths of a degree.

The widening of the capillary at A enables the change in the quantity of mercury in the bulb to be made.

Suppose it is necessary to read a temperature between 100° and 101°C. The thermometer is placed into a bath at a temperature of about 102° to 103°C., and so that the mercury extends from the bulb up to the widened part when at this temperature.

If a quick jerk is made the thread will break at the point where the capillary enters the wider part.

When cooled to about 100°C., i.e. when held in steam, the temperature of which may be determined by observing the atmospheric pressure, and referring to a book of tables, the top of the thread should sink to the lower end of the capillary. It will not lie exactly at zero, and the position occupied by the end at this known temperature is recorded. It may be necessary to vary the upper temperature when the thread is continuous before the end will sink conveniently towards the zero at the temperature of steam.

When this condition has been attained the small rise of temperature is readily recorded.

It is not necessary to bring the mercury thread exactly to zero before reading the small temperature change, and if the correction is to be applied it may be made by giving the value to t which corresponds to the temperature for which the apparatus is set, just as if it had been set at the zero.

If it is more convenient to observe a diminution of temperature below a certain point, it is only necessary to bring the top of the mercury thread at this temperature to a point high up on the scale. A slight depression will cause the top of the column to sink, and the interval may be measured as before.

In such accurate temperature measurements it is necessary to make a correction for the emergent column. This is done by reference to Grutzmacher's table taken from the *Zeitschrift für Instrumentenkunde* (1896), p. 220.

A reference to this table will show how important such a correction is.

GRUTZMACHER'S TABLE

Temperature interval	Value of one scale division in degrees cent. with thread totally immersed	With the thread all outside and at mean temperatures below	Corresponding value in degrees cent. per scale division
°C.	°C.	°C.	°C.
−35 to −30	0·982	0	0·977
0 to 5	0·997	15	0·995
45 to 50	1·011	26	1·015
95 to 100	1·021	32	1·032
145 to 150	1·027	38	1·045
195 to 200	1·028	44	1·053
245 to 250	1·024	50	1·055

The second column gives the correction due to the variation of mercury within the bulb, and it shows that the thermometer reads correctly at a temperature between 5° and 45°, for in this region lies the point where the value of one scale division is 1° C.

The fourth column gives, in addition, the correction for the fact that part of the mercury is at a different temperature from that in the bulb.

In order to make a correction, choose the part of the table which describes most nearly the condition of the experiment, e.g. at a temperature in the neighbourhood of 100° with a mean thread temperature about 30°, one degree recorded by the apparatus is approximately 1·032°.

Two thermometers will be required, one adapted for use in the neighbourhood of 0° C. and the other adapted for use near 100° C.

Molecular Weight by Depression of Freezing-point

The thermodynamic formula for the depression of freezing-point is

$$T = \frac{P\theta}{L\rho},$$

where L is the latent heat of fusion of the solvent, ρ its density, and θ is the temperature of fusion.

P is the osmotic pressure.

Let W denote the weight of solvent used, and let w gm. of solute be added of molecular weight, M.

Let P_0 denote the pressure for 1 gm. molecule per litre, i.e. $22{\cdot}4 \times 10^6$ dynes/sq. cm. By using this value and $\rho = 1$, $L = 80 \times 4{\cdot}2 \times 10^7$ ergs, $\theta = 273\,°K$, calculate the corresponding depression.

In the present case the volume of solvent is $\frac{W}{1000\rho}$ litres, and the number of gm. molecules dissolved is $\frac{w}{M}$.

The corresponding depression is T′, where

$$T' = \frac{1000\rho}{W} \cdot \frac{w}{M} \cdot \frac{P_0\theta}{L\rho} = \frac{w}{WM} \cdot 1000 \cdot \frac{P_0\theta}{L}.$$

It will be the object of the experiment to calculate M.

The vessel, D, with the stirrer, S, is weighed and a quantity of the solvent introduced, and the vessel again weighed.

This gives W.

The thermometer is placed with its bulb in melting ice, and the top of the mercury thread adjusted so that it stands near the upper end of the scale. This temperature is 0°C. and small differences from this point may be read from the thermometer graduations. These may be corrected by Grutzmacher's table.

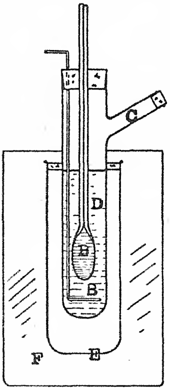


FIG. 140

A weighed quantity of the solute is introduced into the solvent through the side tube, C, and solution brought about by help of the stirrer.

The vessel is placed in an enclosure, E, and surrounded by a freezing mixture, F.

The thermometer, B, must be kept in ice until D has cooled down to 0°, and then the transference must be quickly made to D. Otherwise the thread will rise beyond the scale, and mercury will flow into A.

During solidification the mercury will stand at a definite mark if the solvent is pure, and the difference, T′, can be measured.

In order to set the thermometer ready for use in the experiment, it must be placed in water cooled to within a degree or two of the freezing-point. The thread of mercury which must be continuous from B to A is broken by a quick jerk at the top of A, when the temperature of the water is attained.

In this way the thread will extend nearly to the top of the scale, when the bulb is at the temperature of melting ice.

For the temperature at which the solution freezes the fall below the freezing-point of water is obtained by subtracting the second reading with the bulb in the solution from that with the bulb in ice. The degrees thus read will require correction on account of the emergent column.

In order to make the correction, use Grutzmacher's table, e.g. if a temperature difference recorded is 1·5° C. with an average room temperature 15° C., the degree as read on the scale is really only 0·995°. Thus the difference is

$$1{\cdot}5 \times 0{\cdot}995 = 1{\cdot}492°.$$

The thermometer reads to $\frac{1}{100}$°, so that we can estimate $\frac{1}{1000}$°, and it is therefore necessary to take account of such an error as that arising above, viz. of 0·008° C.

Determination of the Mechanical Equivalent of Heat by Friction Cones

The apparatus consists of two vessels of gun-metal of the shape of truncated cones, one fitting within another. The inner cone contains water, and is held in position by a measured couple applied by means of a weight hanging over a pulley and connected to a disk attached to the cone.

The outer cone is rotated by means of a cord to a large wheel rotated by hand.

It is possible by steady turning of the handle to cause the weight to hang almost steady, so that a steady couple is applied to the inner cone by friction, and its amount is measured by multiplying the force and the diameter of the wheel to which the cord is attached.

A counting device is fixed to the outer cone so that the number of rotations is recorded.

If n is the number of rotations, w the weight in dynes applied to the disk, and a its arm, the work done is

$$2\pi nwa \text{ ergs.}$$

The rise in temperature of the water is measured by means of a thermometer, so that if M represents the total water equivalent of the

cone and contained water, and if the rise in temperature is T, the heat developed is MT calories.

If no heat is lost to the surroundings, the mechanical equivalent, J, or the number of ergs necessary to produce one calorie is given by

$$J = \frac{2\pi nwa}{MT}.$$

A common form of apparatus in use in laboratories is shown in the figure; it was designed by Dr. G. F. C. Searle.

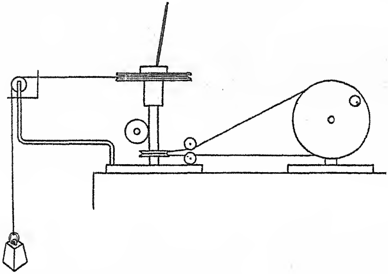


FIG. 141

An improved design is described in the *Philosophical Magazine*, Sept. 1920, by H. P. Waran.

In order to prevent radiation losses it is a good plan to cause a rise in temperature of not more than 10° C., and to cool the water about 5° below the room temperature initially. The two cones are mounted in a metal case lined with cork to diminish conduction losses.

The heat is conducted through the walls of the inner cone to the water, and it is necessary to stir the water during the experiment to assist the flow of heat throughout its extent. This may be done by means of the thermometer.

One of the features of Waran's improvement is the automatic stirring of the liquid which consists of an oil with good conductivity and known specific heat.

The Determination of Joule's Equivalent of Heat by Callendar and Barnes's Electrical Method

The principle of the experiment is to supply electrical energy to a wire surrounded by water, and to measure the heat developed in the water by noting the rise of temperature.

If we express the electrical energy in ergs or joules, we can thus deduce that required to generate one calorie.

Inside the glass tube, H (fig. 142), is fixed a helix of manganin wire of resistance about 9 ohms. The ends of the wire are joined to the terminals, C and C′.

Water from the tank, B, enters H by a side inlet tube at one end and after flowing round the wire comes out from a similar outlet tube at the other. Thermometers, which enter the ends of H as shown, penetrate the inflowing and outflowing water, and serve to record

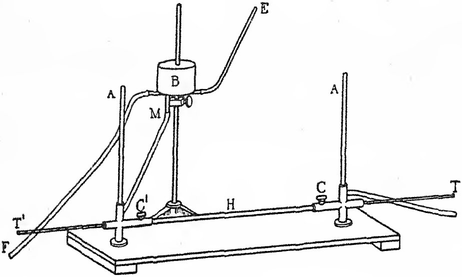


FIG. 142

the temperatures of the water before and after it has received heat from the wire.

The vertical tubes, AA, allow air bubbles which may flow in with water, to escape. The level of B is variable for the tank is movable up and down its stand.

The makers of the apparatus recommend that the rate of flow should lie between 55 c.c. and 65 c.c. per min., or about 1 c.c. per sec. B should be adjusted to produce this rate.

The tank consists of an inner and outer chamber.

Through the tube, E, water enters the outer chamber from the supply, and comes out through M to enter H.

The water flows over from the outer chamber to the inner and escapes to the drain by means of F. Thus water is supplied at a constant pressure to H, and the flow is made steady. The outflowing water may be collected in a measuring glass, or it may be collected in a weighed beaker and the mass per second flowing out deduced.

The apparatus gives best results for a current of approximately 2 amperes, so that a voltage of from 20 to 25 should be applied. To

obtain a steady source, ten or twelve accumulators should be used and connected through an adjustable resistance to C and C′.

In this case, with the rate of flow indicated, a difference of temperature of the ingoing and outgoing water of about 8° C. is maintained.

If possible, the mean temperature of the water should be that of the room, to avoid errors due to radiation.

When it is not possible to arrange this exactly, it is necessary to apply a correction. Let the temperature of the room round the apparatus be t_0, and the mean temperature of the water at entrance and exit be t.

Then the number of calories lost per second is

$$m\,(t - t_0) \times 0{\cdot}05,$$

where m is the outflow per second, and we may make the correction by adding $0{\cdot}05\,(t - t_0)$ to the difference of temperature recorded by T and T′. This correction has been obtained experimentally by the designer.

Measure the resistance of the spiral by means of a Post Office box.

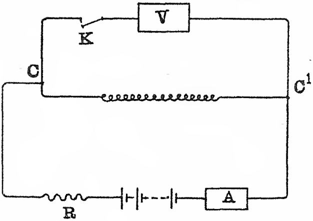


FIG. 143

Connect the resistance, R (fig. 143), in series with the supply of current and the ammeter, A, which has a range up to 3 amperes.

If a voltmeter is available there is no need to measure the resistance of the helix, for the difference of potential between C and C′ may be measured by connecting it through the key, K, in parallel as in the diagram.

Adjust the current to about 2 amperes, and if necessary make slight variations from this to adjust the temperatures so that heat losses are made as small as possible.

Let m denote the rate of outflow per second, I the current in amperes, E the voltage supplied between C and C′, and T the difference in temperature of the two thermometers, corrected if necessary according to the rule given above.

The supply of electrical energy is EI joules per second, and the heat developed is mT calories per second.

Thus, J, the heat equivalent, is given by

$$JmT = EI$$

or

$$J = \frac{EI}{mT} \text{ joules}$$

$$= \frac{EI}{mT} \times 10^7 \text{ ergs.}$$

If the resistance of the wire is measured, instead of EI we must write I^2R. The accuracy to be expected is between $\frac{1}{2}$ and 1 per cent.

The following is an example of an experiment carried out in the laboratory, and indicates the order of the quantities:

Temperature of room, 17·3° C.
Temperature of water at inlet, 16·8° C.
Temperature of water at exit, 25·35° C.
Mean temperature of water, 21·07° C.
Temperature difference of water in the two cases, 8·55° C.
Radiation correction = 0·05 × (21·07 − 17·3) = 0·19° C.
Corrected temperature difference, 8·74° C.
Volume of water flowing out in two minutes, 115 c.c.

$$m = \frac{115}{120} \text{ gm. per sec.}$$

Current, 2 amperes. Resistance of wire, 8·92 ohms.

$$\therefore \quad J = \frac{4 \times 8{\cdot}92}{\frac{115}{120} \times 8{\cdot}74} = 4{\cdot}260 \text{ joules.}$$

An Experiment on the Exchange of Heat of a Body with the Enclosure

The loss or gain of heat of a body to or from its surroundings is under certain conditions described quantitatively by Newton's law. According to this law the rate of loss of heat from a body is proportional to the difference of temperature of the body and of the surroundings.

One of the conditions which must be satisfied, if this law is to be applied, is that the difference of temperature should be small. It is difficult to say definitely what is to be understood by a small difference in this case, but in practice it should certainly not exceed 20° C.

The law is often applied and appears to be fairly accurate if the enclosure consists of the walls and the various objects in the laboratory, all at some average temperature. Such conditions are, however, indefinite, and in any investigation in which the law is to be applied, a definite, constant-temperature enclosure should be provided, for example, by surrounding the space about a suspended calorimeter by

a large metal can which itself is maintained at a constant temperature. For laboratory exercises, the maintenance at constant temperature by elaborate means need not be undertaken since a metal can standing on the table of the laboratory will usually remain at an almost constant temperature. Nevertheless, this condition should be checked by suspending a thermometer so that its bulb is in contact with the can and by observing the readings from time to time.

A most important condition is that there should be no general movement of the surrounding air such as results from draughts. The only movement of air should be that which results from the body itself. This is convection arising from the difference of temperature of the body and the surroundings. It is by this process that the loss of heat chiefly occurs although there will be some loss by the process of radiation, and to a very small extent by conduction.

The total of these processes produces a loss of heat subject to Newton's law.

If the temperature difference is denoted by $\theta°$ and the heat of the body by Q calories, the law states that

$$\frac{dQ}{dt} = -K\theta.$$

The constant K depends on the extent and character of the surface of the body. It will depend on the degree of polish of the surface, and in any experiment the whole surface should be uniform in this respect. If the surface is highly polished the loss will be small, and in order to get effects suitable for some tests the surface should not have too high a reflecting power. It is usually sufficient to clean the surface, such as that of a copper calorimeter, by careful rubbing with a duster.

In special cases the surface is uniformly metal coated or varnished. More accurately, the law is according to the equation

$$\frac{dQ}{dt} = -K\theta^n,$$

where n is an index differing from unity.

In the first experiment on this law, suppose that a calorimeter contains water and that its total water equivalent including that of the water is W.

Let it be suspended in a constant-temperature enclosure and let θ denote the temperature difference. Thus, if T denote the temperature of the calorimeter and T_0 that of the enclosure,

$$T - T_0 = \theta.$$

Let the water be heated electrically by a coil of wire placed in it, and suppose that the potential difference between the ends of the wire is V, the current being A, measured in volts and amperes respectively.

It must be supposed that the water equivalent of the wire is included

in W, but this is as a rule small enough to be neglected in comparison with that of the water and calorimeter.

The rate of supply of energy to the heating coil is $\frac{VA}{4{\cdot}2}$ cal. per sec.

Thus
$$\frac{VA}{4{\cdot}2} = \frac{dQ}{dt} + K\theta,$$
and
$$Q = WT,$$
so that
$$\frac{dQ}{dt} = W\frac{dT}{dt} = W\frac{d\theta}{dt}.$$
Thus
$$W\frac{d\theta}{dt} = \frac{VA}{4{\cdot}2} - K\theta,$$
or
$$\frac{W}{K}\frac{d\theta}{dt} = \frac{VA}{4{\cdot}2K} - \theta.$$
For convenience, let
$$x = \frac{VA}{4{\cdot}2K} - \theta,$$
so that the equation takes the form
$$\frac{W}{K}\frac{dx}{dt} + x = 0.$$
The solution of this equation is
$$x = Be^{-Kt/W},$$
where B is a constant.

It is supposed that initially the calorimeter and the enclosure are at the same temperature, i.e. $\theta = 0$ when $t = 0$.

Thus $B = \frac{VA}{4{\cdot}2K}$, whence, giving x its value in terms of θ,
$$\theta = \frac{VA}{4{\cdot}2K}(1 - e^{-Kt/W}).$$
If at any time the electrical supply is cut off, the equation describing the temperature change becomes
$$W\frac{d\theta}{dt} = -K\theta,$$
or
$$\theta = Ce^{-Kt/W}.$$
Suppose that the heat supply is cut off at a time, t_1, when the value of θ is θ_1, then
$$\theta = \theta_1 e^{-K(t-t_1)/W}.$$
These two equations in θ form the basis of the calculations to be made in the experiment.

Let the current be switched on at a time recorded as zero. Take readings of the temperature until it has risen about 15° C. The current must be adjusted so that the rise is at a convenient rate not too rapid

and not too slow for the time to be spent over the experiment. For this purpose place a quantity of water in a similar calorimeter and connect the heating coil in series with a variable resistance, e.g. a carbon rheostat. Place the heating coil in the test calorimeter and

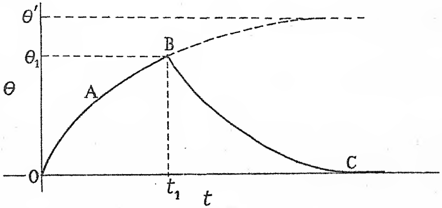


FIG. 144

examine the rate of rise of temperature, adjusting the resistance to a convenient value. It is advisable to place an ammeter in the circuit in order to measure the current. When a suitable value has been decided upon, reproduce the conditions in the circuit of the experiment.

The curve resulting on a graph of θ against t will resemble OAB (fig. 144).

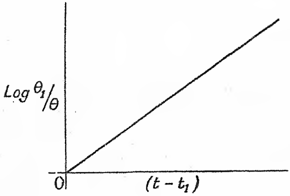


FIG. 145

If the heating were to continue sufficiently long the temperature would rise to θ', which according to the equation, when t is made very large, is

$$\theta' = \frac{VA}{4{\cdot}2K}.$$

After a rise of temperature of about 15° the point B is reached and the current switched off. The subsequent graph of θ against t is BC.

A linear graph of this portion of the curve is obtained by plotting $\log \frac{\theta_1}{\theta}$ against $(t - t_1)$, since the equation for θ in this region gives

$$\log \frac{\theta_1}{\theta} = \frac{K(t-t_1)}{W}.$$

The slope of this line is $\frac{K}{W}$.

The former relation applied to the temperature θ_1 gives

$$\theta_1 = \frac{VA}{4{\cdot}2K}(1 - e^{-Kt_1/W}).$$

The values of all the quantities on the right-hand side of this equation are known except K, for V and A are obtained from meters and $\frac{K}{W}$ is now found.

Thus, K is determined and from this W is found.

The value of W should also be determined directly from the calorimeter and contents. This gives a quantitative check on the law applied, and the linearity of the graph obtained is a check upon the validity of Newton's law, i.e. upon the value of unity for the index, n.

The conditions here are those usually prevailing in calorimetric experiments. The original statement of the conditions for the validity of Newton's law included the presence of a draught.

CHAPTER XI

REFLECTION OF LIGHT

The Sextant

THE instrument consists of a graduated arc, SS (fig. 146), with two radial arms, A and C.

A third arm, B, moves about an axis through one of its ends at right angles to the plane of SS. It is fitted with a clamp and tangent screw, so that it can be accurately adjusted, and carries a vernier at its end which moves over the scale of SS.

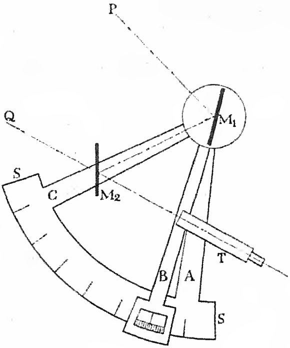


FIG. 146

A plane mirror, M_1, is attached to B, lying with its surface in the direction of B, and in a plane normal to that of the scale. The axis about which B turns lies in the surface of this mirror, which is called the index glass.

The second mirror, M_2, fixed to the arm, C, is called the horizon glass, and its plane must also be perpendicular to the scale. It consists of a plate of glass only one-half of which is silvered.

At T, on the arm, A, is fixed a telescope with its axis parallel to the plane of SS and passing through the centre of M_2.

Suppose the movable arm is turned so that the mirrors are parallel, and T directed towards a distant object, on which it is focused.

Only one image will be seen of the object, for the light from M_1 and

M_2 is brought into the telescope in the same direction, M_2T. The rays by the two reflections may not lie along the same line as those seen directly through M_2, but since they are parallel there is only one image formed by the telescope (see fig. 147).

If the index glass be rotated through an angle, A, by turning the arm, B, then the rays reflected by it into the telescope no longer come from the same object as that which supplies the ray, QM_2. Two superposed images are seen in the telescope, and the angle between PM_1 and QM_2 is 2A, since on turning a reflector through any angle a beam of light is rotated through double this angle. Thus every degree on the scale, SS, corresponds to a difference of direction of two degrees. The scale is marked to give directly the angle between the rays, PM_1

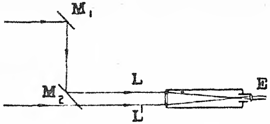


FIG. 147

and QM_2, i.e. to measure the angle subtended at the instrument by two distant objects.

In order to measure this angle one of the objects is observed directly through M_2, while the other is made to produce a superposed image on that of the first by rotating M_1 into the proper position. The angle through which the index glass is turned from the parallel position is then one-half of the angle subtended. Before making any measurement the instrument must be tested to see that it satisfies the following conditions:

(1) The plane of the index glass must be normal to the plane of the scale,
(2) the axis of the telescope must be parallel to the plane of the scale, and
(3) the index and horizon glasses should be parallel, and at the same time the vernier should read zero.

It will not be necessary, as a rule, to adjust for the first of these, but in order to see that the instrument is satisfactory in this respect look at the image of the scale in M_1. Since M_1 passes through the centre of this scale the latter and its image will appear to intersect at the edge of the mirror and, if the adjustment is satisfactory, to lie in the same plane.

Both M_1 and M_2 are attached to frames which can be turned through

small angles by means of screws. If necessary, M_1 may be adjusted until the test is satisfied.

The second condition is tested by observing two objects and causing them to coincide at the centre of the field of view. The axis of the telescope is a line joining the centre of the object glass to the centre of the eyepiece, or to the centre of the field of view. Perpendicular to this axis lies one of the cross-wires. Tilt the instrument until the images lie near the edge of the field, and note if they still coincide. Then tilt it so that they lie near the opposite edge. If coincidence persists the axis is correctly adjusted. If this is not the case the telescope can be adjusted by means of the screws.

Observe an object through the telescope and make it also appear in the field of view by reflection in M_1. If it is possible to bring about coincidence, the two mirrors are parallel, and on account of the first adjustment this will mean that the third condition is partly satisfied. By means of the screw attached to M_2 the two images can be made to coincide if they do not do so at first.

When these conditions are satisfactorily arranged it will probably happen that the pointer does not read zero when a distant object is viewed. To correct for this it is only necessary to note the zero error and apply it in all the observations. Coloured glasses are provided for diminishing the brightness of any object such as the sun. These can be made to intercept the light immediately before falling on the mirrors.

Experiment 1

Place two candles at as great a distance as is convenient, and measure the angle they subtend at the instrument.

Also find the angle by measuring the distance to each candle and their distance apart.

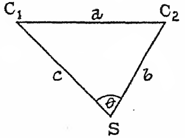


FIG. 148

Let the candles be C_1 and C_2, and let S denote the sextant.

The distances to be measured are a, b, and c, and if $s = \frac{1}{2}(a + b + c)$

$$\tan\frac{\theta}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Check the values of θ obtained by the two means.

Experiment 2

Let a trough of mercury or a carefully levelled mirror be placed so that the image of a lamp can be seen directly, and also by reflection, and measure the angle subtended at the sextant by the object and its image by holding the plane of the instrument vertically, at as great a distance from the lamp as possible.

The elevation of the lamp is half this angle.

The diagram (fig. 149) illustrates that the angle, CBD, is measured since the instrument is of necessity above the surface, AE.

Actually, we require the angle, LAL_1, but since we use a distant lamp, the two angles do not differ appreciably.

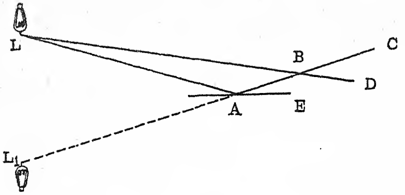


FIG. 149

Thus, the elevation may be measured by half the angle CBD.

Measure the horizontal distance between L and A, and deduce the height of L above the floor.

Check the result by actual measurement.

Measurement of the Angles of Crystals by Wollaston's Goniometer

The goniometer is a convenient instrument for measuring accurately the angles between the faces of small crystals which are too small to be examined by means of a spectrometer.

It consists of a scale, S, which may be rotated by the large milled head, B, and its position read off by means of a fixed vernier, V.

The crystal is fixed by soft wax to a plate, P, carried by an adjustable support, D, which may be rotated by the smaller head, A.

The edge of the crystal formed by the two faces between which it is desired to measure the angle is adjusted so that it lies parallel to the axis of the circle.

This adjustment is first made approximately by eye. In order to make the adjustment accurately, view the upper corner of a distant window in both faces. On rotation the images will move in a vertical plane if the edge is parallel to the axis. This may be tested by noting

if each image moves in a direction parallel to the edge of the window as seen directly by the eye.

Place the instrument so that the axis is parallel to a tall, distant window, and turn the screw-head, B, until the graduated circle comes against the stop.

The eye is placed close to the crystal so that an image can be seen by reflection in one of the two faces.

The axis is then rotated by the smaller screw-head, A, until the top of the window, as seen by reflection, appears to coincide with the bottom, as seen directly.

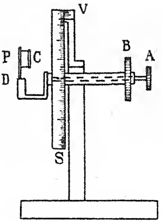


FIG. 150

When the adjustment has been made the angular position of the circle is noted.

By means of B, the circle is now turned until the top of the window, as seen by reflection in the other face, coincides with the bottom, as seen directly. The second face now occupies a position parallel to the first, and if θ is the angle between them, the circle has been rotated through its supplement. This angle $(180 - \theta)$ is read off from the circle and θ deduced.

It should be noted that the crystal must lie as close to the axis of the goniometer as possible, for the motion of the crystal from the first position with reflection in one face to the second position, with reflection in the adjacent face, consists both of translation and rotation, unless the crystal is on the axis. The amount of translation may be sufficient to cause an error in the angular measurement.

If the window is a long way away the error is only small.

The Determination of the Radii of Curvature of Spherical Mirrors

(A) *Concave Mirrors*

The most convenient method of determining the radius of curvature of a concave mirror is to place a pin-point in front of it and to locate

the position in which the image of the pin appears to coincide with the pin itself. The method of parallax is employed to ascertain when coincidence is attained.

The rays from the point of the pin falling on the mirror are reflected back from the surface along their original paths and must therefore strike it normally; consequently, the pin-point lies at the centre of curvature of the surface.

Another method consists in locating a series of pairs of conjugate points for the surface and using the formula:

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r}. \qquad \ldots(1)$$

A pin is set up as object and another pin adjusted until the image of the first coincides with it. We can then measure a pair of values, u and v.

Several pairs of values are obtained, and the above formula then gives r. Take the average of four or six observations.

(B) *Convex Mirrors*

Method 1

In the case of a convex mirror the image is virtual, and it is not convenient to locate it by a pin placed in a particular position since the image lies behind the mirror.

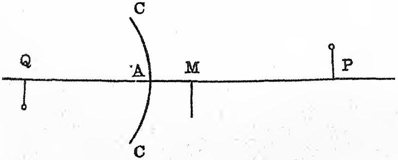


FIG. 151

A pin, P, is set up in front of the mirror, CC, and in between them is placed a plane mirror, M, so that the image of P in both can be observed. The mirror is adjusted until the two images coincide (fig. 151).

By the simple law of reflection in the plane mirror, M, we know that the image of P in M lies at the same distance from M as P does, but on the other side of it. If the image be Q, we can calculate the distance, AQ, for

$$AQ = MQ - MA = MP - MA.$$

Then $v = -AQ$, adopting the usual sign convention, viz. directions measured from A towards the object are positive, and in the opposite direction they are negative.

Thus, using the formula (1) we can again deduce r by measuring AP and AM.

Several pairs of values should be obtained, and they should give the same value of r.

Method 2

Another method is to set up a pin and form a real image of it by a convex lens. The image is located by placing a second pin, Q, so that there is no parallax between it and the image. Then place the convex surface between the lens and second pin, and move the surface until an image is formed coincident with the first pin, P. The rays after passing through the lens are directed to the point, Q, but strike the surface normally, and are therefore reflected back along their path. The radius of curvature of the convex surface is MQ (see fig. 152).

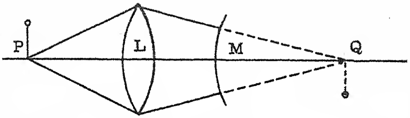


FIG. 152

Method 3. By means of a telescope, metre rule, and small millimetre scale

The diagrams (figs. 153, 154, 155) show the arrangement of apparatus. S is a small scale placed horizontally in contact with the surface of a convex mirror, along a line dividing it into two equal parts.

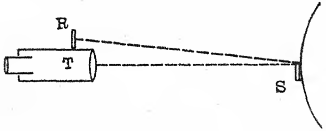


FIG. 153

The scale, RR′, is mounted at some convenient distance away, usually about 60 or 70 cm., and below its middle point is fixed a telescope, T, focused on the image of the scale, RR′, in the mirror.

The apparent length of the image is read off by means of SS′, which will be sufficiently well focused to make this possible.

The distance from the centre of the mirror, P, to the middle of RR′, is measured, say, d, and from these two measurements, together with the length of RR′, $2l$, it is possible to calculate the radius of curvature of the surface of the mirror.

Let rays from R and R′ strike the mirror at L and L′, and be reflected down the telescope at O. Then LL′ will denote the extent of the image, and the point, B, at which these two lines meet OP will

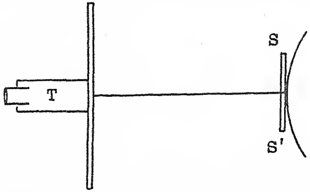


FIG. 154

be conjugate to O for reflection in the mirror. We may say that a point source at O will have a point image at B, so that if PB $= x$, we have

$$-\frac{1}{x}+\frac{1}{d}=-\frac{2}{r},$$

r denoting the numerical value of the radius of the mirror.

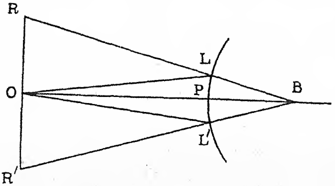


FIG. 155

But if we take LL′ as approximately straight, since the image is of small dimensions, and denoting LL′ by $2c$

$$\frac{l}{c}=\frac{d+x}{x}.$$

Hence
$$\frac{1}{x}=\frac{1}{d}\left(\frac{l}{c}-1\right);$$

so that
$$\frac{2}{r}=\frac{1}{d}\left(\frac{l}{c}-1\right)-\frac{1}{d},$$

or
$$r=\frac{2dc}{l-2c}.$$

The result may be verified by means of a spherometer.

The Focal Lines formed by a Concave Mirror

When light diverges from a point and falls on a mirror, it is supposed in the elementary theory that after reflection the rays pass through a single point or appear to proceed from a point. This is approximately true if the dimensions of the mirror are small compared with the distance from the source. A closer approximation to the truth is that the rays after reflection pass through two lines or appear to come from two lines, situated in parallel planes and perpendicular to one another.

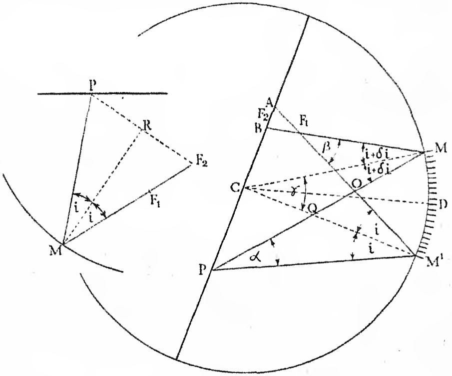


FIG. 156

These are the focal lines, and it will be shown how to calculate their positions theoretically, while it will be the object of experiment to verify the result obtained.

We shall take the case of light falling obliquely on a concave mirror.

If the point source lies on the axis of the mirror we have symmetry about this axis, and the two lines degenerate into a point or circle through which the rays pass.

In the diagram, MM′ denotes the concave mirror, and C its centre of curvature. The complete circle of which the section, MM′, forms a part is drawn for convenience.

P denotes the position of the point source of light, and the diameter is drawn through P.

The extreme rays, PM and PM′, are drawn and the reflected rays, MB, M′A, are drawn intersecting at F_1, and cutting off from the diameter the strip, AB.

The mirror is a part of a sphere, so that rays falling on the mirror from P, whether in the plane of the figure or not, will pass through AB. AB is thus a focal line, and is denoted by F_2.

If we imagine the figure to be rotated about the diameter, the rays, M′A and MB, will still intersect at a point, but now out of the plane of the diagram. For a small rotation the point of intersection would be on a line through F_1 normal to the figure.

Thus, a second focal line is through F_1 perpendicular to the plane of the figure.

Let the angle of incidence at the point M′ of the mirror be i, and let PD $= \rho$.

Denote the distances of the focal lines from D by ρ_1 and ρ_2, i.e. $DF_1 = \rho_1$, and $DF_2 = \rho_2$.

Let the radius of curvature, CD, be R.

Let the mirror subtend an angle, r, at C, α at P, and β at F_1. We shall regard its dimensions as small in comparison with R, ρ, and ρ_1, so that these three angles will be small and may be measured by drawing perpendiculars from M on to the corresponding lines, and dividing this perpendicular by the distance from the point concerned,

e.g.
$$\alpha = \frac{MM' \cos i}{\rho},$$

since the normal from M, on the line, PM′, makes an angle very nearly equal to i with MM′, and MM′ is small and is regarded as straight.

In the same way
$$\beta = \frac{MM' \cos i}{\rho_1},$$

(for $\cos i = \cos (i + \delta i)$ when δi is small),

while
$$r = \frac{MM'}{R}.$$

From the triangles OMF_1 and POM′
$$\angle OPM' + \angle OM'P = \angle OF_1M + \angle OMF_1.$$
$$\therefore \quad \alpha + 2i = 2(i + \delta i) + \beta,$$
or
$$2\delta i = \alpha - \beta;$$
and in the same way from triangles, CQM and PQM′,
$$\delta i = \alpha - r,$$
$$\therefore \quad 2r = \alpha + \beta,$$
or
$$\frac{2MM'}{R} = \cos i \,.\, MM' \left(\frac{1}{\rho} + \frac{1}{\rho_1}\right),$$
i.e.
$$\left(\frac{1}{\rho} + \frac{1}{\rho_1}\right) \cos i = \frac{2}{R}.$$

This is the formula concerning the position of the first focal line.

In order to determine ρ_2, we note that

$$\triangle \text{PM'A} = \triangle \text{PM'C} + \triangle \text{CM'A},$$

i.e. $$\tfrac{1}{2} \cdot \rho\rho_2 \sin 2i = \tfrac{1}{2}\rho \text{R} \sin i + \tfrac{1}{2}\text{R}\rho_2 \sin i,$$

which may be rewritten as

$$\frac{2\cos i}{\text{R}} = \frac{1}{\rho} + \frac{1}{\rho_2}.$$

So long as i remains constant, ρ_1 and ρ_2 vary with ρ just as u and v vary together in the formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$$

When i is made zero we obtain the usual formula,

$$\frac{1}{\rho} + \frac{1}{\rho_1} = \frac{2}{\text{R}},$$

for both ρ_1 and ρ_2, so that the two lines coincide.

To find ρ_1 and ρ_2 experimentally, use a small hole in a screen, with a lamp behind as a source of light.

Find R first by adjusting the mirror so that an image is formed on the screen at the side of the hole as in the experiment on concave mirrors. The distance from screen to mirror will then be R.

Allow the light to fall on the mirror at angles of about 20°, 30°, and 40°, and find the positions of the focal lines by means of a sheet or card or white paper held in a clamp.

Measure the distances from the mirror to these lines, thus obtaining ρ_1 and ρ_2.

In order to find i, it is convenient to mount M on a stand carrying a pointer moving over a scale of degrees.

Read off the mark against the pointer when the image of the hole is thrown back near the object, and turn from this position to any required incidence.

If no scale of this kind is provided, measure the distances, PF_1 and PF_2, say, a and b.

We then know three sides of each of the triangles, PMF_1 and PMB, and can determine i from the usual trigonometrical formula for the tangent of half the angle of a triangle.

Compare the values calculated in this way with those deduced from the formula derived from the above theoretical considerations.

Searle's Method of Determining Optical Constants

Accurate methods for the measurement of radii of curvature of polished surfaces, of focal lengths, and for the localization of the cardinal points, have been described by Dr. G. F. C. Searle. For the original accounts the reader is referred to the *Philosophical Magazine*,

Feb. 1911, pp. 218–24, or to the *Proceedings of the Optical Convention*, 1912, pp. 161–72.

The Determination of the Curvature of Spherical Surfaces

For this purpose a table is mounted on a tripod stand (fig. 157), two of the feet of which carry screws for levelling. The table is horizontal, and can rotate truly about a vertical axis. It carries a millimetre scale on the top which can be clamped in any position, and a carriage bearing the surface slides along it. The figure shows a lens system in the place where the carriage slides.

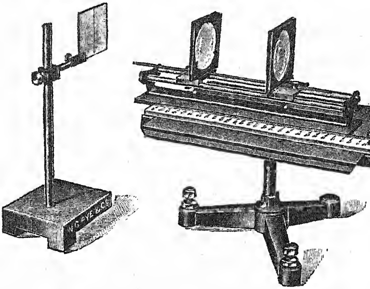


FIG. 157

It is essential that, as the carriage slides along the scale, the centre of curvature of the surface should move along a line which intersects the axis of rotation of the table.

In the figure the scale is shown, and also the wooden slider which acts as the carriage. On the slider is a metal mount which may be screwed to the carriage. This mount carries a horizontal spindle, each end of which is turned to a conical point. One of these ends is provided with a screw thread, so that it will fit a brass plate carrying three screws.

The screws fit into a second brass plate to which the lens or mirror to be examined is fixed by a little wax. The arrangement thus provides a convenient means of adjusting the surface.

The edge of the carrier is smooth and straight, so that it can slide along the scale, and an index mark serves to record the position.

In preparing the apparatus for use the spindle is first set accurately parallel to the edge of the carrier.

The tip of a pin is held just in contact with one of the conical points of the spindle, from which the brass plate is removed. The carrier is

taken from the table top and replaced so that the other conical point lies near to the pin.

If it is possible to bring this point and the pin into contact, the spindle is parallel to the edge of the board and scale.

The spindle is rotated until this adjustment is possible, and the mount then firmly clamped to the carrier by means of the screw. It is also necessary that the axis of the spindle should intersect the axis about which the table turns.

The scale is adjusted and the spindle moved along it until one point lies as nearly on the axis as can be judged by eye.

A microscope is then brought up and focused on the point, with the scale lying normally to the axis of the microscope. The table top is turned through 180°, so that if the axis of the table passes through

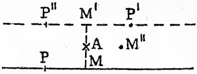


Fig. 158

A (fig. 158), and P denotes the first position of the spindle point, the second position will lie at P′, where $PM = M'P'$. The slider has to be moved through the distance, P′P″, in order to bring the point once more on to the cross-wires. If this distance is recorded and the slider moved back half this amount, the point will be at M′, the foot of the perpendicular from the axis to the line P′P″.

The microscope is then traversed so that it is focused on the point. Its axis is thus directed along MA. Turn the table through 90°, so that M′ moves to M″; and move the scale along a direction perpendicular to itself until the point once more comes on the microscope cross-wire. The point then lies on the axis of rotation of the table, and the axis of the spindle will intersect that of the table as it slides along the scale.

Mount the surface to be examined on the brass plate, and attach it to the spindle. If the surface is one of the faces of a lens, the side not under examination should be blacked by covering it with vaseline and lamp-black to absorb rays striking it.

Set up an object in some convenient position and view its image in the surface. If on rotating the spindle about its horizontal axis the image does not move, the centre of curvature lies on this axis, and the preliminary adjustments are complete. Since the image remains stationary the rotation of the surface serves to bring up fresh parts of the sphere exactly into the place occupied by the part moved away, i.e. the centre lies on the axis of rotation.

The object may conveniently be a fine, brightly illuminated line drawn on a piece of ground glass.

Set up this object again so that rays fall nearly normally on the surface, and examine the image formed by reflection in it by means of a microscope. By moving the carriage a position can be found in which a rotation of the table to and fro produces no displacement of the image. The position of the index mark is noted.

If the lens is moved until the vertical axis of the table is a tangent to the surface at some point, a rotation to and fro will not displace the part of the surface close to this point. Place a few grains of lycopodium powder at this point, and focus the microscope on one of these. The lens is moved and the focusing repeated until the movement ceases. The index reading is again noted. The difference of the two gives the radius of curvature.

Preliminary observations may be made by eye until the motion appears to cease in the two cases, and only the final exact adjustments need be made by the aid of the microscope.

CHAPTER XII

REFRACTION OF LIGHT

Introductory Remarks

RAYS of light in passing from one medium to another usually undergo a deviation from their course in the first medium.

On the wave theory this is accounted for by the fact that the light travels with different velocities in the two media. The refractive index with respect to two media is defined to be the velocity of light in the first divided by the velocity in the second, and we denote this by ${}_1\mu_2$.

When light travels from air to another medium, say glass or water, we shall write ${}_a\mu_g$ or ${}_a\mu_w$ unless there is no doubt that we are dealing with air and some other stated medium when we may write simply μ.

We have by definition,

$${}_1\mu_3 = {}_1\mu_2 \times {}_2\mu_3,$$

and

$${}_1\mu_3 = \frac{1}{{}_3\mu_1}.$$

In particular

$${}_g\mu_w = \frac{{}_g\mu_w}{{}_a\mu_g},$$

a result which will shortly be found useful.

In many experiments in this and following chapters it will be necessary to furnish a bright source of monochromatic light. A simple way to provide such a source is to use a Mecca burner, which consists of a Bunsen burner rather larger than the ordinary type of burner provided with a wide end over which is stretched a gauze with a wide mesh.

If a small bead of soda glass is placed on this gauze and the Bunsen made to roar as much as possible, a quite satisfactory yellow flame will be produced.

It is a great advantage that there is no crackling in the flame as in the case of the use of common salt, when small pieces of hot salt are thrown about falling on the bench and on the slits of spectrometers. In the case of the latter serious damage to their shape may result.

The slit may be illuminated directly, or better still, an image of the brightest part of the flame may be thrown on to it by a short focus convex lens.

The student will do well to pay attention to the small point of illumination of the slit. It is important always, but assumes greater importance in the case of experiments on interference and diffraction which will be described in the next chapter. The difficulty of discovering Newton's rings, interference fringes, and diffraction bands is

almost always due to a lack of care to obtain the best possible illumination of the slit or whatever may be used as a source of light.

The best source of yellow light is a sodium vapour electric lamp. This source is intense and very convenient.

Determination of the Refractive Index of a Plate by its Apparent Thickness

The apparatus necessary is a good travelling microscope and a plane glass plate with parallel sides (e.g. a cover glass).

Set the microscope with its axis vertical and focus it on the metal platform.

Note the reading on the vertical scale.

Insert the glass over the point on which the instrument was focused, again focus on the metal and observe the scale reading.

Usually the metal surface, though dark, is easy to observe, but if desired a thin sheet of white paper may be placed over it and the surface of the paper used instead of the metal surface.

Raise the microscope until the upper surface of the glass is sharply focused. There will usually be specks of dust on the surface to assist this setting of the microscope, but if any difficulty arises place a small drop of ink on the surface and focus the extreme edge of the drop. The drop need be no larger than that made by a sharply pointed pen.

Note the scale reading when this third adjustment is made. If we describe the scale readings by (1), (2), and (3) respectively, the difference between (1) and (3) gives the actual thickness of the glass and that between (2) and (3) the apparent thickness.

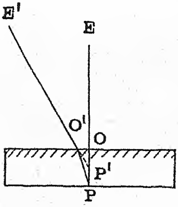


FIG. 159

Consider a point, P (fig. 159), situated at the lower surface of the glass and from which rays of light originate. Only those making small angles with the normal to the surface will enter an eye or microscope placed above P.

If one of these rays, PO′E′, makes an angle, i, with the normal in the air and an angle, r, in the glass,

$$\mu = \frac{\sin i}{\sin r} = \frac{OO'}{O'P'} \cdot \frac{O'P}{OO'} = \frac{O'P}{O'P'}.$$

For small values of i and r we may write:

$$O'P' = OP', \quad O'P = OP,$$

so that
$$\mu = \frac{OP}{OP^1} = \frac{\text{thickness of glass}}{\text{apparent thickness}},$$

for all rays inclined at such small angles appear to come from P′, so that OP′ is the apparent thickness.

Determination of the Refractive Index of Liquids by Total Reflection

When a ray of light passes from a medium of refractive index, μ_1, to another of refractive index, μ_2, with an angle of incidence, i, and of refraction, r, we have

$$_1\mu_2 = \frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r}.$$

In the case when $\frac{\mu_1}{\mu_2} > 1$, the value $\frac{\mu_1}{\mu_2} \sin i$ must not exceed unity. In the limiting case when

$$\sin i = \frac{\mu_2}{\mu_1}$$

the corresponding value of r is 90°, and i then measures the critical angle. For values of i greater than this critical value the surface acts as a perfect reflector.

If i is slowly increased a value is finally attained when the refracted ray suddenly disappears.

In this case if the second medium is air so that $\mu_2 = 1$, we have

$$\frac{1}{\mu_1} = \sin i.$$

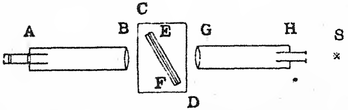


FIG. 160

This formula may be used to determine μ, by mounting a small rectangular trough, CD (fig. 160), with sides of plane glass on the table of a spectrometer, so that a parallel beam of light may be passed through it from a collimator, GH, and received in the telescope, AB.

The light is suddenly cut off when the air cell, EF, consisting of two plates of glass mounted parallel to one another and cemented together with a thin air space between, is turned so that the light falls on the air at the critical angle.

It will be noted that the critical angle is that for air and glass, but the apparatus is used to determine the refractive index of the liquid in the trough.

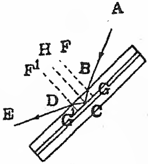


FIG. 161

Let the ray, ABCDE, be incident from the water on the glass and be totally reflected at the glass-air surface at the critical angle. Then if FBG and F^1DG^1 be the normals to the glass at B and D, we have

$$\frac{1}{{}_a\mu_g} = \sin \angle \text{BCH},$$

where CH is the normal at C and ${}_a\mu_g$ denotes the refractive index from air to glass.

Also $$ {}_w\mu_g = \frac{{}_a\mu_g}{{}_a\mu_w} = \frac{\sin \text{ABF}}{\sin \text{CBG}} = \frac{\sin \text{ABF}}{\sin \text{BCH}}.$$

$$\therefore \quad {}_a\mu_w = \frac{1}{\sin \text{ABF}}. \qquad \ldots(1)$$

We have thus to measure the angle ABF, and from it we deduce the value ${}_a\mu_w$, the refractive index from air to water.

As a source of light use a Bunsen flame containing sodium, and illuminate the slit of the collimator, which must be adjusted for parallel light (p. 276). Focus the telescope on the slit and turn EF (fig. 160), until the light just appears. EF is attached to a pointer which moves over a scale of degrees. Note the position of the pointer. Turn EF from this position into another where the image again disappears.

Let AA^1 denote the axis of the telescope and EF the first position. The second will be E^1F^1 if $\angle$ AOE $= \angle$ AOE^1. The second position is reached either by turning through the angle, EOE^1 or EOF^1, and $\angle EOF^1 = \pi - \angle EOE^1$.

When the apparatus is used there is no uncertainty concerning which angle is measured.

The angle we require is that between the ray and normal to EF or

E^1F^1, i.e. the angle between OA and the normal to EF or E^1F^1. This angle is half the angle between the two normals, and this is the same as half the angle E^1OF.

Thus we have to note the angle through which EF turns and take one-half of it to find the angle ABF of formula (1).

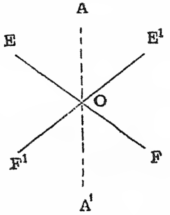


FIG. 162

To Find the Refractive Index of a Liquid, using a Lens and a Plane Mirror

In the experiment a plano-concave lens is formed of the liquid under examination (fig. 163), and its focal length found experimentally. The refractive index and radii of curvature of the two surfaces enter into the formula for the focal length, so that it is possible to deduce the index, μ, from a determination of the focal length, f, and the radius

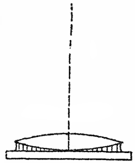


FIG. 163

of the curved surface of the lens. The liquid lens is made by placing a drop of the liquid on a plane mirror and laying a convex lens of from 10 to 20 cm. focal length on the drop. The liquid is squeezed into the space between the mirror and lens and we have a combination of two lenses—one of glass and the other of liquid, giving a combined focal length of, say, F.

If the convex lens has a focal length, f', then

$$\frac{1}{F} = \frac{1}{f'} + \frac{1}{f}.$$

We may therefore deduce f from a knowledge of F and f'. To determine f', place the lens horizontally on the mirror and adjust a pin, held in a stand above the lens, until the inverted image and object appear together and there is no parallax between them. The distance from lens to object gives f'. Now place the liquid and lens on the mirror and again find the position of coincidence of object and image. This gives F, so that we now have f.

For the purpose of substituting in the above formula, F and f' must be given their appropriate negative signs.

If r is the radius of curvature of the curved liquid surface, and μ the refractive index of the medium,

$$\frac{1}{f} = \frac{\mu - 1}{r}, \qquad \ldots(2)$$

since the second surface has zero curvature.

r can be found by measuring the radius of the surface of the lens in contact with the liquid by means of a spherometer, or by any of the methods described in the last chapter, so that μ can be calculated from the equation (2).

If it is preferred, the determination of r may be avoided.

If water be used and its index regarded as known and having the value, 1·33, we may make a water lens as above, and calculate its focal length, f''.

But
$$\frac{1}{f''} = \frac{0{\cdot}33}{r},$$

$$f = \frac{0{\cdot}33}{\mu - 1} \cdot f'',$$

or
$$\mu = 1 + \frac{0{\cdot}33 f''}{f},$$

where μ is the refractive index of a liquid other than water.

A convenient liquid to use for the experiment is aniline. Care must be taken to prevent it from getting at the back of the mirror, since it dissolves the varnish protecting the silvering.

Determination of the Refractive Index of a Lens by Boys' Method

This method of finding the refractive index of a lens consists in measuring its focal length and the radii of curvature of both surfaces. The value of μ may then be determined from the equation:

$$\frac{1}{f} = \left(\mu - 1\right)\left(\frac{1}{r} - \frac{1}{s}\right),$$

where f denotes the focal length of the lens and r and s the radii of curvature of its surfaces, with the usual convention regarding the signs of the quantities in the formula. In a convex lens let r_1 and r_2 denote

the radii of curvature numerically and F the numerical value of the focal length. Then

$$\frac{1}{F} = \left(\mu - 1\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Determine F by any of the methods described below, p. 297. In order to find r_1, set up a pin with its point on a level with the centre of the lens. Two images will be seen by reflection in the faces of the lens, one erect from the front surface which acts as a convex mirror and one inverted by the concave back surface. It is the latter which is required for the experiment. Move the pin until its inverted image is coincident with it, as judged by the method of parallax. When this is the case the rays must strike the back surface normally and be returned along their incident course.

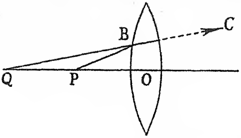


FIG. 164

If an eye be placed on the side of the lens remote from P (fig. 164), it will receive the transmitted part of the ray, PB, and will see the image of P in the direction, CBQ.

Q will thus be the image of P in the lens, and OP and OQ are conjugate distances. Thus, writing u = OP and v = OQ, we have

$$\frac{1}{OQ} - \frac{1}{OP} = -\frac{1}{F}. \qquad \ldots(3)$$

This enables us to determine OQ from OP and F.

Q is the centre of curvature of the right-hand surface of the lens since QBC is normal to this surface. Thus, OQ gives the value of the radius, say, r_1.

Turn the lens round and repeat the process to obtain r_2, the radius of curvature of the other face of the lens.

Let OP = d_1 in the first case, and let d_2 denote the corresponding distance in the second. Then from (3):

$$\frac{1}{r_1} = \frac{1}{d_1} - \frac{1}{F}, \quad \frac{1}{r_2} = \frac{1}{d_2} - \frac{1}{F}.$$

Hence $$\frac{1}{F} = \left(\mu - 1\right)\left(\frac{1}{d_1} + \frac{1}{d_2} - \frac{2}{F}\right),$$

so that we may calculate μ from the experimental determination of F, d_1, and d_2.

It is sometimes difficult to see the image by reflection at the back surface of the lens, but by holding it over the surface of mercury or floating it in the mercury, the image may be made to stand out and be easily located.

The Spectrometer

The spectrometer consists essentially of a telescope and collimator. The latter is a system of lenses mounted in a telescopic tube with an adjustable slit at one end, and it serves the purpose of rendering rays from the illuminated slit parallel on emergence. Both are mounted on a rigid stand, the collimator being fixed, and the telescope rigidly attached to an arm which rotates about the centre of the stand. Both are mounted horizontally with their axes in the same place.

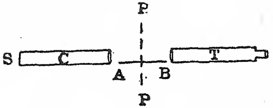


FIG. 165

The centre of the instrument is occupied by a table provided with screws for levelling. Underneath the table is a metal scale of degrees on which can be read off the positions of the telescope and of the table. In the diagram, T denotes the telescope, AB the table, and C is the collimator. PP denotes the axis of the instrument, and the table and telescope rotate round it. The telescope is fitted with a Ramsden eyepiece carrying cross-wires.

Before using the spectrometer for any experiment certain adjustments have to be made in addition to those made in the construction of the apparatus by the maker.

In the first place, the eyepiece of the telescope is adjusted so that the cross-wires are distinctly focused. The telescope should then be taken to an open window and focused on a distant object such as a distant telegraph post, care being taken that there is no parallax between the image and the cross-wires.

When this is done the telescope must not be readjusted again during an experiment or it will be necessary to repeat this process. It may happen that a second observer whose sight differs from that of the first is unable to focus the cross-wires easily. He may readjust the eyepiece provided that he does not alter any other part of the telescope, for then he is not causing it to be out of focus for parallel rays. He merely

gives himself convenience in focusing easily and leaves the cross-wires and image without parallax.

The telescope is now turned towards the collimator, the slit being made vertical and illuminated with monochromatic light, and the collimator is adjusted until a distinct image of the slit falls on the cross-wires.

The apparatus is now adjusted so that parallel rays pass from collimator to telescope.

Schuster's Method of Focusing a Spectrometer for Parallel Light

When a distant object is not available for the purpose of focusing the collimator and telescope for parallel light, Schuster's method may be employed.

Illuminate the slit of the collimator with yellow light and without any focusing, place the prism approximately in the position of minimum

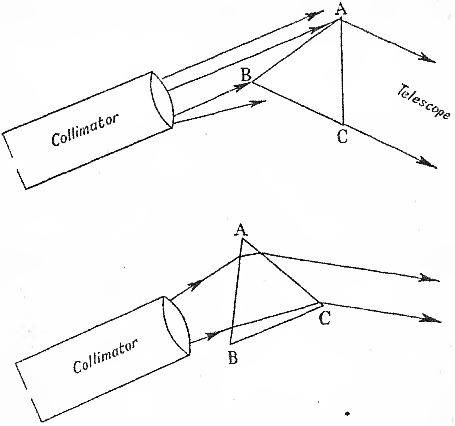


FIG. 166

deviation. Turn the prism slightly away from this position, bringing the refracting angle towards the telescope. Focus the telescope on the image as distinctly as possible, making the slight rotation of the telescope which may be necessary to keep the image in the field of view. Rotate the prism slightly to the other side of the minimum

position and focus the collimator until on looking into the telescope the image is again as distinct as possible. Repeat this process of alternately focusing the collimator and telescope until the rotations of the prism do not cause the image to go out of focus. When this is the case the rays entering and leaving the prism are parallel.

If the prism is first turned so that the refracting angle moves towards the collimator, then the first focusing must be made by means of the collimator. Should any mistake arise at this point the image will rapidly become more and more indistinct, and will call attention at once to the mistake. Usually only a few alternate focusings are necessary.

A simpler method for focusing for parallel light can be carried out by placing the prism on the table of the spectrometer and allowing light from the collimator to fall at grazing incidence on one of the faces of the prism, as in the upper part of fig. 166.

The narrow beam of rays which strikes the face, AB, is very nearly a parallel beam, so that by focusing the telescope on the slit of the collimator it is focused approximately for parallel light. The prism

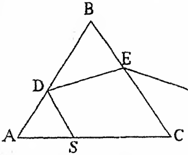


Fig. 167

should then be turned to receive rays from the collimator, as shown in the lower part of the figure, and the telescope turned to receive rays leaving the face, AC, at nearly grazing incidence. The collimator should be focused to bring the image of the slit on to the cross-wires of the telescope. This procedure should be repeated, if necessary, until repetition makes no improvement in the sharpness of the image.

It will be noted that the rays entering the prism in the first instance make approximately the critical angle with the normal to the face, AB. This suggests another way of making the adjustment of the telescope for parallel light.

If the ground surface, AC, of a prism be illuminated and a telescope placed to receive rays emerging from the face, BC, the field is divided into two parts separated by a sharp boundary. On one side the field is illuminated by light which has been totally reflected at the surface, AB, and on the other by light which has been partly reflected and partly refracted. Thus, the former is much brighter than the latter.

The sources of the rays are the points of the surface, AC, and the boundary is formed by rays such as SD, which fall on AB at the critical angle of incidence. All points, such as D, make a contribution to the boundary line so that it is formed by parallel rays leaving BC. If the telescope is sharply focused on this line, it is focused for parallel rays. When the telescope is adjusted in this way it is directed at the collimator which is then adjusted to bring the image of the slit in sharp focus on the cross-wires of the telescope.

When a prism is used in the spectrometer it is necessary to adjust it so that its refracting edge is vertical, i.e. parallel to the vertical slit of the collimator.

The screws on the table enable this to be done. They are shown at D, E, and F, placed at the corners of an equilateral triangle, and the

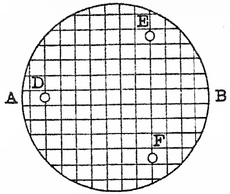


FIG. 168

table is usually ruled with lines as shown (fig. 168), to assist in setting the prism with one face normal to one of the sides of the triangle, for example, EF.

The table should be made as nearly horizontal as possible by the use of a spirit-level, and the prism then placed with one face perpendicular to EF.

Now suppose light from the collimator falls on this face and is reflected into the telescope. If this face is vertical the slit will now appear to lie in the same part of the field of view of the telescope as when it is seen directly.

The three screws should be adjusted to restore the image to its direct position if necessary.

Let the light be reflected into the telescope by the other face bounding the refracting edge. If further adjustment is necessary it must be done by the screw, D, for this will not disturb the previous adjustment, since it does not turn the face perpendicular to EF out of its vertical plane.

The two faces are now vertical and the instrument is adjusted.

It is sometimes necessary to arrange one face of a prism so that it lies normal to the collimator or telescope.

This may be done by turning the telescope from the position in which the slit is seen directly without the prism, through a right angle, so that the axes of the telescope and collimator are perpendicular to each other.

The prism is now placed on the table of the instrument and the table rotated until, by reflection in the face concerned, an image of the slit is thrown on the cross-wires. The face now lies at 45° to the axes of the collimator and telescope and a further rotation of 45° will bring it either perpendicular to the collimator or telescope.

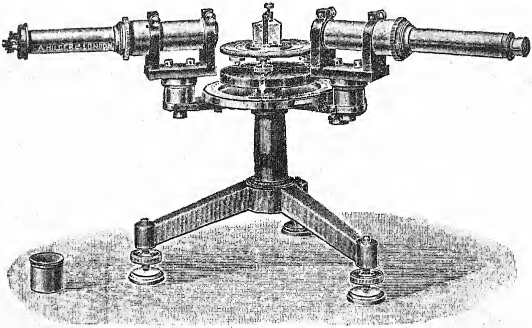


FIG. 169

In making measurements with the spectrometer the slit should, as a rule, be narrow, and the cross-wire should lie accurately down the centre of the image of the slit before the position of the telescope or table is noted on the metal scale.

Tangent screws help in the accurate final setting of the telescope and table, and verniers on this scale serve to measure the angles to an accuracy of one minute of arc.

A modern form of the apparatus is illustrated in fig. 169*a* (at p.1).

The Refractive Index of a Prism by the Method of Minimum Deviation

When a prism with a refracting angle, A, causes a minimum deviation, D, in light passing through it the refractive index is measured by the formula:

$$\mu = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2}A}.$$

Thus the determination of μ consists in measuring A and D. There are two methods in common use for measuring an angle of a prism.

One of these consists in allowing the light from a narrow slit to fall partly on one of the faces of the prism bounding the angle and partly on the other. An image of the slit can be seen in the telescope when the latter lies on either side of the prism.

Thus, in the case illustrated (fig. 170), the telescope will receive rays from the direction AO on one side and from AK on the other. The telescope has to be turned through the angle KAO in turning from one direction to the other.

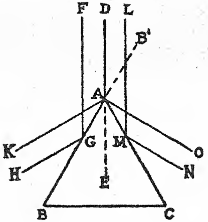


FIG. 170

Since DA and AO make equal angles with AC, we have

$$\angle \text{EAC} = \angle \text{CAO},$$

and similarly,

$$\angle \text{EAB} = \angle \text{BAK}.$$

Thus

$$\angle \text{OAK} = 2 \angle \text{BAC}.$$

To measure A we need only focus the image of the slit on the cross-wire in the two positions and halve the angle through which it is rotated.

Suppose AB is the face which reflects the light into the telescope along AK. If the prism is rotated until the face, AC, now lies parallel to AB the rays will once more be reflected in the direction, AK, but now by AC. Some of these will enter the telescope if there is a sufficiently broad pencil of them. But the prism has been rotated so that AC moves round through the angle CAB'.

Thus we measure, by means of the table, the angle $(180° - \text{A})$, and A can be deduced.

Measure the angle of the prism by both methods.

It now remains to find D.

Set the prism so that with A as refracting angle, light is refracted through it and received in T.

It will be found that as the prism is rotated the telescope has to be rotated to keep the image of the slit in the field of view. Rotate the prism until T is as close to the position directly opposite the collimator as possible, with the slit in the field of view. When this is the case the angle between the telescope and direct position is as small as possible.

Note the position of T and then remove the prism and observe the slit directly. Again note T's position, and hence find D.

It is, of course, necessary to adjust the spectrometer and edge of prism in the way described in the section on the adjustments of the spectrometer.

The final movements of the telescope or table must be made carefully with the tangent screws.

The Dispersive Power of a Prism

The dispersive power of a medium is measured by

$$\omega = \frac{\mu_B - \mu_R}{\mu - 1}.$$

μ_B and μ_R are the refractive indices for blue and red rays and μ has the value, $\frac{1}{2}(\mu_B + \mu_R)$.

The refractive indices may be found by the method of minimum deviation.

As a source of blue and red rays a discharge tube containing hydrogen may be used.

The tube should be held vertically, and the slit illuminated by it directly, or an image of a bright part of the tube thrown on to it by means of a short focus lens.

Three well-marked lines can readily be seen, one in the red, a second of blue-green colour, and a third in the violet. Use the first and third of these—they are known as the C and H_γ lines respectively, while the second is the F line.

The Refractive Index of a Liquid by Total Internal Reflection within a Glass Prism

A glass prism with one face unpolished is mounted on the table of a spectrometer. The table is levelled and the edge between polished surfaces set normal to its plane (see p. 279). The angle between these faces is measured in the usual way by allowing light from a wide collimator slit to fall on both faces and by measuring the angle between the two reflected beams.

Light from a sodium flame is allowed to fall on the unpolished surface, or an image of the flame is thrown on it by means of a lens to cover the whole matt surface.

It is first necessary to find the refractive index of the glass prism and then to coat one of the bright faces with a thin layer of the liquid.

Glycerine is a very convenient substance with which to carry out the experiment.

The theory of both parts of the experiment is the same.

Theory

ABC denotes the prism, of which AB is the unpolished side. This side acts as a collection of point sources of light of which S denotes one. Rays from it strike AC and are reflected and refracted there. Those like SD falling at an angle of incidence less than the critical angle are partially reflected and partially refracted so that the ray, EF, issuing from BC is less intense than the incident ray, SD.

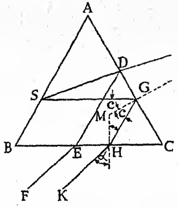


FIG. 171

Rays falling at angles greater than the critical angle suffer no refraction at AC, so that the emergent ray from BC is scarcely less bright than the incident ray. In any case there will be a marked difference between the former and latter group. The ray, SG, is drawn for critical incidence so that the direction of the emergent ray, HK, stands between those for the faint and bright rays.

We get a similar state of things for any other point in AB, and for each point the rays striking AC at critical incidence give rise to rays parallel to HK on emergence.

Such a group of rays is brought to a focus in the focal plane of a telescope placed to receive them.

Similarly, a group of rays, parallel to EF, will correspond to all rays falling on AC parallel to SD, and these will fall on the telescope in a direction different from that of HK, and will form a line in the focal plane not coincident with the former. This is true for all the directions of rays from BC. We shall thus have a multitude of parallel lines in the field of view divided into two groups of different intensity by the critical direction, HK. The effect will produce a field sharply divided into bright and dark halves by the direction, HK, making α with the normal to BC.

If the telescope is turned to face AC the field will be similarly illuminated on account of the reflections that have taken place on BC. The issuing critical rays will make an angle, α, with the normal to AC.

If the cross-wires of the telescope are set on the dividing line of the field when it is directed towards BC, and again when towards CA, we can deduce the value of α by observing the angle through which the telescope has been turned, since the telescope is rotated through

$$180^\circ - C + 2\alpha = \theta \text{ (say)}.$$

Thus
$$\alpha = \frac{\theta + C - 180^\circ}{2},$$

where it is assumed that the telescope is moved from side, BC, towards CA in a counter-clockwise direction, as seen in the diagram.

From C and α we can calculate μ from the formula given below.

Suppose that AC is coated with a substance of refractive index, μ_1, and that the index of the prism is μ.

We have, on referring to the diagram,

$$\mu_1 = \mu \sin c,$$
$$\beta + c = C, \text{ where } \beta = \angle \text{MHG}$$

since the points C, G, M, H, are concyclic,

$$\sin \alpha = \mu \sin \beta,$$
$$\mu_1 = \mu \sin (C - \beta) = \mu \sin C (1 - \sin^2 \beta)^{\frac{1}{2}} - \cos C \sin \alpha$$
$$= \sin C (\mu^2 - \sin^2 \alpha)^{\frac{1}{2}} - \cos C \sin \alpha.$$

When $\mu_1 = 1$, i.e. when there is no layer on face, AC, we have

$$\mu^2 = \left(\frac{1 + \sin \alpha \cos C}{\sin C}\right)^2 + \sin^2 \alpha.$$

This is the formula from which μ may be calculated.

Fig. 172 shows the state of affairs when the critical angle is large. This will be the case when the media on either side of AC are nearly of the same optical density.

If one side were coated it would be necessary to clean that side and coat the other when the telescope is turned. This would make it difficult to keep the prism fixed and would be inconvenient.

It is therefore best to use an eyepiece, such as the Gauss eyepiece with cross-wires that can be illuminated.

The light from these may then be reflected in the face opposite the telescope and the image made to coincide with the object. When this is the case the telescope stands perpendicularly to the face, and the angle between this direction and that in which the division of the field is viewed measures α.

After finding μ for the glass, place a few drops of the liquid on one of the polished faces and press over it a thin plate. This ensures that the face is covered with liquid.

It is better for the sake of definition of the two halves of the field

to allow light to fall at grazing incidence on the prism surface, say, AC. Then the rays entering the prism make angles less than c with the normal so that the field is now only half illuminated and the edge corresponds to the direction, HK. AB should be kept dark by covering with a sheet of dark paper.

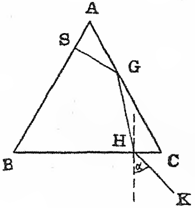


FIG. 172

If the light is incident externally on the liquid film, it must enter by the edge, AD (fig. 173), any ray such as P would not reach AC at grazing incidence.

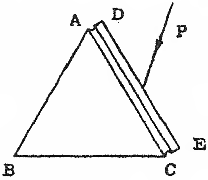


FIG. 173

Calibration of a Spectroscope

The spectroscope already described had only one prism, but it is an advantage sometimes to use two mounted together on the spectrometer table. The experiment may be carried out with one only, but a wider spectrum is obtained with two on account of the greater dispersion.

If the prisms, P_1 and P_2, are used to produce a series of spectral lines due to some source, S_1, the telescope, T_1, will have a definite position on its scale corresponding to each particular wave-length. To calibrate a telescope is to find the values of the wave-lengths corresponding to the different parts of the scale. If a curve is drawn whose co-ordinates are the wave-lengths and corresponding scale readings, such a curve is a 'calibration curve', and may be used to determine any wave-length from the division of the telescope scale at which it is seen.

First make the usual adjustments for parallel light. Place one prism, P_1, in the position corresponding to minimum deviation and then put in P_2 also in the position of minimum deviation.

Examine the spectrum of a Bunsen flame containing sodium light. Fix the telescope so that the sodium line lies on the cross-wire and note the position on the metal scale of the spectrometer.

A graduated scale is provided consisting of close rulings cut on an opaque screen so that the lines are transparent.

It is fitted at the end of a second collimator, C_2, which is adjusted to direct rays down the telescope after reflection on one face of P_2.

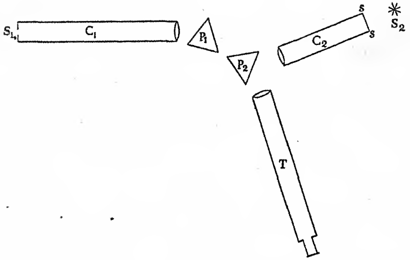


Fig. 174

The telescope is already set for parallel rays so that by adjusting C_2 so that an image of the scale appears at the cross-wires, it is ensured that parallel rays emerge from C_2.

The position of the sodium line is taken as a point of reference and scale divisions are read so many to the right or left. If the scale is moved accidentally, no error is then caused and the calibration curve will still be of use.

In a good spectroscope the sodium lines will be separated and will appear as two very close together. Use one of these as the reference line.

Measure the scale positions of a series of lines of known wave-length extending over the visible spectrum.

Use only sharply defined lines.

The table gives the wave-lengths of lines which may be produced conveniently in the laboratory.

In order to produce the lines from the metals use a spark between poles made of these metals connecting them to opposite terminals of a Leyden jar charged by an induction coil.

The salts are volatilized in a Bunsen flame as with sodium. Examine the inner cone of a Bunsen flame; it is due to carbon monoxide, and contains several series of lines. The spectra of light from discharge tubes containing various gases should be examined as well.

Lines from a neon lamp should be examined also. The field will be observed to contain many lines, the yellow line of wave-length 5853 units is a bright line and it is a good exercise to determine from the curve the wave-lengths of the other lines, afterwards comparing with standard tables.

Plot the curve on a large scale on squared paper.

The Auto-Collimating Spectrometer

In this instrument the telescope acts also as the collimator. The apparatus is almost identical with the ordinary spectrometer, except that there is no separate collimator and the telescope is modified.

TABLE OF WAVE-LENGTHS

Salt or metal	Description of Line	Wave-length (tenth metres)
Lithium Chloride or any salt of Lithium	Red	6708
Any Salt of Sodium	Double Yellow	5890 5896
Salt of Potassium	Red Extreme Violet	7668 4044 4047
Strontium Chloride	Blue (Not to be confused with the bands)	4607
Thallium Chloride	Green	5351
Helium	Red Yellow Violet	6678 5876 4471
Hydrogen	Red (C) Blue-Green (F) Violet (H_γ)	6563 4861 4340

On looking into the eyepiece of the instrument, the field is seen to be divided into two parts, the lower half is dark and the upper bright, but crossed by a pin which extends from the upper edge downwards across about half of the bright part of the field.

Near the eyepiece end the tube of the telescope is provided with an opening by means of which a slit just within may be illuminated.

Just below the dividing line of the two halves of the field of view a small right-angled prism is placed which deviates the light from the slit down the centre of the telescope tube. Its position is denoted by the dotted rectangle, P, in fig. 175.

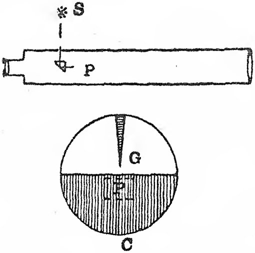


FIG. 175

If a polished surface, such as the face of a prism, is placed in front of the object glass and set at right angles to the axis of the telescope we may, by focusing, make the rays leaving the object glass parallel, so that they will be returned by reflection at the face of the prism and come to a focus in the focal plane of the telescope.

Focus the eyepiece carefully on the pin. Illuminate the slit by means of a small electric lamp—a four-volt lamp, supplied by two accumulators, is convenient for the purpose. The slit will be observed to be of the shape of an inverted T. The image must be brought so that the horizontal bar lies along the edge bounding the two halves of the field, when the end of the vertical bar will coincide with the point of the pin. In order to obtain this result the face of the exterior prism must be accurately normal to the emergent rays. The table is provided with three screws in order to level the prism, and the process described on p. 279 must be followed.

The positions of the slit and pin are such that when the image of the slit lies at the same distance from the object glass as the pin, the latter is at the principal focus.

The prism is slightly tilted to throw the image of the slit on to the upper half of the field of view.

To Find the Refractive Index of glass by means of the Autocollimating Spectrometer

Place the prism on the table and adjust the faces bounding the refracting angle as described above, so that the light reflected normally by both throws an image of the slit into the field of view just below the pin.

When this is so the faces are vertical and consequently so also is the edge of the prism.

Approximately monochromatic light may be obtained by placing a sheet of yellow glass between the bulb of the electric lamp and the slit.

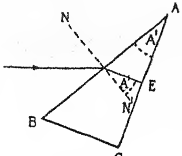


FIG. 176

Observe the position of the table when the face, AB, reflects the light normally. Rotate the table until the light falling on AB and refracted there strikes AC normally and is returned along its path. The table has been turned so that the face, AB, has turned from a direction normal to the rays from the telescope into the position at which refraction takes place, i.e. it has turned through the angle of incidence, i. The angle of refraction in this case is the same as the angle, A, and is so marked in the diagram.

This angle may be measured by setting the face, AB, normal to the rays and reading off the position of the table. Then by rotating the table until the rays strike AC normally, we turn the table through an angle $(180 - A)°$.

We may therefore calculate μ, since we know the angle of incidence and refraction in a particular case and

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin i}{\sin A}.$$

The Hilger Constant Deviation Spectrometer

This apparatus, illustrated in fig. 177 (and 177a at p.1), consists of a cast-iron stand with two arms at right angles on which are held rigidly the telescope and collimator.

The telescope may be fitted with a high-power eyepiece with adjustable cross-webs or with a shutter eyepiece which can be adjusted to cut

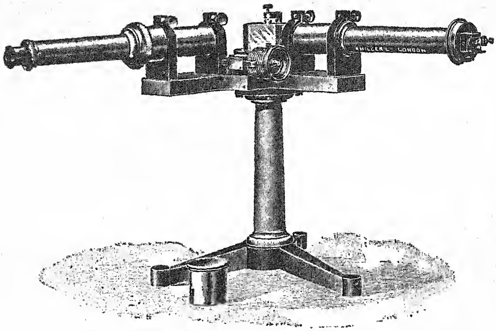


FIG. 177

out any part of the field except that under examination. An adjustable metal pointer is provided which has a brightly polished fine point, and is illuminated by reflecting light from outside by means of the mirror

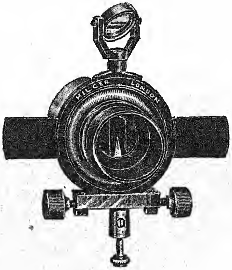


FIG. 178

shown in the figure in position above the eyepiece. Thus any point in the field may be taken as a reference point by setting the pointer upon it (fig. 178 and 178*a*).

In addition, light filters provided for the purpose may be used which impart any desired colour to the bright point. This adds to comfort in reading and consequently to accuracy.

The vertical collimator slit may also be reduced in length by means of a cross horizontal slit, so that a small rectangular source is obtained.

The principle of the apparatus is based on the constant deviation prism which is illustrated in fig. 179. The faces particularly concerned

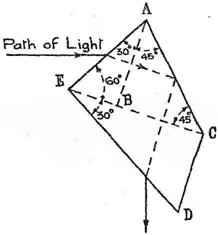


FIG. 179

in the deviation of the ray are inclined at the angles marked in the figure, and total reflection occurs at the face AC.

The prism is mounted on a turntable in a position marked for it. ✓ The mean deviation of the rays is a right angle, and in order to pass

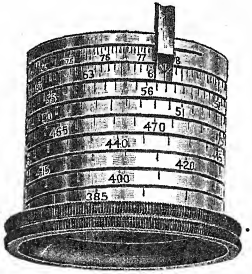


FIG. 180

through the spectrum the table is turned by means of a screw to which a drum is attached provided with a milled head (fig. 180). On the drum is a scale so that the wave-length of any line under observation

and appearing in the field of view of the telescope may be read off directly.

Before taking any observations of wave-length it is necessary to adjust the prism accurately so that the correct wave-length is indicated when the corresponding line appears at the eyepiece indicator.

To make this adjustment, illuminate the slit with light of a standard wave-length, set the drum so that the appropriate wave-length is indicated at the index of the drum, and adjust the prism so that the line appears under the eyepiece index.

Clamp the prism in position with the screw provided.

Other wave-lengths of light illuminating the slit may then be determined by rotating the prism by means of the drum until the line

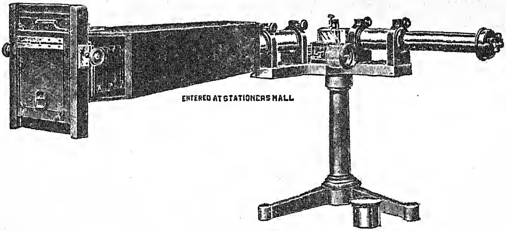


FIG. 181

appears at the eyepiece index, and reading off the number against the drum index.

Fig. 179 indicates the course of a ray in the prism.

In the apparatus designed for use with certain accessories—the Lummer-Gehrcke Parallel Plate, the Fabry-Perot Etalon, or the Michelson Echelon Grating, the collimator arm is of greater length than that illustrated in the figure to permit of interposing the accessory between the prism and collimator.

The method of fixing the collimator and telescope and of obtaining different parts of the spectrum by rotating the prism is very convenient and accurate; moreover, the drum can be rotated while looking through the eyepiece, and one is saved the inconvenience of moving round with a rotating telescope.

A suitable standardizing wave-length is the red line of the helium spectrum, which has a wave-length of 6678·1 Ångström units (1 Ång. unit $= 10^{-10}$ m.) and is the shorter of the two red helium lines.

The sodium lines may also be used. These are separated by the prism and have wave-lengths 5890·2 and 5896·2 Ångström units respectively. It is a good plan to set the instrument on one of these lines and check the setting by turning to the other and noting if the reading gives the correct wave-length.

The slit may be illuminated by throwing an image of the source on it. In the case of the sodium lines the source is obtained in the usual way, and for a helium line throw the image of the bright part of a helium discharge tube on the slit.

A protective metal cover for the prism table is provided.

When a photograph of the spectrum is required the eyepiece is removed and replaced by a camera with a lens of suitable focus. This is shown in fig. 181 (and 181*a* at p. 1). It is capable of adjustment by tilting so that the whole spectrum can be photographed and a vertical displacement enables the same plate to be used, giving photographs one above the other. There is, of course, also a shutter for exposure.

In addition to the usual vertical slit, there is a horizontal adjustable slit attached to a hinge so that it can be swung out of the way or in a position covering the vertical one.

Thus we have crossed slits and a small rectangular source can be obtained by closing down both the slits as much as is required.

The Pulfrich Refractometer

This apparatus is shown in fig. 182. It is designed to measure refractive indices of solids and liquids to an accuracy of about $\frac{1}{10}$ per cent.

The principal part of the apparatus consists of a prism having two plane polished faces at right angles to each other. One of these is horizontal and the other vertical. On the horizontal face is placed the substance whose refractive index is required. If this substance is a solid it must have two faces cut perpendicularly to one another, both of which are cut accurately plane, so that one may rest on the horizontal surface of the prism and the other stand vertically. Optical contact is brought about by placing a few drops of a liquid on the horizontal surface of the prism, which has a refractive index higher than that of the solid to be experimented upon, and standing the solid on it. The makers recommend monobromonaphthalene for this purpose. In the case of a liquid, it is contained in a glass cell cemented to the prism.

Light is directed into the liquid in a direction almost parallel to the horizontal surface so that light entering the prism makes the critical angle, c, with the normal.

Let it emerge from the prism at an angle, i (fig. 183). Suppose that the refractive index of the substance to be examined is μ and of the material of the prism, μ_0.

Then
$$\sin c = \frac{\mu}{\mu_0},$$
$$\frac{\sin i}{\sin\left(\frac{\pi}{2} - c\right)} = \mu_0,$$
i.e.
$$\cos c = \frac{\sin i}{\mu_0},$$
$$\sin^2 c + \cos^2 c = 1 = \frac{\mu^2}{\mu_0^{\,2}} + \frac{\sin^2 i}{\mu_0^{\,2}},$$
$$\mu = \sqrt{\mu_0^{\,2} - \sin^2 i}.$$

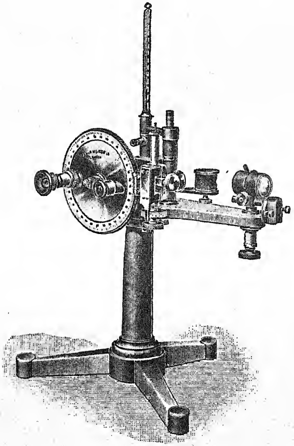


Fig. 182

The substance must have a refractive index less than that of the material of the prism, if it is to be examined by this method.

The apparatus measures the angle, i. For this purpose a telescope is attached to a circular scale and the rays are received by it. Since no rays within the prism make an angle greater than c with the normal on entrance to the prism, the angle, $\left(\frac{\pi}{2} - c\right)$, measures the minimum angle at which rays strike the vertical face. Corresponding to this, i measures the minimum inclination of the emergent rays to the

normal. Thus, in the telescope the rays emerging in this direction bound the field of view.

The apparatus is arranged so that the rays are deflected down the telescope, and when its cross-wires lie on the dark edge of the field the scale reads off the angle, i, to an accuracy of one minute of arc. μ_0, is an instrument constant and has the value 1·74.

The telescope is autocollimating, so that it is a simple matter to set it normally to the face of the cube (p. 288).

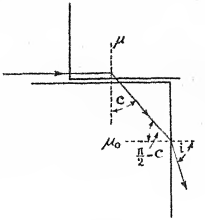


FIG. 183

The prism and specimen are surrounded by a metal water jacket and thermometers are provided for reading the temperature. A table is supplied with the instrument giving the value of μ corresponding to different values of i.

The Abbe Refractometer

This instrument, like the Pulfrich refractometer, makes use of a standard prism made of a medium of high refractive index, in this case dense flint glass.

It measures refractive indices of solids and liquids over the range 1·3 to 1·7 and is provided with a scale which can be read to 0·0001 and is accurate to 0·0002.

The solid should be prepared in the form of a small block with a flat polished surface. This surface is placed in contact with the face, AB, of the prism with a very thin film of a highly refractive liquid between them. The faces at the side of the block should be narrow and normal to the polished surface. A ray of light incident in the medium at grazing incidence with the standard prism enters the latter at the critical angle. It emerges from the face, AC, at an angle, i. This forms the boundary between the light and dark parts of the field of view seen in a telescope directed towards AC.

The cross-lines of the telescope are set upon the dark edge along the direction OE. The scale is calibrated to read refractive indices directly.

The most important use of the instrument is for the measurement of the refractive indices of liquids. A second prism of dense flint glass is hinged so that the long faces of the two prisms can be brought into close contact. The second prism is mounted below the first, and its face instead of being polished is left grey to prevent image formation. A drop or two of liquid is placed on the grey surface, which is then brought up to the polished surface, and the two prisms are clamped together. The function of the lower prism is to direct light at grazing incidence into the liquid film. It directs rays of light such as PQRSTU,

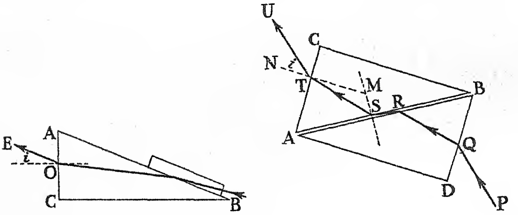


FIG. 183 (*a*) FIG. 183 (*b*)

so that RS in the liquid grazes the standard prism surface. The refractive index of this prism must be higher than that of any liquid to be examined.

In fig. 183 (*b*), SM is normal to AB, and TM to AC. The angle TSM is the critical angle, c, and

$$\sin c = \mu/\mu_0,$$

where μ is the refractive index of the liquid and μ_0 that of the material of the prism.

If $\angle$ TMS$=\alpha$ and $\angle$ NTU$=i$,

$$\sin i/\sin(\alpha+c)=\mu_0.$$

From these equations c could be eliminated and a relation between μ and i obtained. The instrument, as has been stated, is calibrated to read refractive indices directly.

The instrument can be used with daylight illumination. This is made possible by the introduction of a compensator consisting of two direct vision prisms. They can be rotated in opposite directions at equal rates, and form together a system of variable dispersion. In this way an amount of dispersion can be introduced to counteract that of the specimen under examination.

The compensator is mounted at the base of the telescope, and the prisms are rotated by a milled head. The compensator carries a scale

by means of which a measurement of the 'dispersion' of the material under examination may be obtained. The term 'dispersion' in this case means the difference in refractive index for the C and F lines of the hydrogen spectrum which have wave-lengths of 6563 Å and 4861 Å respectively. Tables are supplied from which this difference may be obtained from the compensator reading.

In the measurement of the refractive index, light is directed into the prisms and along the axis of the telescope by means of a mirror attached to the instrument. The eyepiece is focused on the cross-line and the reader on the scale.

The appearance in the field of view is usually a coloured patch bounded by a dark edge. The compensator should be adjusted by turning the milled head until the colour disappears as nearly as possible, leaving a double field half bright and half dark with a sharp dividing edge. The index arm should be moved until the cross-lines cut on the dividing edge and the refractive index read from the scale. It is a good plan to repeat the observation several times, including resetting of the compensator.

The instrument gives the refractive indices for sodium D light, and it is graduated to read accurately at 20° C. For the purpose of a laboratory exercise, temperature changes, which affect the prisms slightly, may be neglected. Water jackets are provided by means of which the prisms can be maintained at a required temperature. A thermometer is also provided to measure the temperature, and after it has remained steady for a few minutes the liquid may be introduced and its refractive index measured at the particular temperature.

It is an interesting exercise to measure the variation of refractive index in the case of some suitable liquid, such as a mixture of carbon bisulphide and alcohol, benzene, or toluene.

A book of instructions on the use of this instrument, provided by Messrs. Hilger and Watts Ltd., should be studied for further and more detailed information.

The Determination of the Focal Lengths of Thin Lenses

Students at this stage will be familiar with simple experiments for the determination of focal lengths of thin lenses. It is a good plan to revise the earlier work and to practise the use of the convention of signs, which it is more convenient to use in the more advanced parts of geometrical optics and which tends now to be generally adopted.

The positions of objects and of images are made independently, that is to say, there is one co-ordinate system for objects and another for images. These systems are described as the object space and the image space, although it is not to be understood that these represent mutually exclusive regions. The object co-ordinate system can consist of axes with one point as origin and the image co-ordinate system can consist

of other axes with another origin. In practice, the axis of the lens or other optical system is taken as the axis for locating the position of the object or image, i.e. both spaces have a common axis. The optical systems employed are almost always symmetrical about this axis, so that only one axis perpendicular to this axis of symmetry is required for locating positions not on the latter. It is usually necessary to consider only line objects with the foot on this axis and standing normally to it.

Although it is possible to choose any two points as origins, one for objects and one for the images, it is convenient to choose two special points. These are points on the axis of symmetry such that if an object lying normally to the axis is placed at one of them, an image of equal size lying on the same side of the axis is formed at the other. These points are called the principal points and are usually denoted by (H, H′). The planes through them perpendicular to the axis of symmetry are called principal planes. In simple optical systems these points coincide, e.g. with the pole of a mirror or a thin lens, for if an object be placed close to the central part of a thin convex lens, an image is formed coinciding with the object. There is a rule of signs for distances in association with these systems. The positive direction for images is that in which the rays of light are travelling, while the positive direction for objects is opposite to this. Thus the rule for the construction of diagrams which will be adopted in the description of experiments is that objects will be placed on the left; directions to the left are positive for the object space and directions to the right are positive in the image space.

FIG. 184

In fig. 184, H and H′ are the origins, O and O′ denote the feet of an object and of an image respectively. HO is a positive direction, as also is H′O′.

Directions perpendicular to the axis, HH′, are positive in the upward direction and negative downwards.

Another sign convention refers to the curvature of surfaces which make up the optical systems. This is quite independent of the convention just described which concerns the co-ordinates. It is contrary to that which is often taught in the early stages in optics and which was adopted in the chapter on reflection (pp. 260–1). The rule is that when rays of light are incident upon a convex surface, the radius of curvature of that surface is described as positive. If the light falls on a concave surface, the radius is said to be negative.

The formulae resulting from these conventions will only give correct results if the conventions are kept. Thus, if a concave mirror of radius 10 cm. comes into consideration, the value of the radius of curvature which must be introduced into the formulae is -10. This convention for curvature has been adopted so that the focal length of a converging lens turns out to be positive. This is not merely in agreement with the convention already used by opticians, but corresponds with what appears to be the obvious use of the word 'positive' in this connexion, for the term 'positive' must surely be given to the lens which converges rays of light and not to that which diverges them.

Instead of the letters u and v to denote object and image distances, l and l' will be used, and the familiar formulae will transform into the new ones by writing $u = +l$, $v = -l'$, since the old system makes use of the convention that the positive direction is against the incident light. Thus the lens formula,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

becomes

$$\frac{1}{l'} + \frac{1}{l} = -\frac{1}{f}.$$

Note the negative sign on the right-hand side. Since the old formula corresponds to the negative sign for f in the case of the converging lens, the new formula can be written in the form

$$\frac{1}{l'} + \frac{1}{l} = \frac{1}{\mathrm{F}},$$

where F $(= -f)$ is the focal length now described as positive for the converging lens. Those who prefer the notation commonly used in the case of simple optical systems can translate the formulae given here by writing $l = u$, $l' = -v$, and by changing the sign in the case of radii of curvature.

Thin lenses will be considered first, so that H and H′ coincide at the pole of the lens and l and l' are measured from the same origin but in opposite directions.

Converging Lens

Method 1

Support the lens vertically and place a vertical pin behind it so that its point lies on a level with the centre of the lens and is situated on its axis. The adjustments are made easier by placing a sheet of white paper behind the pin.

Set up another pin so that it coincides with the image. When this is the case the image and the second pin will not appear to move relatively to each other when the eye is moved horizontally from side to side.

Measure the object distance, l, from the lens and the distance between the object and image $(l + l')$. In fig. 185, $l = O_1P_1$, $l + l' = P_1P_2$.

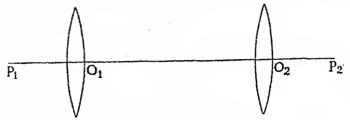


FIG. 185

The conditions of formation of a real image may be studied from this experiment by taking several positions of the object, i.e. for several values of l, and by measuring the resulting distances between object and image.

Write $(l + l') = Y$. From the formula for l and l', viz.

$$\frac{1}{l'} + \frac{1}{l} = \frac{1}{f}$$

it follows that $Y(l - f) = l^2$, f being now used with the sign convention adopted here.

This is the equation of a hyperbola with asymptotes:

$$l - f = 0,$$
$$Y = l + f.$$

The part of the curve of interest is that for which Y and l have positive values, and is that shown in fig. 186. The minimum value of Y occurs when $l = 2f$, $Y = 4f$.

The curve should be drawn from a number of values of Y and l obtained experimentally and the asymptotes drawn. The intercepts, OA, OB, and OC, each equal the focal length, and AD is twice this length. The co-ordinates of the minimum, M, have already been stated. It appears that the distance between the object and image cannot be less than $4f$.

It will be noted that for a particular value of Y, i.e. for a particular distance between the object and the image, there are two values of l. This means that, if a lens is placed at O_1 (fig. 186) and the image of P_1 is formed at P_2, the image will still be formed at P_2 when the lens is at another position, O_2. In the first case the image is magnified, and in the second case it is diminished.

Suppose that the distance through which the lens is moved, O_1O_2, is denoted by d and that the distance, P_1P_2, is D,

$$l = O_1P_1 = \tfrac{1}{2}(D - d),$$
$$l' = O_1P_2 = \tfrac{1}{2}(D + d).$$

The formula for l and l' gives

$$f = \frac{D^2 - d^2}{4d}.$$

In drawing the graph it is convenient to take pairs of values of l for each value of Y and from them f may be deduced by this formula.

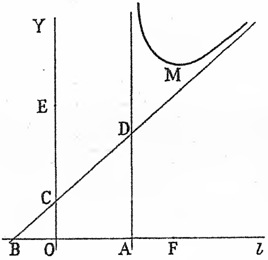


FIG. 186

Method 2

Put up a plane mirror immediately behind the lens and parallel to it. Then place a pin in front of the lens as in the first case and adjust it to make it coincide with the image formed by refraction in the lens and reflection in the mirror. The distance from the pin to the lens is

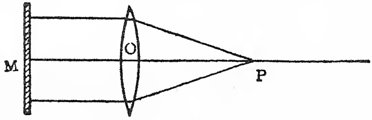


FIG. 187

equal to the focal length of the lens, for the rays are reflected back along their path and must therefore strike the mirror normally. They leave the lens as parallel rays, and must therefore originate from the principal focus.

Method 3

Set up a plane mirror, P^1Q^1, at some distance from the lens, and a pin on the other side of the lens. The pin should be mounted so that its centre is on a level with the centre of the lens and should be adjusted until the position of the image of the pin is made to coincide with the

pin. The way this is brought about may be seen from the diagram. The image formed by the lens evidently lies on the surface of the mirror.

If the distances be measured we have the positions of a pair of conjugate points.

Make several observations for different distances and calculate f.

In making experiments with a convex lens it is useful to know the focal length approximately before making an accurate determination of it. Sometimes time is wasted in trying to locate a real image when the object is so placed that a virtual one is formed. It should be noted that if the object is at a distance from a lens which is less than the focal length, the image is virtual.

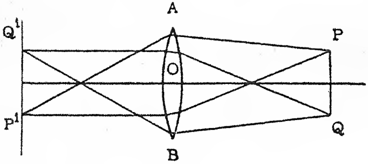


FIG. 188

Focus a distant object—a lamp or window, if not too close, is suitable—and measure the distance from the lens to a well-defined image thrown on to a sheet of paper.

Diverging Lens

Method 1

The focal length of a thin diverging lens is measured most accurately by the help of a thin converging lens used as an auxiliary. If a converging lens is available which is stronger than the diverging lens the two, when placed in contact, will form a converging system which may be regarded as a single converging lens. The focal length of the combination may be found by one of the methods described. Let F denote the focal length of the combination, f_1 that of the converging lens, and f_2 that of the diverging lens.

From the formula

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

the value of f_2 may be deduced from the values of F and f_1.

In order to discover whether the convergent and divergent lenses together form a convergent system, place them with their faces in contact and try to throw the image of a bright distant object on a screen. If it is possible to do this, the convergent lens is stronger than

the divergent. It may happen that the former is only a little stronger than the latter. In this case the screen must be held at a considerable distance from the combination and care must be taken not to overlook its formation on that account. If it is impossible to produce a convergent combination in this way, the following method may be adopted.

Method 2

Throw the image of a small source of light, such as that provided by a hole in a screen placed in front of a bright sodium flame, by means of a converging lens on to a screen and note its position.

Let the source be S_1 and let the image formed by the converging lens at O_1 be at S_2. Place the divergent lens at a point O_2 so that the image is displaced from S_2 to S_3, where it is again located on the screen.

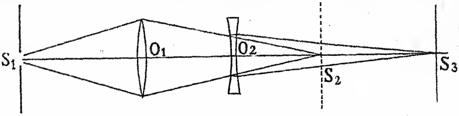


FIG. 189

The distances required are O_2S_2 and O_2S_3 for S_2 and S_3 are conjugate points for the diverging lens. The rays are directed to S_2 so that S_2 is a virtual object and in accordance with the notation described, $l = - O_2S_2$, $l' = O_2S_3$.

The formula
$$\frac{1}{l'} + \frac{1}{l} = \frac{1}{f}$$
then gives the value of f, which in this notation will be of negative sign.

The experiment should be repeated for different positions of O_2, while S_1 and O_1 remain fixed, and also for different positions of the convergent lens relative to S_1.

Method 3

Another method for the determination of the focal length of the diverging lens requires the above apparatus with a plane mirror in addition.

The diagram (fig. 190) illustrates the method. If the rays from the concave lens strike a plane mirror placed at any point, M, to the right of it at normal incidence, they are returned to form an image at the source, S_1. In these circumstances the rays from the convex lens are directed towards the principal focus, F, of the concave lens. The procedure is to focus the rays from the source by means of the lens, O_1, and to locate the image, F. The concave lens is then placed between

O_1 and the mirror and moved along the axis until an image of S_1 is thrown back as close as possible to S_1 itself. For this purpose it is convenient to place a source of light behind a small hole on a white screen, and to use the hole as a source so that the image may appear on the screen close beside the hole. The mirror, M, may be placed in contact with the concave lens and they may be held together by means of an elastic band. From the diagram it is clear that the focal length of the concave lens must be less than the distance, O_1F. Thus the return of rays in this way cannot be obtained for every position of the convex

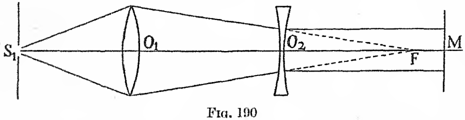


FIG. 190

lens with respect to the source. It is a matter for trial in the first instance, to ensure that the image distance, O_1F, is sufficiently great and the object distance, O_1S_1, should be made a little greater than the focal length of the convex lens in order that the image may be thrown on a screen placed at a sufficient distance to the right of O_1. The concave lens and mirror should then be moved from this screen towards O_1 until an image appears on the screen at S_1. This will then give the magnitude of the focal length required, and a series of measurements may be obtained by variations of the distance, O_1S_1, within certain limits.

Method 4

Set up a concave mirror behind the lens and a pin in front. An inverted image of the pin will be seen on looking through the lens,

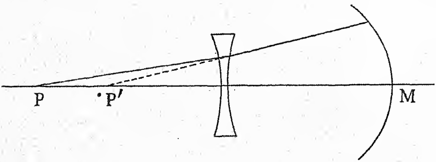


FIG. 191

and it may be made to coincide with the pin by suitably adjusting the mirror.

The diagram illustrates that this is brought about by the normal incidence of the rays emerging from the lens on the mirror.

The virtual image of P is thus at P′, the centre of curvature of the mirror.

Remove the lens after noting its position and the distance of P from it, and again adjust the pin until it coincides with its image formed by the mirror. We thus locate P′ and can measure l', the distance from the position of the lens.

On substitution in the formula:

$$\frac{1}{l'}+\frac{1}{l}=\frac{1}{f},$$

we calculate f.

Repeat this for several different cases.

Experiments on Complex Optical Systems

The mirrors and lenses which have been used in the foregoing experiments are examples of simple optical systems. These have been described as systems in which the principal planes coincide at the poles of the surfaces. Thus, in these cases the positions of the principal points are known and the determination of the focal length consists in locating the focal point and in measuring its distance from the pole. In complex optical systems the focal lengths are the distances measured from the appropriate principal points, the position of these being at first undetermined. Thus the determination of focal lengths in these cases requires a different approach to the problem. This will be illustrated in the following experiments in which methods are described for the determination of focal lengths without a knowledge of the positions of the principal points. The focal points are easily located and, from a knowledge of their positions and of the focal lengths, the principal points can be determined.

The optical systems which can be used in a laboratory in order to study the practical methods of determining their constants are most conveniently the thick lens and two separated thin lenses.

Method 1. The Determination of the Focal Length and the Principal Planes of a Thick Lens

In the diagram (fig. 192) ABDE represents a thick lens and FABF′ is its axis. F denotes the object principal focus, i.e. an object placed at F gives rise to emergent parallel rays. F′ is the image principal focus, i.e. parallel rays from a distant object give rise to an image at F′. These focal points can be used as origins from which to measure the positions of objects and images. Thus an object, O, may be described as lying at a distance x from F and the corresponding image, O′, lies at x' from F′. The relation between these co-ordinates is

$$xx'=ff'.$$

f and f' denote the focal lengths, i.e. the distances, HF and H'F', respectively, where H and H' are the principal points. In the present case their positions are not yet known. f and f' are equal when the media on both sides of the lens are the same.

Thus $$xx' = f^2.$$

The determination of f resolves itself into determining the positions of F and F' and of pairs of conjugate points.

The sign convention remains the same and in the diagram FO and F'O' are both positive.

In order to determine the positions of the focal points, rays from a distant object may be brought to a focus by exposing alternately the

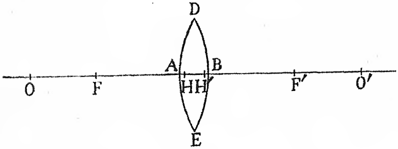


Fig. 192

two faces of the lens to them, but it is usually more convenient in the laboratory to place a plane mirror behind the lens, e.g. on the side of B and to adjust a pin-point on the opposite side so that its image coincides with it. This image is formed by the reflection of rays emerging from the lens which strike the mirror normally and are thus parallel on emergence. The pin-point is thus at the principal focus, F. By turning the lens round, the point F' is located in the same way. The distances of these points from A and B respectively should be recorded.

By finding the positions of pairs of conjugate points conveniently measured from A and B, the distances, FO and F'O', may be deduced, and hence f determined from the formula.

By measuring distances f from F and F', the points H and H' can be found and their positions with respect to the lens faces should be recorded.

Method 2. The Determination of the Focal Length by the Method of Magnification

An important property of the principal planes is that, if a ray is incident on the system towards a point on the principal plane of the object space, the emergent ray is directed from a point on the principal plane of the image space which lies on the same side of the axis and at the same distance from it.

It is thus possible by a geometrical construction to determine the position of an image corresponding to a given object. This is illustrated in the diagram (fig. 193).

The ray, BL, parallel to the axis, must pass on emergence through the image principal focus, F′, and must appear to emerge from L′ where HL = H′L′, where H and H′ are the principal points and HL and H′L′ are traces of the principal planes.

Again, the ray through F emerges parallel to the axis from M′, where HM=H′M′. Thus the point, B′, is determined and A′B′ can be drawn perpendicular to the axis. Thus, AB and A′B′ represent an object and its image.

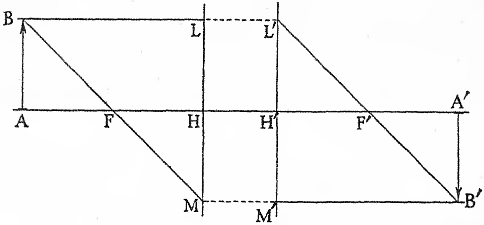


FIG. 193

In the experiment to be described the magnification, i.e. the ratio $\frac{A'B'}{AB}$, will be measured.

It is to be noted that in the diagram A′B′ and AB lie on opposite sides of the axis and, taking account of the sign convention, the magnification, m, is given by

$$m = -\frac{A'B'}{AB} = -\frac{HM}{AB} = -\frac{HF}{AF}.$$

In terms of the notation used here

$$m = \frac{f}{f-l}.$$

In the same way it may be shown that

$$m = \frac{f'-l'}{f'},$$

and in the present case $f = f'$.

These formulae for magnification suggest a method of carrying out the experiment. From the former

$$\frac{1}{m} = 1 - \frac{l}{f}.$$

Thus, if $\frac{1}{m}$ be plotted against l, a linear graph is obtained with a slope equal to $\frac{1}{f}$.

The determination of l requires a knowledge of the position of H, but it is not necessary to know the value of l to obtain a linear graph with the same slope, $\frac{1}{f}$.

Suppose that the object distance is measured from any fixed point which lies at an unknown distance, c, from H. Then, denoting the distance from the point by z, we have $l = z + c$, and the equation becomes

$$\frac{1}{m} = 1 - \frac{z + c}{f}.$$

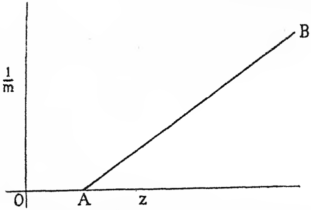


FIG. 194

The graph is represented by the line AB, the slope of which gives $\frac{1}{f}$ and the intercept OA $= f + c$.

It is thus possible to determine both f and c, which means that the focal length is determined, and also the distance of H from the fixed point chosen as the origin. A similar method is suggested by the relation between magnification and image distance. The relation is

$$m = 1 - \frac{l'}{f},$$

and since the position of the image principal point, H′, is unknown its position with respect to an arbitrarily chosen origin, from which the images can be measured, can be determined as before.

The experiment can be carried out conveniently by using a transparent scale as object and by illuminating it with monochromatic light. A similar glass scale may be used as a screen so that the apparent length of a certain number of scale divisions seen in the image can be read off. The adjustment is made by noting when there is no parallax

between the image and scale. For accurate work a low-power microscope may be used to measure the image.

It is also instructive to adjust the positions of object and image to obtain definite magnifications, e.g. of one or two. In the former case the distances of object and image from the principal points are $2f$ in each case.

If the screen is kept fixed and the object and lens moved until a magnification of 2 is obtained, it can be shown that the object is at a distance $\frac{3f}{2}$ from H and the image $3f$ from H′. The lens has thus moved through a distance f and this may be measured directly.

It will probably be found convenient, when the series of values of the magnification is being determined, to keep the lens system fixed and to move object and screen. In this case a suitable origin from which to measure the object or image distances is some mark on the holder. The graph of fig. 190, or a similar one for m and the image distance, can then be drawn and used to give the focal length and positions of the principal points.

Determination of the Principal Points by Rotation of a Lens

This experiment is for the purpose of illustrating and making use of the properties of the nodal points of an optical system. In any such system there are two nodal points. These lie on the axis and are such that if an incident ray is directed towards one of them the emergent ray is parallel to it and appears to come from the other. The points can be described as points of unit angular magnification, for the incident ray and its conjugate emergent ray make the same angles with the axis.

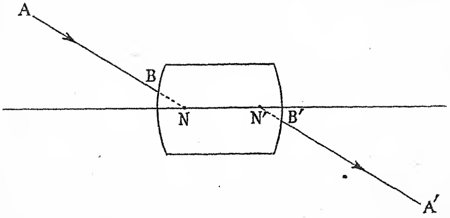


FIG. 195

Suppose that an optical system is mounted so that it can turn about a vertical axis passing through the object nodal point, N.

The emergent ray, A′B′, will pass through the image nodal point, N′, and be parallel to the incident ray, AB.

If the system is slightly rotated about N, the ray, AB, is still directed to the object nodal point. Consequently the emergent ray, although slightly displaced from its original position, will still emerge parallel to AB from the new position of N′. This is the property of which use is made in this determination of the principal points for, when the medium is the same on both sides of the lens, the nodal points and principal points are coincident.

Plano-Convex Lens

In this case any ray, R, striking the lens at H in the figure, where H is the point of intersection of the axis and curved surface, is not deviated, but gives rise to the parallel emergent ray, R′, for it is just as if refraction took place in a slab of glass bounded by the face, A, and the tangent at H. R′ appears to come from H′, so that H and H′ are the principal points. These may be located as above.

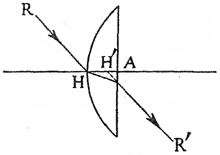


FIG. 196

We may also determine the distance AH′.

For AH′ is the apparent thickness of the widest portion of the lens.

$$\therefore \quad \mu = \frac{AH}{AH'} = \frac{t}{d}.$$

This gives d if μ and t are known.

It forms a useful exercise to determine H and H′ by the method described above, and then to deduce μ from the observations.

If $$HH' = \delta,$$

$$\mu = \frac{t}{t - \delta}.$$

Thick lens

If the lens is mounted on a stand which can be rotated about a vertical axis we may use the property of the nodal points to locate H and H′.

It is convenient to place the lens in a holder which can slide along a scale fixed to the rotating stand. The arrangement of the apparatus is illustrated in fig. 180.

A mirror is fixed on one side of the lens normally to the axis and does not rotate with the lens.

On the other side a pin is set up and moved until its image, by two refractions through the lens and a normal reflection at the mirror, coincides with the object.

The pin then lies at a principal focus.

The lens is moved along the scale and the pin adjusted, so that image and object coincide, until a position is found when slight rotations of the stand fail to cause displacement between the pin and its image. When this is the case the rotation takes place about a vertical axis through the nodal point nearer the pin.

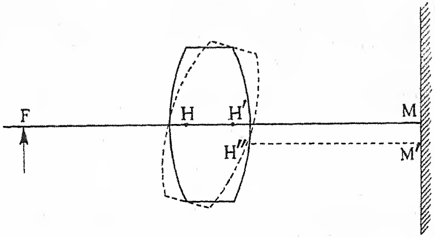


FIG. 197

The figure explains this, for if H′ denotes the other nodal point in the symmetrical position and H″ the position of this nodal point in a slightly displaced position, a ray from F falling on H first passes along HH′M and is reflected back along its path after incidence on the mirror at M.

When the lens is rotated into the position indicated by the dotted lines the ray, FH, is refracted and emerges from the lens parallel to its original direction, i.e. it is directed along H″M′, and is reflected back to H″, emerging once more along PF. The image of the point therefore keeps the same position.

The position of the axis thus fixes the nodal point, and the axis is usually clearly indicated on the stand. This finds the point, H, and, by turning the lens, H′ may be found in the same way.

The Measurement of the Focal Length of an Optical System by means of a Goniometer (Searle's Method). (*Proceedings of the Optical Convention*, 1912, p. 165)

A simple form of goniometer devised by Dr. G. F. C. Searle, in conjunction with Messrs. W. G. Pye and Co., provides a very instructive

method of determining the focal length of an optical system by means of the properties of the nodal points.

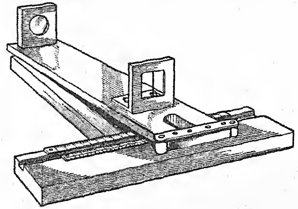


FIG. 198

The goniometer is illustrated in fig. 198, and consists of a wooden base provided at one end with a spherical pivot, consisting of a ball of phosphor-bronze, and carrying at the other a scale marked in millimetres.

A movable arm rotating about the pivot carries an achromatic lens of focal length about 35 cm., a vertical adjustable frame, across which a vertical wire is tightly stretched, and a fine horizontal wire passing across an opening which serves as a scale index.

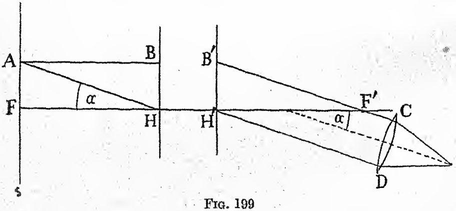


FIG. 199

The ball is adjusted at a distance of 40 cm. from the scale, which can be read to an accuracy of one-tenth of a millimetre, so that a rotation of the arm through one-seventieth of a degree is able to be measured. This is about $\frac{1}{4000}$ radian.

In fig. 199, let HB and H′B′ denote the principal planes of an optical system, the points, H and H′, denoting the first and second principal points.

In practice the medium is the same on both sides of the system, so

that the nodal points coincide with the principal points, HH′. Any ray incident towards H thus gives rise to a parallel emergent ray from H′. Thus, in fig. 199 the ray, AH, gives rise to H′D. (The rays do not necessarily pass actually through H and H′) since refraction may occur at lens surfaces upon which the rays are incident. The figure serves to show directions of rays. If the point, A, lies in the focal plane of the system, it gives rise to a pencil of rays which become parallel on emergence and, since the ray H′D of the emergent beam is parallel to AH, the direction of the emergent rays is that of AH. Thus the angles marked α in the figure are equal. If AF be measured the focal length can be determined by measuring α for

$$f = \mathrm{HF} = \mathrm{AF} \cot \alpha,$$

and for small angles

$$f = \mathrm{AF}/\alpha.$$

The goniometer is a very convenient apparatus with which to carry out this experiment, and in the figure the lens of this instrument is represented by CD.

The instrument is first adjusted so that the vertical wire lies in the focal plane of the goniometer lens. This may be done by adjusting the wire so that the image of a distant object seen through the lens falls upon the wire. An alternative method is to place a plane mirror behind the lens and to adjust the wire until there is no parallax between it and its image in the lens and mirror. The lens and wire are then kept in fixed positions on the arm of the goniometer.

The optical system is now brought up so that its axis lies collinear with that of the goniometer lens when this lens lies so that the horizontal wire index is central.

A glass scale is placed in the focal plane of the optical system and may be represented by AF. A denotes one of the marks on the scale. The scale may be adjusted in the focal plane by looking through the goniometer lens and the system, and by placing it so that its image is seen coincident with the vertical goniometer wire. The method of parallax is used to obtain an accurate adjustment. When this is made as closely as possible by means of the unaided eye a lens should be used as a magnifying glass to make it still more exact.

Note the reading on the glass scale when the apparatus is in the symmetrical position, so that the point corresponding to F is observed. Note also the indication of the goniometer scale.

Turn the goniometer through some convenient small angle and note the mark of the scale seen coincident with the vertical wire. Again note the goniometer reading. We thus have the values of AF and of the angle, α, and can consequently deduce F. By turning the system end for end, so that H′ lies to the right of H, we again determine the focal length.

We can thus locate the positions of H and H′ with respect to the outer surfaces of the system. These should be recorded.

In order to carry out this experiment it is convenient to employ two thin lenses situated at a known distance apart. These form an optical system of the type described, and having the same medium on each side.

Hence H and H′ are both nodal and principal points, and f and f' are equal.

Find the focal length and the position of the principal points for this case.

It is shown in treatises on Optics (see, for example, Flint's *Geometrical Optics*, p. 108) that for two lenses separated by a distance, d, the position of the first nodal point is at a distance from the lens on which light is incident and whose focal length is f_1, given by

$$\frac{f_1 d}{d - f_1 - f_2},$$

while the second nodal point is at a distance

$$\frac{f_2 d}{d - f_1 - f_2}$$

from the second lens, f_2, denoting its focal length, while the focal length of the system is

$$\frac{f_1 f_2}{f_1 + f_2 - d}.$$

Measure f_1, f_2, and d, and verify these results.

f_1 and f_2 may be readily measured by the goniometer by replacing the optical system by the lenses in turn.

A useful exercise can be performed by plotting $\frac{1}{F}$ against d in the case of two thin lenses. The formula for F shows that a linear graph results, since

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}.$$

The slope of the graph gives $\frac{1}{f_1 f_2}$ and the intercept on the axis of d gives $\frac{1}{f_1} + \frac{1}{f_2}$. Thus the focal lengths of the lenses can be obtained. It should be noted that the sign of the focal length of the combination depends on the distance apart of the lenses.

The Focal Length of a Microscope Objective

The goniometer provides a useful means of measuring a short focal length, such as that of a microscope objective.

The screen, AF, is replaced by a micrometer slide provided with a magnifying reading glass. The slide is provided with a small engraved scale with divisions at each tenth of a millimetre. The objective is placed to receive parallel rays emerging from the goniometer lens and focuses them on the scale. Thus an image of the goniometer vertical wire is received and is viewed by the magnifying glass.

The position is read for the symmetrical position and again after a slight displacement of the goniometer arm. The image is now focused at a new point corresponding to A (fig. 199). The distance between the two images is read off and, from a knowledge of the angle of rotation of the arm, the focal length is calculated as before.

The Determination of the Focal Length and Principal Points of an Optical System by means of the Revolving Table

The principle of the method is the same as that described on p. 309, for it depends on the same property of the nodal points. The revolving table affords, however, a much more convenient and accurate means of carrying out the experiment. (See figs. 157 and 200.)

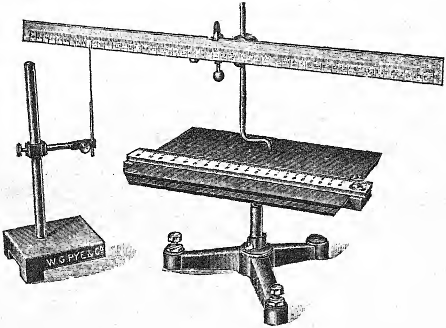


FIG. 200

Each nodal point is determined by the principle that when a small rotation is made about a vertical axis through the second nodal point, the image produced of a distant object is not displaced. Thus the position of the axis of rotation of the table locates the second nodal point.

The lens system is mounted on a slider with straight edges, and is placed on the table so that the edge slides along the scale. An index mark on the slider serves to locate positions with respect to the scale.

The design secures that there is no side shaking.

It is essential in this experiment as in the former (p. 309) to make the axis of the system intersect the axis of revolution.

In order to arrange this the image of a distant object is made to coincide with a pin held in a clamp and the system is moved on the table until, with this focusing exactly made, a slight rotation of the table causes no displacement between the image and pin.

Let the nodal points, which are in this experiment coincident with the principal points, be represented by HH′ (fig. 201). The line HH′ represents the axis of the system and O denotes a distant source of light producing an image, O′.

Let A denote the point of intersection of the axis of revolution and the horizontal plane. When AH′ is normal to the direction of the

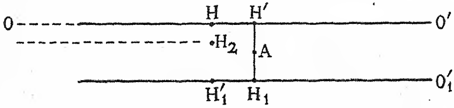


FIG. 201

light a small rotation does not displace H′ in a direction at right angles to the axis of the optical system.

In the rotation H moves to a point, H_2, but the object, O, is distant, so that a ray falling on H_2, as shown by the dotted line of the figure, is parallel to the direction OH. The emergent conjugate ray leaves H′ in the same direction and consequently the image, O′, is not displaced.

With this setting of the apparatus, which must be made by trial, the nodal point, H′, lies in a plane normal to the rays and containing the axis of revolution.

The table top is now turned through two right angles, and the lens system moved along the scale until the distant object is focused on a second pin as before.

In general, this pin will not coincide with the first, but will be displaced in a direction normal to the direction of the light. This is illustrated in the figure. H in the second case becomes the second nodal point.

If O_1' denotes the second pin, it is clear that the axis lies on a line drawn parallel to the light and passing midway between the pins.

The scale on the table top is unclamped and moved through half the distance between O′ and O_1', so as to carry the axis of the system into the correct position.

The apparatus is now once more carefully adjusted so that the image does not move for slight rotations.

The axis then passes through the nodal point. The system is then turned end for end and the second nodal point found.

In order to find the focal length readily, a clamp is fixed to the table which carries a scale (fig. 200).

The pin is left in position and the table turned until the scale just touches it, the scale reading being noted. The table is turned so that the other end of the scale just touches the pin and the scale reading is noted for this case also. The distance between the two marks on the scale is determined by subtraction; one-half of this gives the focal length.

A diagram is then made showing to a convenient scale the positions of the cardinal points with respect to the first and final surfaces of the optical system.

The apparatus may be employed to examine such a system as that described in the previous experiment with the goniometer. The apparatus is not very convenient for the measurement of short focal lengths. It is better to use the goniometer for this purpose.

When a distant object is not available in the above experiment, it is only necessary to place a plane mirror on the side of the lens system away from the pin and adjust the pin until image and object coincide.

The focal length of a thick lens, the radii of the surfaces being r and r', and the thickness t, is given by

$$f = \frac{\mu r r'}{(\mu - 1)\ \{(\mu - 1)\ t + \mu\ (r' - r)\}}.$$

Here r and r' must be given their proper signs.

It is instructive to determine the value of the refractive index of the glass forming a thick convex lens.

Determine r and r' by the revolving table method (p. 315), and f by the method just described, while t may be measured by means of callipers.

Solve the resulting equation for μ.

The distance between the principal points is given by

$$\frac{t\ (\mu - 1)\ (r' - r + t)}{\mu\ (r' - r) + (\mu - 1)\ t}.$$

The student should verify the two expressions given above.

From the positions of the principal points the value of μ can be again determined.

Spherical Aberration with a Thick Lens

In the simple theory of lenses it is assumed that all the rays starting out from a point on the axis are brought to a focus after refraction through the lens to another point on the axis.

This is only true for rays lying very close to the axis. A ray, PA,

(fig. 202), will be made to cross the axis at P′, and if PA_1 makes a small angle with the axis, P′ will be the focus for all such rays. But a ray, PA_2, will cut the axis at C_2, and PA_3 at C_3, after refraction, these points lying closer to the lens as the incident ray is more inclined to the axis.

The distances, $P'C_2$ and $P'C_3$, are called the longitudinal aberrations of the corresponding rays.

The distance, $P'C_2$, depends on the inclination of the emergent rays to the axis. If this distance is x and the tangent of the acute angle at C_2 is m, we may say that x is some function of m. We may write x in powers of m, or

$$x = a + bm + cm^2 + dm^3 + \ldots,$$

where a, b, c, d, etc., are constants.

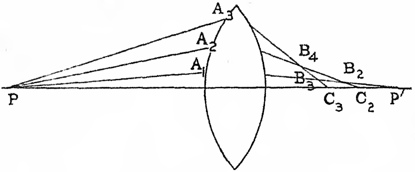


FIG. 202

Since the lens is symmetrical about the axis, $P'C_2$ is the same for positive and negative values of m.

Thus, if we write $-m$ instead of m in the above equation we have the same value of x, so that x depends only on even powers of m.

$$\therefore \quad x = a + cm^2 + em^4 + \ldots.$$

When the emergent ray has a very small value of m, the point, C_2, is at $P'C_2$, or x vanishes. Thus a vanishes and we have

$$x = cm^2 + em^4 + \ldots.$$

If m is not very big, m^4 and higher powers of m, will be very small, so that we may write

$$x = cm^2.$$

Let P′ be taken as origin and PP′ as axis of x.

Let y denote distances measured at right angles to PP′.

The line, B_4C_2, has an inclination to the x axis whose tangent is m, and it passes through the point, C_2, with co-ordinates $(cm^2, 0)$.

Thus its equation is

$$y = m\,(x - cm^2).$$

The emergent rays cross one another at points B_2, B_3, B_4, etc., and the points at which consecutive rays cross lie on a curve known as the caustic of the lens.

Suppose that to a ray consecutive to that through C_2 the corresponding tangent is m'.

Its equation will be

$$y = m'(x - cm'^2).$$

If we solve these two equations we can find the co-ordinates (x, y) of the point common to both lines. We do not actually require these, but only the relation between these two co-ordinates, which will give a locus of the intersections of consecutive rays or the equation to the caustic.

Subtracting one equation from the other:

$$x(m' - m) = c(m'^3 - m^3),$$
$$x = c(m'^2 + mm' + m^2)$$
$$= 3cm^2,$$

since m and m' are nearly equal.

$$\therefore \quad m = \left(\frac{x}{3c}\right)^{\frac{1}{2}};$$

$$\therefore \quad y = \frac{x^{\frac{3}{2}}}{(3c)^{\frac{1}{2}}} - c \cdot \left(\frac{x}{3c}\right)^{\frac{3}{2}} = \frac{2}{3} \cdot \frac{x^{\frac{3}{2}}}{(3c)^{\frac{1}{2}}},$$

or $$27cy^2 = 4x^3.$$

Thus, for values of m, such that m^4 is negligible, the form of the caustic is a semi-cubical parabola.

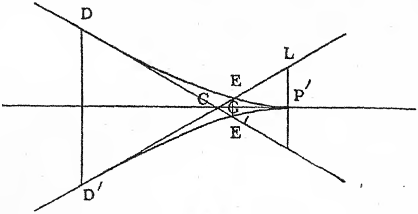


FIG. 203

In the figure, DCE′ and D′CE are the extreme rays from the lens and DEP′, D′E′P′ the two branches of the caustic curve.

All rays inside the extreme rays touch this curve at points which are nearer to P′, the smaller the inclinations of the rays to the axis. The rays are consequently crowded together just inside the caustic curve, and a screen placed between C and DD′ will show a circular patch of light with a brighter illumination round the circumference.

At points between C and EE′, a brighter patch appears in the centre which increases to EE′ where the illuminated area is a minimum. EE′ is called the circle of least aberration.

Beyond EE′ the first circle diminishes until at P′ it becomes a point.

Set up a large lens on the optical bench with a small hole illuminated by a Bunsen flame containing common salt as a source of light. The light may be concentrated on the hole by means of a small condensing lens.

It is best to have the distance from the hole to the large lens about twice the focal length of the lens.

Examine the image by means of a micrometer eyepiece, when the appearances described will be seen if the source lies on the axis of the lens.

Find the position P′, and read off the position of the eyepiece on the bench scale. Move in towards the lens and measure the diameters of the circular rings at a series of distances from P′. Plot the results on squared paper and verify the equation of the caustic curve.

The radius of the circle of least aberration is $r = \frac{1}{4} cm_0^3$, where c is constant in the equation and m_0 is the value of m for the extreme rays.

m_0 is nearly proportional to the aperture of the lens. Vary the aperture by putting diaphragms over the lens and measure the corresponding diameters of the least circles. Verify that the diameter is proportional to the cube of the aperture.

The above value of r is readily calculated by remembering that it is the value of the ordinate of the caustic at the point where it is cut by the extreme ray.

It is therefore necessary to solve the equations

$$y = m_0 (x - cm_0^2), \qquad \ldots(4)$$

$$27cy^2 = 4x^3.$$

The solutions are

$$x = \tfrac{3}{4}cm_0^2, \qquad y = -\tfrac{1}{4}cm_0^3.$$

In order to find the value of P′L, we note that by equation (4), which is the equation of the line, CL, the value of y corresponding to $x = 0$ is $-cm_0^3$.

This negative sign occurs because we have measured the angle of which m is the tangent from the x axis in a clockwise direction.

In the case of the line, CL, the value of m_0 is negative since the angle is obtuse.

The lateral aberrations of the extreme ray, i.e. the length of P′L, is thus $-cm_0^3$.

But the radius of the circle of least aberration is $\frac{1}{4}cm_0^3$ (numerically).

Thus the ratio $\dfrac{\mathrm{P'L}}{\mathrm{GE}} = 4.$

Verify this result.

CHAPTER XIII

INTERFERENCE, DIFFRACTION, AND POLARIZATION

Introduction

THE theory that light consists of waves leads us to expect, by the principles of superposition, that class of phenomena described by the term 'interference'. Particles of a medium, when simultaneously displaced by the arrival of several disturbances, have a resultant displacement obtained by adding together the vectors representing individual displacements. In particular, if a particle is subject to displacements in directly opposite directions, it will be displaced a smaller amount than if it were subject to either separately, and if both the displacements are equal, but oppositely directed, there will be no displacement of the particle. On the other hand, displacements arriving at a particle in the same direction will cause it to be moved a distance equal to the sum of the separate displacements.

The arrival of oscillatory disturbances at any point may, therefore, cause larger or smaller displacements than would occur as a result of each separately.

The intensity of illumination at any point of a medium is proportional to the square of the amplitude of the vibrations at that point.

It can thus be seen how it is possible to produce places of large or small intensity as a result of the arrival of two trains of waves. It may even be possible that there will be darkness at certain points since it is possible that as a consequence of adding the displacements vectorially we get no net effect.

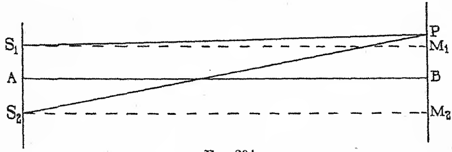


FIG. 204

Let S_1 and S_2 denote two sources of monochromatic light. From each is thus emitted a train of waves of a particular wave-length, the same for each. In addition, suppose that the disturbance starting from S_1 is in the same phase as that starting from S_2. These disturbances are propagated with a certain velocity, and at points equidistant from S_1 and S_2 the displacements will be in the same phase when they arrive

there. In other cases where points are at different distances from S_1 and S_2, it may happen that the displacements are in different directions on reaching the point. Draw AB at right angles to the middle point of S_1S_2 and let it cut a screen, M_1M_2, normally at B.

We shall inquire as to the illumination at different points on the screen. The disturbances will reach B in the same phase, the displacements will be in the same direction, and will thus unite in increasing the illumination at B, which will always be bright.

Of course, we think of the waves as transverse, i.e. the vibration is normal to the direction of propagation.

Take any other point, P, distant x_n from B.

The light from S_2 travels over the path, S_2P, and that from S_1 over S_1P. Thus those disturbances, which reach P simultaneously, started at different times from their sources.

The vibrations will thus be in different phases on reaching P. If, however, the difference between S_2P and S_1P is a whole number of wave-lengths, the disturbance from S_2 left that point a complete number of periods before the disturbance from S_1 set out towards P. The displacements are thus in the same direction, and will unite at P to give brightness, just as they do at B.

If the difference of path is an odd number of half wave-lengths, the displacements on reaching P will be directed oppositely, and will tend *to destroy the disturbance at P. If the amplitudes are equal there will* be no displacement, and consequently no illumination at P. In any case there will be a marked falling-off in brightness. We ought therefore to be able to distinguish on either side of B, as we proceed outwards, alternate bright and dark places.

Suppose the distance $AS_1 = AS_2 = d$, while $AB = D$.

Then
$$PM_1 = PB - M_1B$$
$$= x_n - d,$$

and similarly
$$PM_2 = x_n + d.$$

(M_1 and M_2 are the feet of the perpendiculars from S_1 and S_2 on to the screen.)

$$\therefore \quad S_2P = \{D^2 + (x_n + d)^2\}^{\frac{1}{2}} = D\left\{1 + \frac{(x_n + d)^2}{2D^2}\right\} \qquad \ldots(1)$$

in which we neglect higher powers of $\dfrac{x_n + d}{D}$ *than the square.*

This means that we suppose that x_n and d are of small magnitude relatively to D. This will evidently be the case if we confine our attention to a few only of the alternations in intensity about B (cf. equation (3) below).

In the same way

$$S_1P = D\left\{1 + \frac{(x_n - d)^2}{2D^2}\right\}; \qquad \ldots(2)$$

$$\therefore \quad S_2P - S_1P = 2 \cdot \frac{d}{D} \cdot x_n.$$

Thus, for a bright point

$$\frac{2d}{D} x_n = n\lambda, \qquad \ldots(3)$$

and for darkness

$$\frac{2d}{D} x_n = \frac{2n+1}{2}\lambda; \qquad \ldots(4)$$

or writing $\delta = 2d$, i.e. $S_1S_2 = \delta$,

we have $$x_n \cdot \frac{\delta}{D} = n\lambda \quad \text{or} \quad (n + \tfrac{1}{2})\lambda, \qquad \ldots(5)$$

according as P is bright or dark.

In the experiments to be described (5) is of fundamental importance.

The first three experiments are three examples of obtaining sources S_1 and S_2 of the kind described.

It would be useless to set up two slits and illuminate them by a sodium flame, for we have in such a flame a multitude of sources in different phases.

The method adopted is to cause light from a single source to travel to the points, of which P is typical, by two different routes and to unite on arrival. We thus have the equivalent of two sources emitting vibrations in the same phase.

In making experiments on interference and diffraction accurate measurements have to be made; for this reason the optical bench is used. It consists of a strong, rigid metal frame, provided with levelling screws on which it stands. The frame consists of two metal rails, one of which is graduated accurately in millimetres. Along the rails slide metal uprights, each of which is attached to a vernier at its lower end so that its position on the bench may be accurately read.

The uprights serve to carry a slit, micrometer, microscope, lens, or whatever piece of apparatus is necessary. If it is necessary to move any piece of apparatus transversely across the bench, it is placed in an upright fitted in a support provided with a transverse micrometer screw.

Although it is possible to read off positions of the uprights on the bed of the frame, since the slit or cross-wire may not be exactly above the indicator mark we cannot read off directly the distance from slit to cross-wire. A correction must always be applied. In order to find this correction, a stand carrying a carefully measured rod is placed on the rails, one end of the rod is placed in contact with the slit while

the other end is viewed in the micrometer. Let the length of the rod be l, and suppose the distance between the slit and micrometer uprights is l', as observed on the scale. Then to convert the readings as obtained from the upright to the distances required we must add to the observed readings the quantity $(l - l')$.

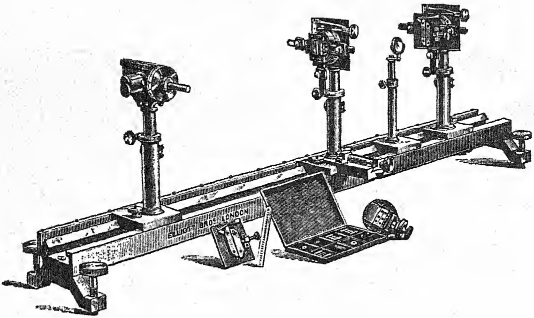


FIG. 205

Determination of the Wave-length of Sodium Light by means of Lloyd's Single Mirror

The simplest way of obtaining interference bands is by means of a mirror silvered on the front surface or blackened at the back in order to avoid multiple reflections.

The diagram illustrates how the interference is brought about.

Light from a slit, S_1, travels directly to a screen, PB, and also by the alternative path after reflection at the mirror, MM. For example, a ray may reach P by the direct path, S_1P, or by the path, S_1CP. The latter ray produces the same effect at P as would arise from a ray, S_2P, S_2 denoting the position of the image of S_1 in the mirror. We thus have the equivalent of two sources, S_1 and S_2, emitting vibrations in the same phase, except for any change of phase that may occur on reflection at MM.

MM is mounted vertically on an upright of an optical bench, and the slit, S_1, is carried vertically on another upright.

MM is mounted as accurately parallel to S_1 as possible, and the fringes are looked for by the micrometer microscope. They will usually come into view, if not already there, after slightly rotating the mirror, and are rendered distinct by adjusting the width of the slit.

The mirror should be parallel to the length of the bench, and we shall see how to make this adjustment accurately.

When the fringes appear as definite and bright as possible, it is necessary before proceeding further to make sure that the distance, S_1S_2, can be found.

The method adopted is to use a lens and obtain the magnified and diminished images of S_1, S_2, but we require a lens whose focal length is less than a quarter of the distance between S_1S_2 and PB.

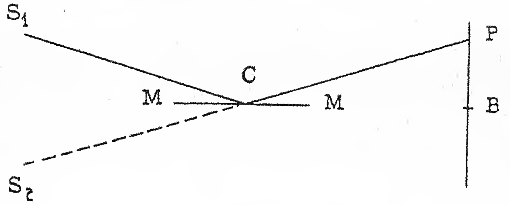


FIG. 206

In Chapter XI, p. 299, we described a method of finding the focal length of a lens in which the object and screen are kept fixed while the lens is moved. In the diagram (fig. 185), suppose that P_1 denotes the position of the object, S_1S_2, and O_1, one of the positions of the lens, while P_2 denotes the position of the image of S_1S_2.

By moving the lens to O_2, the image is formed at the same place, but it now differs from the former in magnitude.

Let d_1, d_2 denote the respective distances between the lines S_1 and S_2 in the image, while δ is the actual distance, S_1S_2 between the slits.

Then $$\frac{d_1}{\delta} = \frac{P_2O_1}{P_1O_1}, \quad \frac{d_2}{\delta} = \frac{P_2O_2}{O_2P_1}.$$

But $$P_1O_1 = P_2O_2 \text{ and } P_1O_2 = P_2O_1;$$

$$\therefore \quad \delta^2 = d_1d_2. \qquad \ldots(6)$$

Thus, by measuring d_1 and d_2 we may deduce δ.

If MM is too close to PB it may be impossible to obtain both images with the lens between the mirror and screen. The mirror must be placed so that this is possible. This is the reason for performing this part of the experiment first, since it is useless to measure the distance between consecutive fringes unless we can find the distance, S_1S_2. The distance apart of the fringes does not depend on the position of the mirror, but if the mirror has to be replaced in another position it is easy to throw out its adjustment. Both images of S_1 and S_2 will be

in focus in the plane of the cross-wires only provided that MM is normal to this plane, i.e. to PB.

It is important that MM should be normal to PB. We have, therefore, to rotate MM slightly until the two are accurately in focus, the lens being adjusted so that its centre lies on a level with the centre of S_1 and opposite the edge of the mirror.

On removing the lens, MM should be rotated slightly about a horizontal axis to get the position where the fringes are brightest, when S_1 and MM are parallel.

Set the cross-wire accurately down the centre of the first bright fringe, and move it always in one direction by means of the screws giving the transverse motion, stopping at every three or four fringes to note the position. From the observations deduce the distance between consecutive fringes.

Finally, measure the distance from S_1 to PB.

From (5) we have

$$x_n = n\lambda \frac{D}{\delta},$$

and proceeding to the next bright band:

$$x_{n+1} = (n+1)\lambda \frac{D}{\delta};$$

$$\therefore \quad x_{n+1} - x_n = \frac{\lambda D}{\delta}. \qquad \ldots(7)$$

Let s denote the distance between consecutive fringes, and we then have

$$\lambda = \frac{s}{D}\sqrt{d_1 d_2}.$$

As a rule it will be best to make use of the full length of the bench, so that D is large, also δ should be as small as convenient. At the same time d_1 or d_2 must not be too small or it will be difficult to make the determination of δ accurately.

In recording the results of this and the two following experiments it is a good plan to begin, as mentioned, at one edge of the field, and make out a table as shown below.

We can then readily obtain a series of independent readings for the determination of the separation of the bands.

In this experiment the first fringe does not lie at the point corresponding to B, fig. 204. This is because the reflected ray undergoes an abrupt change of phase on reflection at the mirror. This has the effect of displacing all the fringes towards B by a certain distance, leaving the separation between successive fringes unaltered.

No. of band	Micrometer reading (a)	No. of band	Micrometer reading (b)	Separation for 15 bands (b) - (a)
1	....	16	....	....
4	....	19	....	....
7	....	22	....	....
10	....	25	....	....
13	....	28	....	....
				....

Mean Separation for 15 Bands ....

Mean Separation for 1 Band ....

Interference by means of Fresnel's Double Mirrors

Interference fringes may also be formed by the use of two mirrors silvered on the front surface or blackened at the back, very slightly inclined to one another.

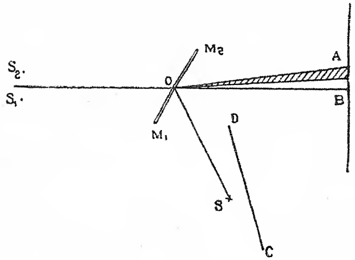


FIG. 207

The diagram illustrates the arrangement. OM_1 and OM_2 denote the two mirrors which may be mounted on the optical bench and placed accurately vertical.

S denotes the source of illumination and may most conveniently be a sodium source.

The fringes are again observed by means of a micrometer, and AB denotes the position of its cross-wires.

The screen, CD, protects AB from direct light from S.

The source is placed on one side of the bench as the diagram indicates, and may be the slit of the optical bench supported in a clamp. S_1 and S_2 are the two virtual images in the mirrors. S_1S_2 and S lie on a circle with centre at O.

The calculation, mode of measurement, and adjustment are identical with those of the previous experiment.

The shaded area of fig. 207 shows where rays from both mirrors interfere, and the section of this region with the plane of the cross-wires of the microscope is the position within which the fringes lie.

Fresnel's Biprism

The biprism is a prism with one of its angles only a little less than two right angles, and with two equal small base angles. The figure illustrates its action. The biprism is represented by CDEF, and it acts like two prisms placed base to base.

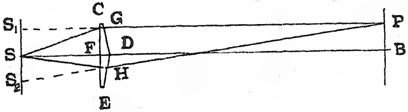


FIG. 208

Rays from a slit, S, are deviated in each part of the prism and unite on the screen, as in the case of the rays, SGP and SHP. We have virtually two sources, S_1 and S_2.

The details of the experiment are very similar to those described in the experiment with Lloyd's mirror.

The biprism is held in a stand which can rotate about a horizontal axis parallel to the central line of the bench. By this means the edge, D, of the prism can be rotated to bring it parallel with the slit.

The adjustments are first made roughly by the eye, and usually the fringes will be observed even with the rough adjustment. Slight rotation of the biprism will, as a rule, improve the appearance of the bands, and still better definition will be obtained by narrowing the slit.

The distance, S_1S_2, is measured with the aid of a lens as before.

In adjusting D parallel to S, the following device is of great assistance. If the eye is placed on the side of the prism away from the slit, and moved across the bench, the slit will appear to cross from one side of the prism to the other. As it crosses D, unless the slit is parallel to the edge of the prism, the top or bottom will cross the edge first, while

if parallel it will appear to make the transition suddenly. The prism must be rotated until this sudden jump occurs.

Be careful after arranging the apparatus to give good fringes, to make the determination of the length, S_1S_2, before measuring the separation of the fringes.

The Determination of the Radius of Curvature of the Face of a Convex Lens by means of Newton's Rings

Newton's rings are formed as a result of interference between the incident and reflected rays from a source of monochromatic light on the air film between a plane glass plate and a convex lens in contact with it.

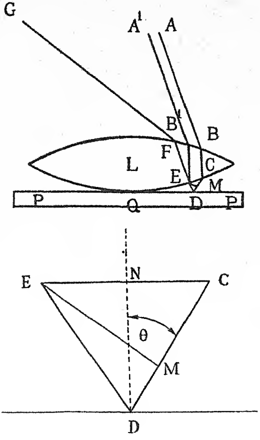


FIG. 209

The diagram shows the lens, L, and the plate, P, in contact at the point, Q.

Two incident rays are drawn, AB and A^1B^1. The first is partially reflected and refracted at the points, B, C, D, E, and F.

A ray is drawn which is represented as suffering refraction, except at D, where it is reflected. The neighbouring ray, A^1B^1, which follows the course, A^1B^1EFG is shown, and these two rays being brought

together along EFG will interfere and may produce greater or less illumination than each separately, according to their phase difference. In calculating this, it is to be remembered that on account of the reflection at D, at a medium optically denser than that in which the ray travels before reflection, a phase change of half a wave-length is imparted to it.

The total phase difference between the two united rays is thus: $(\mathrm{MD} + \mathrm{DE} + \frac{1}{2}\lambda)$ as reckoned in path length, for had the rays both left the lens, the wave-front would have been EM, so that the phases are the same at E and M.

Each ray, such as AB, has a corresponding ray, such as A^1B^1, arising from the same point in the source of light with which it can interfere, so that an extended source may be used and a Bunsen flame containing sodium acts very well; in fact, an extended source is necessary in order to obtain a large area containing rings.

There will be brightness or darkness according as

$$\mathrm{MD} + \mathrm{DE} + \tfrac{1}{2}\lambda = n\lambda \quad \text{or} \quad (n + \tfrac{1}{2})\lambda,$$

where n is a whole number, i.e. according as $\mathrm{MD} + \mathrm{DE} =$ an odd or even number of half-waves.

Let CD fall on the plate at incidence θ.

Let t be the thickness of the film at this point.

The diagram is drawn with t large, or D a long way from Q, for convenience, but in the formation of the rings the film is very thin, and the part of it with which we are concerned is very close to Q.

Just near D we may regard the film as an element with parallel faces separated a distance, t, as in the enlarged element.

$$\begin{aligned}\mathrm{MD} + \mathrm{DE} &= 2\mathrm{CD} - \mathrm{CM} \\ &= \frac{2\mathrm{CN}}{\sin\theta} - 2\mathrm{CN}\sin\theta = 2\mathrm{CN}\frac{\cos^2\theta}{\sin\theta} = 2t\cos\theta.\end{aligned}$$

Thus the condition for brightness is

$$2t\cos\theta = (2n+1)\frac{\lambda}{2},$$

$$2t = \frac{2n+1}{2}\cdot\frac{\lambda}{\cos\theta}.$$

The angle, θ, may be measured by the angle between the normal to the plate and the incident rays before striking the lens, since near Q the lens acts as a parallel plate of glass very approximately.

It is usual to cause the rays to fall normally on the lens, when we have

$$2t = (n + \tfrac{1}{2})\lambda.$$

If we consider a plane normal to that of the paper through Q and D, we can see that the circumference of a circle with centre, Q, and

radius, QD, will pass through points at which the air film has a uniform thickness, and consequently all round this circle rays incident vertically will undergo the same phase change, and thus alternate bright and dark rings are formed about Q.

Let the radius of the ring be $r_n =$ QD, and let the ring be the $(n + 1)$th from the centre.

The centre itself will be black, for the air film is infinitely thin at this point. If this is not the case at first it is because some dust particles lie between the surfaces, and these should be removed.

$$DQ^2 = DN \cdot DK = t\,(2R - t) \text{ (fig. 210)},$$

$$R = \text{radius of lower surface of the lens},$$

$$\therefore \quad DQ^2 = r_n{}^2 = 2Rt$$

approximately, since t^2 may be neglected.

Thus for brightness

$$r_n{}^2 = R\,(n + \tfrac{1}{2})\,\lambda.$$

The first ring corresponds to $n = 0$; the second for $n = 1$, and so on. Thus the radii are proportional to $\sqrt{1}$, $\sqrt{3}$, $\sqrt{5}$, etc.

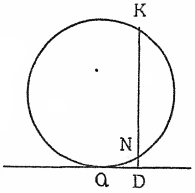


FIG. 210

For the purpose of the experiment a convex spectacle lens of about 100 cm. radius of curvature is suitable, and the light from a sodium flame is reflected down on to it by means of a sheet of plane glass held at 45° to the vertical (see fig. 211).

The rings are viewed by means of a travelling microscope.

In order to focus quickly on the rings, remove the lens and focus on the top of the glass plate. On replacing the lens, and adjusting the microscope over the point of contact, the rings, which lie in the air film, should be distinct. A good bright sodium flame is necessary, and often difficulties disappear if this point is attended to.

Ring	Micrometer reading (L)	Micrometer reading (R)	Diam.	(Diam.)^2
20				
19				
18				
...				
...				
...				
5				
4				
3				
2				
1				

(L) denotes the reading on the left of the centre, (R), that on the right.

Move out the micrometer to about the twentieth ring from the centre, and then, moving back again, turning the screw always one way to avoid any errors due to backlash, set the cross-wire carefully

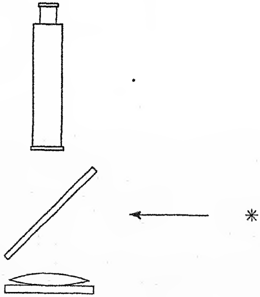


FIG. 211

down the centre of each bright ring and observe the micrometer reading. In this way, by observing the rings on both sides of the central dark patch, the various diameters are determined.

Make a table as shown above.

Draw a curve with the square of the diameter as ordinate, and the number of the ring as abscissa. The graph should be a straight line.

Then if $$P_2N_2 = (D_{n_2})^2,$$
$$P_1N_1 = (D_{n_1})^2,$$
since $$D_n{}^2 = 4R\lambda\,(n + \tfrac{1}{2}),$$
$$\frac{P_2N_2 - P_1N_1}{N_1N_2} = \frac{(D_{n_2})^2 - (D_{n_1})^2}{n_2 - n_1} = 4R\lambda.$$

If sodium light is used, λ may be taken as 5890×10^{-8} cm.

In this description we have not taken account of the fact that the rings are seen, not directly, but after refraction through the lens. They are formed in the air film in the space between the lens and plate of glass. This difficulty is avoided by placing the plate above the lens and in contact with it, for in that case we view the rings through a plane sheet of glass.

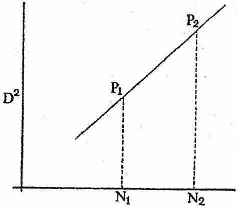


FIG. 212

It is, of course, more convenient to place the glass below the lens, for reasons of ease in keeping them steady. The error in this case is not great if a thin lens be used, for then the object—the rings—is at the surface of the lens and consequently at its principal plane. The image is in the second principal plane, and of the same size as the object. For a thin lens these planes and lens surfaces are nearly coincident. In practice we have an image of magnification slightly differing from unity, or the diameters in the formula above are to be multiplied by such a fraction. For the purpose of the experiment this factor is omitted.

Applications of Newton's Rings

In the experiment just described, Newton's rings were obtained by placing a curved surface upon a plane surface and the separation of the rings was determined, for light of a given wave-length, by the radius of curvature of the curved surface. If the radius increases, the rings grow larger and if it diminishes, they grow smaller. This observation

may be used to detect and to measure changes in radii of curvature. A similar appearance occurs if the lens is kept fixed and the flat plate moved and the amount of the separation can be measured by observations on the rings. This may be used to measure small changes of length in a body to which the plate is attached. An example of this is provided by the experiment described on pp. 446–9, in which the change of length in a bar is measured as a result of magnetizing it. Two experiments will be described in which elastic constants are determined by observations of this kind.

The Determination of Poisson's Ratio and Young's Modulus for Glass in the Form of a Bar

The definition of Poisson's ratio has been given in the chapter on Elasticity (p. 68). From the fact that when a bar of an elastic material is curved longitudinally and, in consequence, all its fibres on one side of the neutral surface are stretched and those on the other side compressed, it follows that lateral changes in dimension occur which lead to lateral curvature. Poisson's ratio can be expressed in terms of the longitudinal and lateral curvatures.

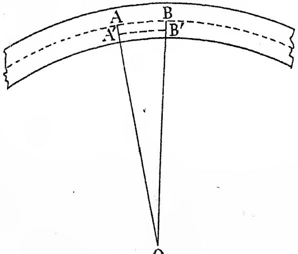


FIG. 213

Suppose that a bar is initially unstrained. Let it then be strained so that the radius of curvature of the neutral surface is r_1.

By referring to p. 72 it will be seen that a filament such as A′B′ at a distance y below the neutral surface which, in fig. 213, is shown to pass through AB, is contracted by an amount $\frac{y}{r_1}$ per unit length. Consequently there will be a lateral expansion at this level in the beam of magnitude, $\frac{\sigma y}{r_1}$, σ denoting Poisson's ratio. Thus a section at right

angles to the bar with CD at the level of AB and C'D' at that of A'B' will have the appearance of fig. 214.

The radius of curvature of the neutral surface in this section is denoted by O'C and, since the expansion per unit length is $\frac{(C'D' - CD)}{CD}$, it follows that the radius of curvature is $r_1' = \frac{r_1}{\sigma}$.

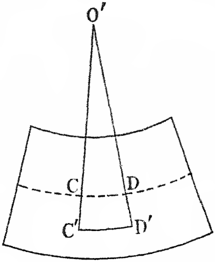


FIG. 214

If the radius of curvature in the longitudinal direction changes from r_1 to r_2, that in the lateral direction will change from r_1' to r_2', and thus

$$\sigma = \frac{\left(\frac{1}{r_2'} - \frac{1}{r_1'}\right)}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} = \text{ratio of changes in the curvatures.}$$

The changes in the curvatures can be investigated by means of Newton's rings.

Let ABC denote the longitudinal curve of a glass beam and DBE the lower surface of a thin convex lens in contact with it at B.

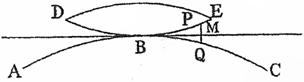


FIG. 215

If light is directed vertically downwards on to the lens, as in the previous experiment, Newton's rings will be formed in the air film between these surfaces. At the point, P, the rays are partially reflected

and partially refracted, and some of the refracted rays are reflected at Q. Thus a change of phase of $\frac{2PQ}{\lambda}$ wave-lengths is introduced between rays reflected at P and those reflected at Q in addition to a change due to reflection at Q. If the total change is equal to the phase change of a whole number of wave-lengths, the point, P, is bright, while if the change is that of an odd number of half-wave-lengths the point is dark. If the radius of curvature of the surface, DBE, is r_0,

$$PQ = PM + MQ = \frac{d^2}{2}\left(\frac{1}{r_0} + \frac{1}{r_1}\right),$$

where r_1 is the radius of curvature of the beam and d is the distance, MB.

Thus the condition for brightness at P is

$$\frac{d^2}{2}\left(\frac{1}{r_1} + \frac{1}{r_0}\right) = (n + c)\,\lambda,$$

where c is a constant to account for the phase change on reflection at Q, where the rays fall upon a denser medium than that in which they are travelling.

A similar relation holds for the transverse section which can be written in the form

$$\frac{d'^2}{2}\left(-\frac{1}{r_1'} + \frac{1}{r_0}\right) = (n' + c)\,\lambda.$$

Note the negative sign of the first term owing to the fact that the upper surface of the beam is hollowed out transversely.

These equations refer to these particular sections of the beam, for a section inclined to them the relation, though similar, is rather more complicated, and it shows that a complete bright curve has the shape of a conic section, the particular type being dependent upon the curvature of the lens.

For the purpose of the experiment, investigations are necessary only in the longitudinal and transverse directions.

The glass beam may be conveniently about twenty inches long by one inch wide and of thickness about one-eighth inch. It should rest symmetrically on two horizontal parallel knife-edges about six inches apart. The beam is bent by applying weights symmetrically over the ends, and these may be applied in a simple way by tapes stuck to the upper surface at each end and hanging over the edges to carry scale-pans or hooks for carrying the weights. Strips of Elastoplast are suitable for these attachments.

With the lens placed at the middle point, B, Newton's rings are produced and the diameters of the rings are measured in both directions by means of a travelling microscope suitably mounted above the beam.

The various attachments and the weight of the beam will cause some initial curvature before the actual loading begins. If the radius of this initial curvature is s, that due to an additional couple, G, is r,

then
$$G = YI\left(\frac{1}{r} - \frac{1}{s}\right),$$

where Y denotes Young's modulus and I the moment of area of cross-section of the beam. The magnitude of the latter is $\frac{bt^3}{12}$, where b is the breadth and t the thickness of the beam.

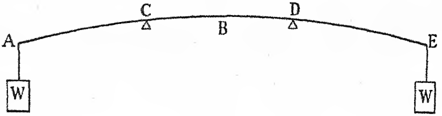


FIG. 216

The couple, G, is that on each half of the beam and is obtained from the product of the load, W, and the horizontal distance between D and E, the weight of the beam is neglected.

Put on a load, W_1, giving a couple, G_1, and let the radius of curvature of the beam become r_1.

By plotting the squares of the radii, d^2, against their numbers, n, as abscissae a linear graph is obtained and, if the slope be ψ_1,

$$\cot \psi_1 = \frac{1}{2\lambda}\left(\frac{1}{r_1} + \frac{1}{r_0}\right).$$

For the transverse section in the same way

$$\cot \psi_1' = \frac{1}{2\lambda}\left(-\frac{1}{r_1'} + \frac{1}{r_0}\right).$$

When the observations are repeated for another load, W_2, the radii of curvature change to r_2 and r_2', and the slopes to ψ_2 and ψ_2'.

It follows that

$$\cot \psi_2 - \cot \psi_1 = \frac{1}{2\lambda}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$\cot \psi_2' - \cot \psi_1' = \frac{1}{2\lambda}\left(\frac{1}{r_2'} - \frac{1}{r_1'}\right).$$

Thus
$$\sigma = \frac{(\cot \psi_2' - \cot \psi_1')}{(\cot \psi_2 - \cot \psi_1)}.$$

The value of Young's modulus may be obtained from the values of the couples applied, for it follows from the formula for G given above that

$$G_2 - G_1 = YI\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = 2YI\lambda\,(\cot\psi_2 - \cot\psi_1).$$

The Determination of Young's Modulus for a Metal Rod by an Interference Method

This is a method described by Searle (*Proc. Camb. Phil. Soc.*, XXII, Pt. 3), and the apparatus described below was designed by him and made by W. G. Pye and Co.

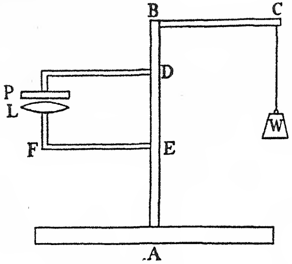


FIG. 217

The rod to be examined is placed in an upright position, AB, on a firm base and a couple, G, is applied at the upper end by means of a horizontal arm, BC, carrying a weight, W. Thus the rod becomes curved to a radius, r, given by the formula,

$$G = \frac{YI}{r}.$$

In addition, the rod is compressed to some extent by the fact that it is carrying a weight, W, at its upper end. If the rod is of circular cross-section of radius a, the contraction on this account is of magnitude $\frac{W}{\pi a^2 Y}$ per cm. The total effect of the curvature and compression can be observed by attaching horizontal arms, PD and FE, to the rod (fig. 217) at a distance apart, DE $= l$. These rods carry respectively a piece of plate glass, P, and a lens, L, so that the plane and curved surfaces come together producing an air film within which Newton's rings can be formed. Any change in the separation of these surfaces

results in a displacement of the rings. Suppose that in the initial condition the two surfaces are just in contact and that they become separated by an amount, x, when a load, W, is added. Let the point of contact, which is the centre of the ring system, lie at a distance, d, from the axis of the rod. The upper horizontal bar will, as a result of the bending, become inclined to the lower at an angle, φ, where $r\varphi = l$, and the surface of the lens and of the plate become separated by an amount, φd. This will become clear by drawing a diagram to illustrate the separation which results from the bending. The two surfaces are thus separated by an amount

$$x = \frac{ld}{r} - \frac{Wl}{\pi a^2 Y}.$$

If the arm, BC, is of length, L,

$$x = \frac{Wl}{\pi a^4 Y}(4Ld - a^2),$$

where I has been given its value $\frac{1}{4}\pi a^4$ appropriate to a circular area.

The form of the apparatus made according to Searle's design is shown in fig. 218. To the right is a microscope fitted with a micrometer

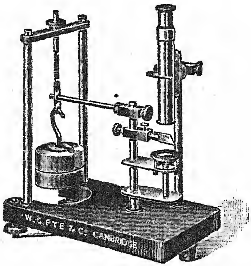


Fig. 218

eyepiece scale for observing and making measurements upon the rings. The light from a sodium flame is directed in the usual way on to the lens and plate. The lens should not touch the plate when the additional load, W, is placed on the cross-bar. If there is contact initially, the plate may be resting on the lens and a definite load may be required to set it free. This may be examined by lightly placing the finger on the unloaded cross-bar, when there may at first be no movement of the fringes.

The horizontal cross-pieces, PD and FE, are rigid brass plates with a pillar attached to the latter supporting the lens. The upper surface of the lens protrudes through a circular hole in the upper plate. This plate carries three levelling screws upon which a flat circular glass plate is carried. To begin with, the plate and lens are in contact and the screws are then adjusted so that the rings move readily when the slightest pressure is placed on the cross-bar, BC. In this position the rings are brought carefully into focus and the diameter of the smallest dark ring is measured. Let this be p_1 in units of the micrometer scale. While the bar is loaded until it finally carries the weight, W, it will be observed that the rings appear to move inwards, shrinking and disappearing at the centre. The number disappearing must be counted and the diameter, q, of the smallest dark ring must be measured when the load is W. If $q \neq p$ the pattern has not exactly repeated itself. This means that the separation of the lens and plate has not been increased by a whole number of half-wave-lengths, but by $(n + f)\frac{\lambda}{2}$, where n is an integer and f a fraction. The increase in the distance between the plates is then

$$x = \frac{(n + f)\,\lambda}{2}.$$

In order to determine f it should be noted that the difference in the squares of ring diameters is proportional to the phase difference at the rings. By means of the micrometer scale the difference in the squares of the diameters of successive rings may be determined. If this is denoted by s^2, the value of f is

$$f = \frac{(p^2 - q^2)}{s^2}.$$

f may be positive or negative.

A special device is fitted to the apparatus in order that the gradual loading may be under control and the rings counted without difficulty.

The formula for the calculation of Young's modulus, Y, is

$$\mathrm{Y} = \frac{2\mathrm{W}l\,(4\mathrm{L}d - a^2)}{\pi a^4\,(n + f)\,\lambda}.$$

Jamin's Interferometer

The apparatus consists essentially of two glass plates, AB and A′B′, of the same dimensions and optical character. The plates are very carefully worked and are of the best optical glass. They are mounted parallel to one another, standing on tables on an optical bench at a distance apart of about one metre.

The first, AB, is set at 45° to the bench with its surfaces vertical, and it is illuminated by rays from a sodium flame.

In the diagram a ray, RS, is shown. It is partly reflected and

refracted at S, and the refracted beam again partly reflected at T. The divided ray takes the paths, RSS′T′U′R′ and RSTUU′R′, so that it is reunited by the second glass plate. This plate is mounted parallel to the first, and set vertically. It rests on a table and may be slightly rotated about a vertical axis. If both plates are exactly similar and are parallel, the lengths of the two paths will be the same for all rays, but by slightly rotating one of the plates a difference in path may be introduced, differing for different directions so that alternate bright and dark bands will be obtained.

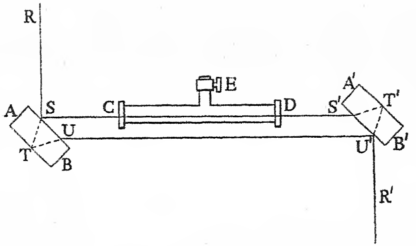


Fig. 219

The tube, CD, is then placed in the path of one of the divided rays between the plates, the other being allowed to pass clear of CD.

CD is a hollow glass tube with ends of plane optical glass. It is fitted with a tap, E, by means of which it may be exhausted or filled with gas.

The tube is filled with air at the atmospheric pressure, and the fringes found by eye. A telescope provided with cross-wires is then focused on the fringes.

In order to ensure that the fringes arise from interference between rays that have passed, one through CD and the other outside it, we may cover up the end of CD, and note if they disappear as they should if due to this cause.

In order to be sure that the rays producing interference do not both pass down CD, it is necessary to intercept the light at the sides of the tube. If fringes are still present they arise from rays passing down CD.

When the correct fringes are obtained the cross-wire of the telescope is focused as accurately as possible down the centre of one bright band, after exhausting the tube as much as possible with an air-pump. The

pressure should be brought down to 1 cm. of mercury at least, and a manometer connected up to read the pressures. Air is now allowed to pass very slowly into the tube by the tap, and the fringes watched. They will appear to move across the field of view, and the number passing the cross-wire must be counted. Allow about five or six to pass, and then stop the inflow and read the manometer. Repeat this, step by step, until the tube is filled with air at atmospheric pressure. Draw a graph showing the relation between the pressure within CD, and the number of fringes that have passed from the initial stage. By an extrapolation deduce the number that pass between the limits of complete exhaustion and the attaining of atmospheric pressure.

Let this number be n.

This means that the difference of optical path in the tube, when completely exhausted and when filled with air at atmospheric pressure, is $n\lambda$.

If the tube is of length, l, and the refractive indices are μ and μ_0, when the pressure is atmospheric and when the tube is exhausted, respectively

$$l(\mu - \mu_0) = n\lambda.$$

But $$\mu_0 = 1,$$

so that $$\mu = 1 + \frac{n\lambda}{l}.$$

A convenient length for the tube is 30 to 40 cm., when for the exhaustion produced by a good air-pump the value of n is of the order 200.

The compensator described in connexion with the experiment with Rayleigh's refractometer may also be used with Jamin's interferometer, and the calculation of the refractive index at normal temperature and pressure may be made with the help of Jamin's interferometer also.

A cylindrical lens with the cylindrical axis vertical is often placed just in front of the point S. This has the effect of widening the source of light in a horizontal direction, leaving it unchanged vertically. A suitable source is a narrow slit illuminated with monochromatic light.

The Refractive Index of Air by means of the Rayleigh Refractometer

A diagrammatic representation of the apparatus is given in fig. 220.

ABCD is an airtight metal box, divided into two separate chambers, each of which may be connected to a manometer. The pressures in the chambers are varied and measured by means of the manometers. The chambers are closed at the ends by means of parallel plates of optical glass.

The collimator is provided with a slit at K, and produces a parallel beam of light which falls on a screen, carrying two fine slits, LL, placed

in front of the air chambers, so that one slit lies adjacent to the end of one, and the other slit adjacent to the end of the other chamber.

These slits are prolonged so as to extend higher than the top of the box, ABCD. Thus, light from the slits passes over the box as well as through it, and finally enters the telescope, T.

These two fine slits produce interference fringes in the focal plane of the telescope, and if the pressures in the chambers are equal, the set of fringes in the lower half of the field appear to be continuations of the fringes in the upper half which arise from rays that have passed over the top of the box.

In order to deviate the upper set of fringes down to make comparison with the lower set easy, a prism, P, is provided which intercepts the upper set of rays, and deviates them downwards.

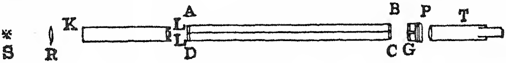


FIG. 220

When white light is used, two sets of coloured fringes are obtained with a white central fringe.

If there is a difference of pressure between the two chambers, there is a displacement of the lower set of fringes owing to the resulting difference of optical path.

G consists of two plates of glass, inclined at a small angle, placed to intercept the lower set of rays. When it lies symmetrically with respect to the rays striking it, it introduces no additional path difference, but on rotating it the rays through one plate traverse a longer path within the glass than do those in the other.

In this way the lower central band can be moved about, and we may, by a rotation of G, bring back the fringe to its central position after it has been displaced on account of the difference in air density in the two chambers. G carries a pointer moving over a scale, and we may calibrate the scale so that the difference of path introduced by setting G is known. Thus, by altering the pressure within the chamber and then moving G to counteract the displacement of the central fringe, we can read directly from the calibrated scale the path difference in the two chambers in wave-lengths, and a graph is plotted with the pressure differences as ordinates and the number of wave-lengths as abscissae.

The calibration is performed by making the pressure in both parts of the chamber equal and illuminating K with monochromatic light, e.g. by a sodium flame.

The scale is set so that the bright bands of the upper set of fringes lie over those of the lower. The pointer is then moved over the scale

until a certain number of bands pass a fixed point in the upper series. The following table will illustrate how this result should be recorded:

CALIBRATION OF SCALE IN WAVE-LENGTHS

Wave-lengths	Scale reading	Wave-lengths	Scale reading	Difference for 40 wave-lengths
0	1·48	40	4·50	3·02
5	1·85	45	4·89	3·04
10	2·24	50	5·24	3·00
15	2·62	55	5·61	2·99
20	3·01	60	5·99	2·98
25	3·38	65	6·37	2·99
30	3·74	70	6·75	3·01
35	4·12	75	7·12	3·00
				24·03
			Mean for 40 wave-lengths	3·004
			Mean value of 1 wave-length in scale divisions	0·0752

After the calibration of the scale the collimator slit is again illuminated by white light, and the central fringes arranged one above the other.

The pressures in the chambers are varied and measured by the manometers.

The central fringes are kept one above the other by means of G, and the pressure differences recorded along with the position of the pointer on the scale.

Draw a graph with scale readings as ordinates and pressure differences as abscissae (fig. 221).

White light is used in order to provide a definite central fringe. The position of this fringe is independent of colour while all other fringes have positions which depend on the wave-length. If monochromatic light is used, all the fringes are alike, and the central fringe is indistinguishable from the others.

Since the screw attached to the compensating glasses has been calibrated in wave-lengths of sodium light, the observations give the path differences in terms of so many wave-lengths of yellow light,

and the refractive index deduced will be that for this particular wave-length.

From the slope of this graph may be deduced the difference in path in wave-lengths for a difference of pressure of 1 cm. of mercury.

Observe the temperature of the air in the tubes by placing a thermometer close to them and noting its indication throughout the course of the experiment. When the pressure is varied by means of the manometers, the changes should take place slowly, and time should be allowed for the air to take up atmospheric temperature.

Fig. 221 gives a general view of the apparatus.

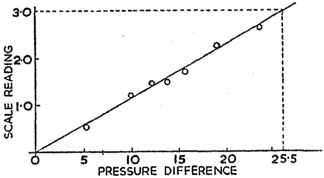


FIG. 221

Theory of the Experiment

For a gas the relation,

$$\frac{\mu - 1}{\rho} = \text{a constant,}$$

is very approximately true. We assume it in this experiment. μ is the refractive index, and ρ its density.

If T is the absolute temperature, we have also

$$\frac{p}{\rho T} = \text{constant.}$$

Thus $$\frac{\mu - 1}{p} . T \text{ is constant.}$$

If μ_0 is the refractive index of air at N.T.P.

$$\frac{\mu - 1}{p} . T = \frac{\mu_0 - 1}{76} \times 273.$$

We shall see that μ_0 can be determined by observations in this experiment.

Suppose the length of the tubes containing the air to be l, and the refractive indices, μ_1 and μ_2, corresponding to pressures p_1 and p_2. Let the wave-lengths of monochromatic light be λ_1 and λ_2 respectively.

The excess of waves in one tube over those in the other is

$$l\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) = \frac{l}{\lambda_0}\left(\frac{\lambda_0}{\lambda_2} - \frac{\lambda_0}{\lambda_1}\right) = \frac{l}{\lambda_0}(\mu_2 - \mu_1)$$
$$= \frac{l}{\lambda_0}\{(\mu_2 - 1) - (\mu_1 - 1)\} = \frac{l}{\lambda_0} \cdot \frac{\mu_0 - 1}{76} \cdot \frac{273}{T}(p_2 - p_1).$$

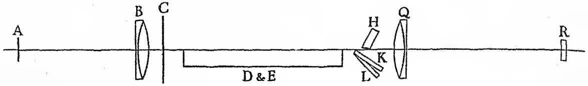


FIG. 222

This number is deduced from the scale over which the pointer moves. Suppose it is m.

Then $$\mu_0 = 1 + m \cdot \frac{76T\lambda_0}{273l\,(p_2 - p_1)}.$$

The pressures are measured in centimetres of mercury, and the ratio, $\dfrac{m}{p_2 - p_1}$, is deduced from the graph as described, while the length of the tubes, which is usually about 25 cm., may be measured by means of a metre rule.

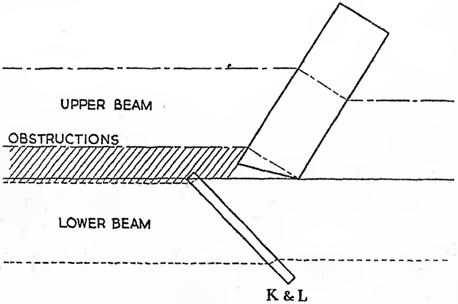


FIG. 223

In figs. 222 and 223 a modern form of apparatus is shown diagrammatically. The plates, L, K, correspond to G, and the prism, P, is here denoted by H, and its action shown in the fig. 223. It can be seen how such obstacles as the upper edge of the tubes containing the gas, which would normally separate the lower ...

FIG. 224

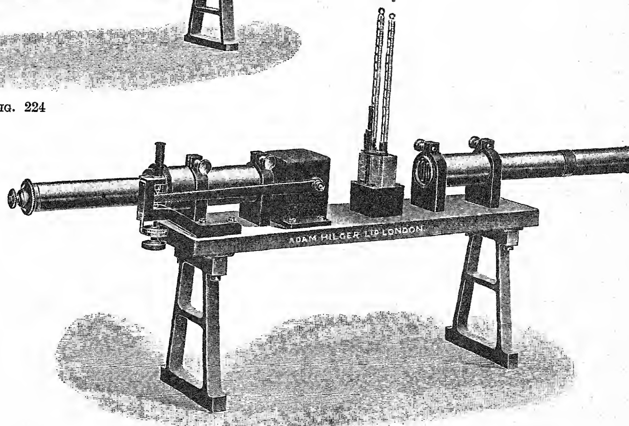


FIG. 225

comparison upper bands, produce no image in the field of view on account of the refraction in H. The two sets of fringes stand one immediately above the other.

They are focused by the achromatic lens, Q, and examined by the cylindrical lens, R. This provides a large horizontal magnification of the fringes, which lie close together because the slits are of necessity rather wide apart (cf. formula (7) of this chapter, δ is large in the present case). The lens however, does not give magnification in a vertical direction, so that the shadow cast by the obstacle is not broadened vertically, and there appears a sharp dividing line between the two sets of bands.

The Rayleigh Refractometer for Liquids

Fig. 225 shows a Hilger apparatus based on the foregoing principles for comparing the refractive indices of liquids.

The apparatus is shorter since the length of liquid traversed need not be so great as that required for a gas.

If a liquid is placed in one of the vessels, and the same liquid containing a solvent in the other, we may determine the effect of the solution on μ, or inversely, we may estimate the amount of solvent contained from its refractive index.

In this instrument the movable plate is in the upper beam instead of the lower, but the method of use is otherwise similar to that of the previous experiment.

Two thermometers project into the liquids for recording their temperatures.

Examine the changes produced by adding small quantities of a salt to pure water, and placing in one cell the solution so obtained and water in the other.

We have, if μ and μ' denote the refractive indices of water and of the solution respectively, d the thickness traversed by the light, and λ the wave-length: $$(\mu \sim \mu')\, d = n\lambda,$$
where n is the number of displacements of the central fringe.

The experiment is performed with white light so as to have a definite central fringe, but λ is the wave-length of the light used to calibrate the scale. By moving the pointer over the scale until the two central fringes are coincident, the scale reading will give the number of wave-lengths which one ray has fallen behind the other in traversing a path of different nature from that of the other.

The calibration is carried out in a preliminary experiment with both liquid cells containing water and with sodium flame illumination, as in the last experiment.

Michelson's Interferometer

The apparatus is illustrated diagrammatically in fig. 226. It consists

mounted vertically on a heavy, firm, rigid stand. The stand consists of a metal bed provided with a large micrometer screw of very fine pitch. Rotation of the screw causes M_1 to slide along the bed and its position may be read off at the screw-head. A general view of the apparatus is given in fig. 227.

The second mirror, M_2, is fixed at the end of a metal arm mounted at the end of the bed, and at right angles to it. This arm also carries

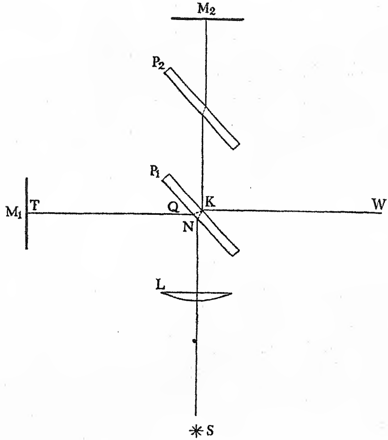


Fig. 226

the two sheets of plane optical glass, P_1 and P_2, which are equally thick, and are mounted at an angle of 45° to the arm.

In order to produce interference fringes, a source of light, e.g. a sodium flame, is placed at the focus of a lens, L, and a parallel beam of light thrown on to the glass, P_1, the direction of the beam being along the arm carrying M_2. One of the rays of such a beam is shown in the figure as SN. On striking P_1 it undergoes partial reflection and refraction at N, and the refracted part is again divided at K, and later further division takes place at P_2.

We consider one of the ways in which division can take place, the recombination of two parts, which have started from one ray and have

A marked difference in this case from that in some of the foregoing is the production of interference between rays differing in phase by very many wave-lengths. In the case of Newton's rings or Fresnel's biprism, the path difference amounts to a few waves only.

The ray, SN, is refracted in P_1 and reaches K, where it is partly reflected so that it gives rise to KQT and partly refracted to give KM_2. These rays are both returned along their paths and reunite to

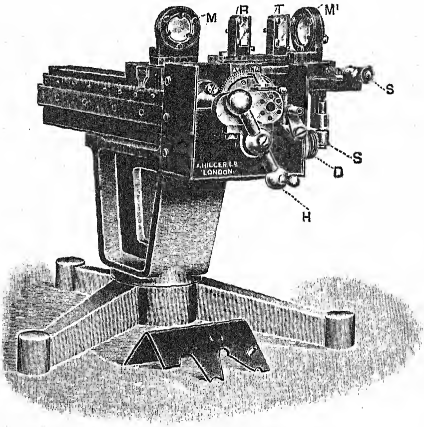


Fig. 227

produce KW. In the course of their journey each passes through the glass a distance equal to 3NK, and change in phase is brought about by the difference in the path in air. This may be varied by altering the position of M_1.

The ray, KW, may be observed by eye, and usually curved interference bands similar to those observed in the experiment on Newton's rings will appear. The curves will not be seen closed, only parts of circles can be seen.

We use here, as in the experiment on Newton's rings, an extended source of light, for the rays from each point of the source are divided and recombination takes place between rays which originally belonged to the same ray. Rays falli...

the same phase change and emerge parallel to KW, so that an optical instrument will focus them together in its focal plane.

Other rays in a direction slightly inclined to SN emerge as a set of parallel rays, slightly inclined to KW. These undergo a different phase change, and also lie in the focal plane displaced from the image due to rays parallel to KW.

It is necessary that the mirrors, M_1 and M_2, should both be vertical, and in order to allow this adjustment to be made, at the back of M_2 are three screws pressing against springs that cause it to rotate.

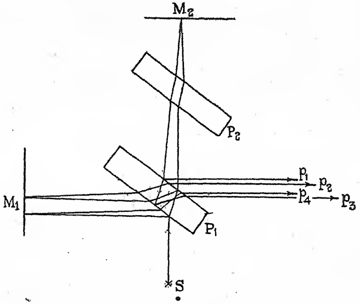


FIG. 228

If a sheet of tin carrying a fine hole is placed in the path of the incident light, four images are seen as a rule, when the eye looks in the direction, WK. The reason for this is best seen by reference to a diagram (fig. 228).

The emergent rays are marked p_1, p_2, p_3, and p_4, and it will be noted that p_2 passes through the glass plates three times; so also does p_1.

Thus, if the rays are made to coincide they will be in a condition to annul or reinforce each other according as the path difference is an odd or even number of half-wave-lengths.

The rays, p_3 and p_4, traverse the plates five times and once respectively.

If a card is placed in front of M_1, the rays p_1 and p_3 still appear, and p_1 will be the brighter since it traverses the plates only three times, so that it can be distinguished.

due to p_1 and p_2 caused to overlap by adjusting the screws behind M_1 and M_2.

When both mirrors are vertical the images coincide in pairs. One constituent of each double image lies behind the other. Those which produce interference lie along the direction, WK.

Fig. 229 illustrates the mode of production of the fringes from another point of view. The interferometer acts as if we had a source,

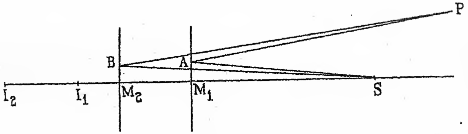


FIG. 229

S, from which rays could be reflected by two parallel mirrors, M_1 and M_2, in which they would produce images, I_1 and I_2. The distance, M_1M_2, is equal to the difference of the distances, NM_1 and NM_2, of fig. 226.

At any point, P, rays, SAP and SBP, would unite, and if their paths differed in length by $n\lambda$, a bright point would arise.

This is equivalent to stating that in this case,

$$PI_2 - PI_1 = n\lambda.$$

Thus we obtain the particular interference band at all points, P, for which this relation holds, i.e. P lies on a hyperboloid of revolution with SM_1M_2 or with WK, of fig. 226 as axis. In a plane perpendicular to that of the figure we have a circular section of the hyperboloid which explains why the fringes are circular as seen on looking along WK.

The Determination of the Wave-length of Light from a Sodium Flame or any Monochromatic Source

A striking feature of the Michelson Interferometer is the screw which displaces M_1. In the apparatus illustrated in fig. 227 the screw has a length of 200 mm. and a pitch of 1 mm. The head of the screw is furnished with a scale divided into one hundred parts, each corresponding to one-hundredth of a millimetre. The screw may be rotated by the handle seen in front of the apparatus. The small lever on the right of the front of the apparatus puts into action the slow-motion screw, one turn of which corresponds to one division on the head of the main screw. As the head of the slow-motion screw is also divided into one hundred parts, it is possible to record a motion of the mirror of one ten-thousandth of a millimetre.

In order to ...

place immediately in front of it a sheet of tin with a small hole in it just opposite the bright part of the flame, and adjust the flame and hole to lie on the level of the centre of the mirrors and plates.

Mount the lens, L, also so that its centre is at the same height as the hole, and place a piece of plane mirror between L and P_1 and move L until an image of the hole is thrown back on to the tin close to the hole. The light is then parallel as it leaves L. Remove the plane mirror and look in the direction, KW, when usually four images of the hole appear. Adjust the mirrors so that these images coincide, two by two.

Remove the tin sheet and allow the light from the flame to fall through L on to the apparatus.

As a rule very slight movements of the mirrors will bring the fringes into view if they are not already to be seen. Sometimes a slight motion of M_1 along the bed of the apparatus helps to discover the fringes.

Set up in front of P_1, on the side towards W, a sharp pin-point to mark the position of the centre of one fringe. Rotate the screw slowly by the slow-motion screw, and watch the movement of the fringes across the field of view, counting the number which pass the point and observing from the scales how far M_1 has moved.

When M_1 moves back a distance $\frac{1}{2}\lambda$ the path difference between the two rays which unite to interfere has been increased by λ, so that where a particular fringe originally appeared the neighbouring fringe now lies.

Thus, if M_1 moves a distance, l, the number of fringes which move past the point is correspondingly $\frac{2l}{\lambda}$. If these are counted, since l can be measured, we can find λ. This experiment should strictly be carried out to determine l in terms of λ, and upon this point read the account of the experiment on the interferometer on p. 355.

Do this for such monochromatic extended sources as are available.

Note also that the intensity of the fringes appears to alternate as the distance of M_1 varies. We shall make use of this fact in the next experiment.

The Determination of the Difference of Wave-length for the Sodium D Lines

Adjust the mirrors M_1 and M_2 so that their distances from N (fig. 226) are equal, as nearly as can be judged by eye.

This adjustment may be brought about accurately by observing the images of the holes formed as above in the two mirrors, and adjusting M_1 until there is no parallax between them. In this case they are equally distant from the observer, and the two mirrors are consequently equidistant from N.

The light from a sodium flame, though for many purposes considered

In the present case both sets of fringes which arise from the two waves overlap, but if M_1 is slowly moved away there is a gradual separation of the two sets, and finally the bright band of the one lies over the dark band of the other. This happens when the distance that M_1 has moved from the first position contains one more quarter of a wave of the one than of the other, for then a difference of phase, corresponding to one-half wave-length, has been added to one more than to the other. In other words, let l denote the distance moved by M_1, the additional air path added to each wave incident on M_1 is $2l$.

Suppose $2l$ contains n_1 waves of length λ_1, and n_2 of lengths λ_2. Then in this case the difference between n_1 and n_2 is one-half.

Suppose that $n_1 > n_2$ and consequently $\lambda_1 < \lambda_2$.

Then, since $n_1 = \dfrac{2l}{\lambda_1}$ and $n_2 = \dfrac{2l}{\lambda_2}$,

we have
$$2l\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = \frac{1}{2},$$

or
$$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{4l}.$$

If the two waves had equal intensity the field would become uniformly illuminated, and the fringes would disappear.

In this case, since one of the lines is more intense than the other, we get an alternation in distinctness, the brighter fringes still stand out in contrast with the adjacent less bright ones.

Note the positions of M_1 at the beginning and successively at positions where the fringes become least distinct and again distinct as M_1 goes farther away and the path difference contains one complete wave more of one colour than of the other. Do this for as many cases as possible, and if d denote the distance between the positions of M_1 in which two successive distinct sets of fringes occur, we have

$$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{2d}.$$

Assume the shorter wave in the sodium light to have a length 5890×10^{-8} cm., and deduce the difference between λ_1 and λ_2 in this case.

The Production of Coloured Fringes

These can be obtained when the mirrors are set for equal paths. Do this as in the last experiment with sodium light and then replace it by a white source. The fringes due to the different colours overlap in this position.

With sodium light, as M_1 is moved, it will be noticed that the circles change the direction of their curvature; the position required is just when the transition takes place. It is difficult to decide just when this

occurs, and it is a good plan to note the two positions of M_1 in which the fringes are definitely curved in one way and then in the other. Put in white light when M_1 occupies the position corresponding to one of these directions of curvature, and then slowly move it back to the other position. The coloured fringes will appear during this movement. M_1 must be moved slowly or the fringes change their places too rapidly to be noted.

Measurement of Refractive Index by means of Michelson's Interferometer

In the experiment on the determination of the wave-length of light from a sodium flame (p. 352) the correctness of the screw was relied upon. This procedure can be criticized on the ground that light waves are standards of length and should be used in the calibration of the screw threads of an optical instrument. This has been the subject of a number of experiments by Michelson and Benoit, who used the red line of cadmium as the standard on account of its homogeneity.

For the present purpose sodium light will be used as a standard, and the first step is to calibrate the screw thread of the interferometer in terms of the wave-length of sodium light by the method of the experiment described on p. 352.

White light fringes are then produced by the method of the preceding paragraph. These fringes are used as an index, for they indicate that the optical paths between the point N (fig. 226) and the mirrors, M_1 and M_2, are equal. If M_1 is displaced, the fringes reappear when M_2 is displaced an equivalent amount. Suppose a thin plate of a transparent substance, such as glass or mica, is placed between N and the fixed mirror. The optical distance between them is altered and the fringes disappear, but can be restored by displacing the movable mirror so that the optical paths are again made equal.

Let the thickness of the plate be d. The time taken for light to traverse this thickness in air is $\frac{d}{c}$, where c is the velocity of light in air. The time taken to traverse this distance in the medium is $\frac{\mu d}{c}$, where μ is its refractive index. Thus the difference in these intervals is $\frac{(\mu - 1)\, d}{c}$ and the insertion of the plate causes an addition to the optical path equivalent to that produced by a distance of $(\mu - 1)\, d$ in air. This can be expressed in terms of sodium light of wave-length λ, since

$$(\mu - 1)\, d = n\lambda,$$

and μ is thus determined for light of this wave-length.

The accuracy of this experiment is limited by the method of measuring the thickness, d, of the plate.

It is convenient to carry out this experiment with a thin plate of glass, such as a cover glass used for the examination of biological specimens, or a thin sheet of mica. The thickness should be measured as accurately as possible by a fine screw gauge.

Calibration of a Spectrometer by the Method of Edser and Butler

In the application of this method a thin film of air is enclosed between two parallel plates of glass upon which light is incident normally. Light is transmitted directly and also after undergoing internal reflections at the boundaries of the air film. Interference takes place between the light transmitted directly and that transmitted after reflection.

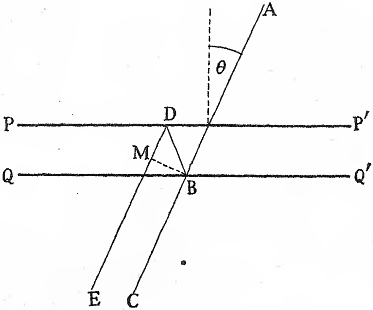


Fig. 230

Let the plates be denoted by PP′ and QQ′, and let a ray, ABC, be incident on the plates at an angle, θ. Let partial reflection occur at B and D to produce an emergent ray, DE. Between this ray and BC there is a lag owing to the path difference, BDM, where BM is perpendicular to DE. If t denote the separation of the plates

$$BD + DM = 2t \cos \theta.$$

This phase difference expressed in wave-lengths is thus $\frac{2t \cos \theta}{\lambda}$, where λ denotes the wave-length of the incident light.

A phase change occurs on reflection at B and D, since reflection occurs with the light incident from a less dense medium. The magnitude of this phase change is the equivalent of $\frac{\lambda}{2}$, but this need not be assumed in the present case, where it will be supposed to be equivalent to a

fraction, f, of a wave-length. Thus the total phase change is equal to $\frac{(2t \cos \theta + 2f\lambda)}{\lambda}$ and the condition for destructive interference is

$$\frac{(2t \cos \theta + 2f\lambda)}{\lambda} = n + \tfrac{1}{2},$$

where n is an integer.

This effect will not be easily observed unless the amplitudes of the rays, DE and BC, are comparable. This means that an appreciable fraction of the light must be reflected. In order to ensure that the surfaces have a sufficiently high reflecting power and at the same time a sufficient power of transmission, a thin film of silver should be deposited on the surfaces. Plates of this kind can be obtained ready for use, but they may also be prepared by lightly silvering.

This principle may be applied in the calibration of a spectroscope by choosing the plates of optically worked glass and separating them by a strip of paper. They should be mounted so that one plate is fixed and the other adjustable by three screws. On looking through the film at a distant light a series of images will be seen in general arising from multiple reflections at surfaces which are not quite parallel. By squeezing the surfaces by means of the screws on one side or the other the images may be made to coincide. When this is the case the surfaces are parallel. A simpler method of mounting may be used by placing a little soft wax between the plates and squeezing them into the position in which they are parallel.

They should then be placed in the path of parallel rays of light from the collimator of a spectroscope, between the collimator and the prism. A source of white light which may be obtained from the white-hot filament of an electric lamp is focused on the slit of the collimator. When the spectrum is viewed through a prism, placed in the position of minimum deviation in order to fix the position of the spectrum, the continuous sequence of colours will be seen to be crossed by dark fringes corresponding to different wave-lengths which satisfy the above relation.

Let λ_1 and λ_2 denote wave-lengths corresponding to the integral numbers, n_1 and n_2, and let the rays fall normally on the film so that $\theta = 0$.

Then $$2t\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) = n_2 - n_1.$$

The reciprocals of the wave-lengths are described as wave numbers, N, so that $$2t\,(N_2 - N_1) = n_2 - n_1.$$

The values for the differences on the right are 0, 1, 2, etc., since the values of n are integers. Consequently, the dark fringes occur at equal steps of wave numbers.

Set the telescope of the spectrometer on some particular fringe chosen as the origin of measurements. Plot on a graph the spectrometer readings corresponding to the various fringes numbered from the origin. Remove the source of white light and replace it by a source giving certain known lines in its spectrum. Two such lines are sufficient for the present purpose, but the use of more is advisable as an additional aid to calibration. The spectrum from the mercury arc is a convenient source, since it gives distinctive and well-known lines. Thus, it produces a yellow line, a green line, and a blue line, whose wave-lengths for this purpose may be taken to be 5770 Å., 5460 Å., and 4358 Å. respectively.

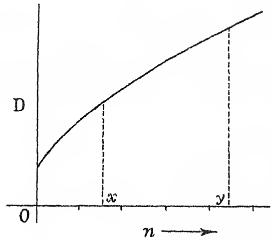


FIG. 231

Note the positions of these lines on the graph, plotting the spectrometer readings, D, as ordinates against the numbers, n, of the fringes.

Let two lines of the known spectrum fall at points whose abscissae lie at distances x and y from the nearest dark fringes, as shown in fig. 231. Suppose that the number of complete integral spaces between the two points is m, then if x and y are expressed as fractions of these spaces $(m + x + y)$ corresponds to a difference $(N_2 - N_1)$ wave numbers, where N_1 is the wave number for one line and N_2 that for the other. Thus the graduation of spaces in wave numbers is known, and from the positions of the known spectral lines the calibration of the spectrometer scale, either in wave numbers or in wave-lengths, is obtained. A graph should be drawn giving wave numbers or wave-lengths for any position of the telescope.

The experiment may be reversed and may be made to determine the thickness of the air film from a knowledge of two wave-lengths.

Suppose that λ_1 and λ_2 are wave-lengths corresponding to the two fringes of order, n_1 and n_2. Then the ratio

$$\frac{(N_2 - N_1)}{(n_2 - n_1)}$$

gives the wave number spacing of the fringes. Any two known wave-lengths, λ and λ', whose separation is known in terms of these spacings, suffice to give the fringe spacing. If the wave numbers are N and N′, the spacing is $(N' - N)/(m + x + y)$, and from this the value of t is known. In an experiment carried out in this way the magnitude of the spacing was approximately $3{\cdot}70 \times 10^{-3}$ cm.

The Fabry-Perot Etalon

This instrument consists essentially of two accurately parallel partially silvered glass surfaces. The distance apart can be varied, the separation depending on the homogeneity of the light used. A suitable distance for the experiments to be described is about 1 cm.

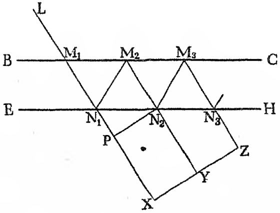


FIG. 232

The fringes produced result from the interference of light reflected and transmitted at the plane parallel boundaries of a thick plate of a medium which, in this case, is air. This type of interference fringes was first described by Haidinger. As will be seen from a study of the theory of the formation of these fringes, the partial silvering of the surfaces has an important effect on the visibility of the fringe system. The broad fringes such as occur in Michelson's interferometer are replaced by narrow, bright ones on a wide, dark background.

In fig. 232, let BC and EH represent the bounding surfaces and let LM_1 represent a ray in a medium above BC, entering the plate between these surfaces and falling at an angle of incidence, i, on EH. Let the amplitude of the light be reduced in the ratio, F, when it is transmitted at N_1, and in the ratio, f, when reflected at N_1, N_2, etc., and M_2, M_3, etc.

Let the light at M_1, when just below the bounding surface, be a, so

that, choosing M_1 as origin, the displacement at any point at a distance, r, along the ray from M_1 is represented by

$$y = a \cos 2\pi \left(\frac{t}{T} - \frac{r}{\lambda}\right),$$

where T is the period and λ the wave-length.

The ray, LM_1, passes from the air to a glass plate with BC as its lower surface emerging into the air between the plates, transmission occurring at the lower plate into the outer air again, so that the original ray and transmitted rays are parallel. A phase difference is introduced between the ray, LM_1N_1X and $LM_1N_1M_2N_2Y$, due to the path difference $N_1M_2N_2$ and N_1P. The thickness of the air film being e, this difference is

$$\delta = 2e \cos i.$$

Thus at the emergent wave-front, XYZ, path differences of δ, 2δ, etc., occur at Y, Z, etc., corresponding to the number of internal reflections and the displacements will contain $(r + \delta)$, $(r + 2\delta)$, etc., instead of r.

It is convenient to use the exponential form for the displacement

$$y = \text{real part of } \exp 2\pi i \left(\frac{t}{T} - \frac{r}{\lambda}\right).$$

Remembering that we are concerned with the real parts of the exponentials, we can write $e^{\frac{2\pi i t}{T}} e^{-\frac{2\pi i r}{\lambda}}$ for the displacement.

The amplitude of the ray emerging at N_1 is Fa and, remembering that two reflections occur before the partially reflected ray arrives at N_2, the amplitude of the ray emerging at N_2 is Ff^2a.

Thus the exponential expression for the total displacement on the wave-front, XYZ, is

$$Fae^{\frac{2\pi i t}{T}} e^{-\frac{2\pi i r}{\lambda}} \left(1 + f^2 e^{-\frac{2\pi i \delta}{\lambda}} + f^4 e^{-\frac{4\pi i \delta}{\lambda}} + \ldots\right).$$

This summation must be made to include all the contributions made by multiple internal reflections. The number of these is probably less than twenty, but the later contributions to the series can be neglected for the present purpose, and the series will be regarded as infinite.

Thus the total displacement is

$$\frac{Fae^{\frac{2\pi i t}{T}} e^{-\frac{2\pi i r}{\lambda}}}{\left(1 - f^2 e^{-\frac{2\pi i \delta}{\lambda}}\right)}.$$

When this is rationalized and the real part taken, we obtain for the total displacement:

$$Y = \frac{Fa \cos\left\{2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) - \theta\right\}}{\sqrt{1 + f^4 - 2f^2 \cos\frac{2\pi\delta}{\lambda}}},$$

where $$\tan\theta = \frac{f^2 \sin\frac{2\pi\delta}{\lambda}}{\left(1 - f^2\cos\frac{2\pi\delta}{\lambda}\right)}.$$

In this calculation no account has been taken of the change of phase which occurs on reflection, and to which reference was made in the experiment on Edser-Butler fringes. The change is $\frac{\lambda}{2}$ at each reflection giving a total difference of λ between each ray and consequently adding nothing to the phase change.

All these rays, N_1X, N_2Y_2, etc., are united in the focus of an optical instrument or by the lens of the eye which receives them, and will thus produce interference effects.

The intensity corresponding to any ray, LM_1, is proportional to the square of the amplitude and is thus proportional to

$$\frac{1}{\left(1 + f^4 - 2f^2\cos\frac{2\pi\delta}{\lambda}\right)}.$$

This has a maximum when $\frac{2\pi\delta}{\lambda}$ has the values 0, 2π, etc., i.e. when

$$\delta = n\lambda,$$
$$2e\cos i = n\lambda.$$

If this formula alone is required, it can be obtained very simply by taking the rays emerging from the plate in pairs and noting that the condition for maximum displacement in the resultant effect is that they should differ in phase by integral multiples of the wave-length.

The effect of these multiple reflections on the partially silvered surfaces can be appreciated from the formula for the intensity.

The reflecting power, R, of a surface refers to the ratio of the reflected to the incident energy, and since this is proportional to the square of the amplitude, $R = f^2$.

The intensity transmitted corresponding to the direction, i, is

$$F^2a^2\left(1 + R^2 - 2R\cos\frac{2\pi\delta}{\lambda}\right)^{-1}.$$

The maximum value of this is

$$I = F^2a^2(1 - R)^{-2}$$

so that the intensity in any direction may be expressed in the form

$$I = \frac{I_0}{1 + \frac{4R}{(1-R)^2}\sin^2\frac{\pi\delta}{\lambda}}.$$

The minimum value of I corresponds to the value of δ, which makes

$$\sin^2\frac{\pi\delta}{\lambda} = 1,$$

and the magnitude is

$$\frac{I_0(1-R)^2}{(1+R)^2}.$$

When R is not very different from unity, this may be a very small fraction of I_0, so that there is a great contrast between the maxima and the minima.

Another point may most conveniently be brought out by a numerical example. Suppose that for convenience of calculation $R = \frac{3}{4}$.

Then $$I = \frac{I_0}{1 + 48\sin^2\frac{\pi\delta}{\lambda}}.$$

A maximum occurs when $\delta = m\lambda$ and the next when $\delta = (m+1)\lambda$, m denoting an integer.

Consider the case when $\delta = (m + \frac{1}{10})\lambda$, i.e. when we have gone over to a direction lying between the two maxima, not very far from one of them.

The value of I for this direction

$$\frac{I_0}{1 + 48\sin^2\frac{\pi}{10}} = \frac{1}{6}I_0 \text{ (approx)}.$$

This means that the intensity falls off quickly as the maximum position is left, so that bright lines occur in the field of view separated by a comparatively long dark interval.

Thus, if there is a second ray in the field of slightly different wave-length, its lines will not overlap those of the first ray unless there is very little difference indeed between the two wave-lengths.

From this theory we see the influence of the partial silvering in producing sharp bright lines on a background that is almost black.

These results can be explained quantitatively by the aid of numerical examples taken from the original paper of Fabry and Perot (*Ann. de Chimie et de Physique*, Ser. 7, p. 459, 1897).

The ratio of the minimum to the maximum intensity is

$$\rho = \left(\frac{1-R}{1+R}\right)^2.$$

For the case of low reflecting power take $R = 0{\cdot}042$, the value of ρ in this case is $0{\cdot}84$, so that for a small reflecting power the maxima and minima have nearly equal intensity. For the case of high reflecting power take $R = 0{\cdot}74$ when $\rho = 0{\cdot}02$, and the minima are thus relatively very feeble.

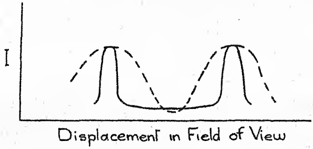


FIG. 233

Fig. 233 shows graphically the difference in two intensity curves. The continuous line is the curve for the Fabry-Perot silvered plate, while the dotted curve is for plates of low reflecting power. Note in the former the broad intervals due to a rapid falling-off of intensity, and contrast it with the latter, where the intensity falls off slowly. In the latter case, lines due to a second ray of slightly different wave-length would produce crests over the hollows and leave the field nearly uniformly bright, so that it would be impossible to distinguish the separate fringes.

In the diagram a single ray is shown which may be regarded as arising from one point of the source. Rays from this point falling on the plate at the same angle would undergo the same partial reflections and experience the same path retardations, so that a circular fringe

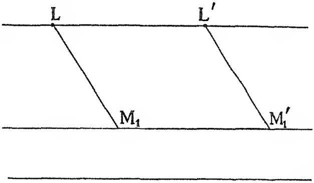


FIG. 234

would be formed. This fringe, which is formed by a single luminous point of the source, coincides with that of any other point, for if L and L′ denote two such points the parallel rays, LM_1 and $L'M_1'$, give rise to parallel emergent rays which have undergone the same path

change, and these are brought together in the focal plane of the observing system.

All rays at different angles suffer different phase changes, but are brought to a focus in the focal plane forming rings of different orders. Thus all points of a luminous source contribute to the ring system and the eye sees the fringes at infinity. For this reason, when looking for them there may at first be some difficulty in seeing them, for the eye tends to focus on the plate. It must be focused as if looking at a distant object.

Details of the Structure and Mounting of the Fabry-Perot Etalon

The plates of an etalon are illustrated in fig. 235, where they are represented by ABCD and EFGH. The faces BC and EH are silvered by cathodic deposition to increase their reflecting power, but so as to leave them partly transparent.

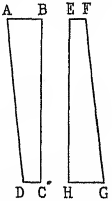


FIG. 235

The fringes are produced by multiple reflections between BC and EH and the faces, AD and FG, are slightly inclined to BC and EF in order to avoid interference effects that would occur if all the faces were parallel. AD and FG are parallel, so that light incident on one side is not deviated by refraction through the plates. The plates are made of selected glass, and there is sometimes an indication by grinding of the edges of the plates to denote which parts of the plates should be placed opposite one another in order to obtain the best parallelism. Separation is maintained in various ways, by the use of a distance piece consisting of a hollow cylinder of fused quartz, by a system of ball-bearings, or by a metal ring carrying three studs of invar. In some cases pads of rubber are provided to avoid local pressure. The plates are held against the distance pieces by springs which may be tightened by screws. In order to adjust the plates to the state of parallelism required, a first rough adjustment may be made by looking at a distant

is very inadequate, and the final adjustment must be made by the fringes themselves. If arcs only are seen, the centre of the system must be brought into view by tightening one or other of the springs. When the centre is visible the rings should keep the same size as the eye is moved; if, for example, the rings expand with new ones coming out from the centre as the eye moves to the right, the plates must be slightly squeezed on the right.

When this adjustment has been made a telescope focused for parallel light will produce a sharp ring system in the focal plane.

Determination of the Distance between the Plates of a Fabry-Perot Interferometer

The formula for this interferometer has been shown to be

$$n\lambda = 2e \cos i.$$

As i increases n decreases, so that the orders of the rings diminish as their radii increase. In general, the value of e will not be such that $\frac{2e}{\lambda}$ is an integer and the first ring will correspond to a small value of i, the rays forming it being slightly inclined to the normal. Let $\frac{2e}{\lambda} = n_0$ and let the order of the first ring be p. The difference between p and n_0 is a fraction, ε,

$$n_0 = p + \varepsilon.$$

For the first few rings from the centre the angle, i, is small enough to permit the use of the relation

$$\cos i = 1 - \frac{i^2}{2}.$$

The rings are formed in the focal plane of the object glass of a telescope, the focal length being f. If the diameter of any ring is d, it is possible to write for the small angle, i,

$$i = \frac{d}{2f}.$$

so that from $n = n_0 \cos i$ it can be deduced that

$$d = f\sqrt{\frac{8\,(n_0 - n)}{n_0}}.$$

For the first ring

$$d_1 = f\sqrt{\frac{8}{n_0}}\,\sqrt{n_0 - p} = f\sqrt{\frac{n_0}{8}}\,\sqrt{\varepsilon},$$

for the second

$$d_2 = f\sqrt{\frac{8}{n_0}}\,\sqrt{\varepsilon + 1}.$$

and in general for the mth,

$$d_m = f\sqrt{\frac{8}{n_0}}\sqrt{\varepsilon + m - 1}.$$

This formula is used to determine the value of the fraction ε and the result should be obtained with an accuracy of at least 1 per cent. There is no difficulty in obtaining any of the quantities in the formula to this degree of accuracy.

The object of the experiment is to determine the value of the separation, e, by an optical method. It is assumed to have been measured as accurately as possible by some mechanical means, and various spacing pieces give separations measured to 0·001 mm. From these values n_0 is determined for a particular wave-length. In the laboratory a convenient source is provided by electric discharge in a tube containing neon, the spectrum of which provides well-known lines. For the purpose of the experiment the wave-length of the lines is required to an accuracy of 0·1 Å.

In a particular experiment the thickness of air film was recorded as 10·040 mm., as the result of a mechanical measurement, and the wave-length of one of the lines used was 4358·3 Å. This gives for the value of n_0 46073, but there is uncertainty in the last figure on account of the corresponding uncertainties in e. In this experiment the fraction ε was found to be 0·85, so that the value of n_0 is given as 46073·85. In this number there is still a doubt about the last figure before the decimal place, which may, however, be correct to one unit. In laboratory practice the preliminary value is often not so good as this, and may be several units out, but if it is possible by further investigation to decide upon the correct digit preceding the decimal point, the value of n_0 is determined by this method to an accuracy of 1 in $4{\cdot}6 \times 10^6$. It is possible to decide upon this unit by repeating the process with another line of the spectrum. Thus, taking the example of a line of wave-length 5852·5 Å., the approximate value of n_0 is 34310. The value of the fraction in this case is 0·24, so that n_0 is given as 34310·24. If the two values are correct for these wave-lengths, the product, $n_0\lambda$, must be the same for each, viz. $2e$. It will be found by trial that this is not the case for the values given above, but that the product is the same for the values 46075·85 and 34312·24 respectively.

This test should be carried out by writing down the values in a column, as follows:

$\lambda = 4358{\cdot}3$ Å.	$\lambda' = 5852{\cdot}5$ Å.
....	
46072·85	34310·01
46073·85	34310·75
46074·85	34311·49
46075·85	34312·24

The values in the second column are obtained from those in the first by multiplying by $\frac{\lambda}{\lambda'}$. When a result is obtained containing the correct fractional part, 0·24, the corresponding pair is that which gives the correct thickness. It is possible that the same fractional part will arise from another number in the first column, but this will always be several units farther away, and the mechanical measurement ought to be accurate enough to remove doubt. Any uncertainty is removed by repeating the process for more lines of the spectrum, and it is recommended for the purpose of this exercise in the use of the instrument that three lines be used. Even this will not always decide this point exactly, and a fourth line may be necessary.

In order to obtain the fringes, the interferometer may be mounted between the collimating lens and prism of a photographic spectroscope. This arrangement of the apparatus is shown in the plate (fig. 236, at p. 1) in which the spectroscope is a direct reading constant deviation instrument. Light from a neon tube should be focused on the slit of the collimator, so that the slit becomes the extended source required to produce a range of angles of incidence on the plates. Each wave-length present produces a series of concentric circles which are deviated and dispersed by the prism. The slit limits the field of view to a diameter or to a chord of these circles and, for a particular setting of the apparatus, the pattern received in the telescope will correspond to a particular wave-length. The direct reading spectroscope will give the wave-length received, but it is better to recognize the lines received from their appearance and to use the standard measurements that will be found in tables of spectral lines.

If the slit is narrow the sections of the circles will appear as a series of dots along the chord or diameter. The dots should appear as sharp as possible, and a final adjustment of the plates to exact parallelism should be made by trial in the process of obtaining the best definition of the dots. When this adjustment is complete, the slit should be made rather wide in order to bring into view small circular arcs with the centre of the pattern along a diameter.

When the adjustments appear satisfactory, use the camera and photograph the system of fringes. The time of the exposure must be decided upon by trial, but as a beginning the time may be taken to be five minutes. This corresponds to experience under laboratory conditions, but the interval clearly depends on the intensity of the source and the sensitivity of the photographic plate to light of the wave-length used. The diameter of the rings must be measured with an accuracy sufficient to ensure a final accuracy of 1 per cent in the value of the fraction. The focal length of the lens of the telescope must be known to 1 per cent, and will usually be given as a constant of the

the fringes on the plate must also be known to the same degree of accuracy. In the derivation of the formula to be used in the experiment, it was assumed that the fringe system was produced by a single lens of focal length, f. In practice this is the lens of the telescope, but the image produced by it is focused on the photographic plate by means of another optical system, and it is thus necessary to know its magnifying power or to measure it. From a knowledge of the magnifying power, the measured diameter of the rings can be reduced to the value required in the formula.

The Determination of the Ratio $\frac{e}{m}$ for an Electron from the Normal Zeeman Effect by means of the Fabry-Perot Interferometer

This experiment depends upon the measurement of the small difference in wave-lengths which occurs in a spectral line when the source emitting it is placed in a magnetic field. An instrument of high resolving power is thus essential and the Fabry-Perot interferometer is well suited for the purpose.

Certain spectral lines undergo a simple type of splitting when the source is placed in a magnetic field. This occurs when the line results from a transition between two single terms, a good example of the phenomenon being given by the yellow mercury line of wavelength 5790·66 Å.

The effect of the magnetic field in this simple case is known as the normal Zeeman effect, and it will be observed in the experiment by receiving the light along a direction at right angles to the field. In this case the single line which appears in the case of no field is accompanied by two lines differing from it slightly in wave-length, one of greater and the other of less frequency, the differences being equal in amount. The magnitude of the difference contains the ratio $\frac{e}{m}$ and from this fact it is possible to determine the ratio from a measurement of the difference in wave-length.

The source of light is a capillary discharge tube containing a mixture of mercury vapour and neon. The presence of the neon makes the running of the tube in a strong magnetic field easier, and with a high current density the neon spectrum is scarcely visible. A heating coil is provided wound round the ends of the tube and lagged with asbestos. This is required to bring out the mercury spectrum at high fields, and the coil should be run in series with a rheostat adjusted to give out only a trace of the neon spectrum at the highest magnetic field available. Pyrex glass is necessary, since soft glass is liable to break under the heat. The capillary part of the tube is placed between the poles

allow in order to obtain fields as strong as possible. A suitable range for the magnetic field for the purpose of the experiment is from zero to 10,000 oersteds. A calibration of the field between the pole pieces is necessary so that the field may be known from the measurement of the current in the exciting coils of the magnet. This is carried out by means of a fluxmeter (p. 521), and it may be necessary to begin by destroying the residual magnetization by passing a current in the appropriate direction of such a magnitude that the fluxmeter reading becomes zero.

The Fabry-Perot plate should be adjusted in the way already described, the spacing of the plates being not less than 5 mm. For the present purpose it is not necessary to measure the plate separation to

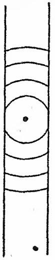


FIG. 237

the degree of accuracy obtained in the method of exact fractions; all that is required is the value given by mechanical measurement.

The light from the capillary part of the tube should be focused on the slit of the collimator of the spectrograph and a photograph should be taken for zero magnetic field. The slit should be fairly wide and set along the diameter of the rings (fig. 237).

Some final adjustment of the focusing lens may be necessary to obtain the best intensity, and this must be made by trial exposures. When the best conditions of sharpness and intensity have been obtained, a series of photographs should be taken with different fields, e.g. (1) zero; (2) 4000 oersteds; (3) 7000 oersteds; (4) 10,000 oersteds. With increasing fields, the fringes appear to broaden and then to separate, and it is possible from the measurement of the diameters to determine exactly the wave-lengths corresponding to each of the rings and their separated parts. A simpler method of reducing the results in order to give the required ratio, $\frac{e}{m}$, will, however, be described

Referring to fig. 238, let MN denote the axis of the lens which produces the fringe system in its focal plane; the axis also lies normally to the interferometer plates. Let one of the rays, AB, which leaves the plates at an angle of emergence, θ, pass through the centre of the lens. It will continue undeviated to the ring at C, which is formed by all the rays emerging at the angle θ. The diameter of this ring is thus DC, and since the rings lie in the focal plane of the lens, $BN = f$ and $\theta = \frac{d}{2f}$.

The rings may be focused on a photographic plate by an optical

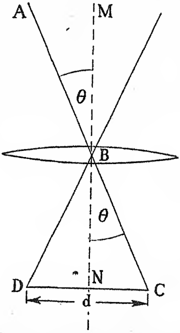


FIG. 238

system of linear magnifying power, μ. In this case if the rings obtained on the plate are of diameter d',

$$d = \frac{d'}{\mu}.$$

The formula for the instrument is

$$2t \cos \theta = N\lambda,$$

where t denotes the separation of the plates and where N is the order for the ring of angular radius θ and is of the order of magnitude 10^4. The angle θ is small enough to permit of the approximation

$$\cos \theta = (1 - \tfrac{1}{2}\theta^2),$$

and this approximation will be made in the calculation. Another approximation which will be made is that $\tan \theta = \theta$, and in the present case this represents a degree of accuracy of 1 in 4000.

Let the first ring of the system measured from the centre have the angular radius θ_0 and the diameter d_0. Let the order be N, so that

$$2t\,(1 - \tfrac{1}{2}\theta_0^2) = 2t\left(1 - \frac{d_0^2}{8f^2}\right) = \mathrm{N}\lambda.$$

A ring of order $(\mathrm{N} - n)$ will have a diameter d_n, such that

$$2t\left(1 - \frac{d_n^2}{8f^2}\right) = (\mathrm{N} - n)\,\lambda,$$

since the orders decrease on passing outward from the centre.

Thus $$d_n^2 - d_0^2 = \frac{4f^2 n\lambda}{t}.$$

With the magnetic field the pattern will consist of the original rings together with satellite rings, which are produced on account of a small change of wave-length, $\delta\lambda$. Let the mth ring, i.e. the ring of order $(\mathrm{N} - m)$ corresponding to the original wave-length, λ, have a diameter d_m.

Then $$2t\left(1 - \frac{d_m^2}{8f^2}\right) = (\mathrm{N} - m)\,\lambda.$$

If this ring is accompanied by a satellite ring of wave-length $(\lambda - \delta\lambda)$, the latter will be of the same order $(\mathrm{N} - m)$, but will have a slightly different radius, d_m', given by

$$2t\left(1 - \frac{d_m'^2}{8f^2}\right) = (\mathrm{N} - m)\,(\lambda - \delta\lambda).$$

Thus $$\frac{t}{4f^2}\,(d_m'^2 - d_m^2) = (\mathrm{N} - m)\,\delta\lambda.$$

Since N is of the order 10^4 and m is a few units only, it is sufficient for the present purpose to neglect m and consequently,

$$\frac{d_m'^2 - d_m^2}{d_n^2 - d_0^2} = \frac{\mathrm{N}\,\delta\lambda}{n\lambda}.$$

It is convenient to adapt this to the usual formula for the change of frequency occurring in the Zeeman effect, which is

$$\delta\nu = \pm\,\frac{e\mathrm{H}}{4\pi mc},$$

where e is the charge in E.S.U. and m the mass of an electron.

When the source is in a magnetic field of H oersteds this gives the change of frequency above and below the undisturbed value appropriate to the satellite lines.

Since $$\delta\nu = -\frac{c\,\delta\lambda}{\lambda^2}.$$

$$\delta\nu = -c\,\frac{n}{\mathrm{N}\cdots}\,\frac{d_m'^2 - d_m^2}{\cdots} = -\frac{cn}{\cdots}\,\frac{d_m'^2 - d_m^2}{\cdots}.$$

The value of $N\lambda$ is sufficiently close to $2t$ to allow this approximation to be made.

It will be noticed that the difference of two quantities, $d_m'^2$ and d_m^2, occurs in the formula.

Thus the formula for the calculation of $\frac{e}{m}$ is

$$\frac{e}{m} = \frac{2\pi c^2 n \; |(d_m'^2 - d_m^2)|}{Ht\,(d_n^2 - d_0^2)},$$

where $|(d_m'^2 - d_m^2)|$ denotes that the numerical value of this quantity is to be used in the formula.

Since they are of magnitudes not greatly different from one another, particular care must be taken in the determination of their difference. The term should be written in the form $(d_m' - d_m)(d_m' + d_m)$ and the difference in diameter should be obtained by setting the cross-wires of a travelling microscope over one fringe and moving them to the satellite fringe thus obtaining $\frac{(d_m' - d_m)}{2}$ directly.

In order to determine the value of $\frac{(d_n^2 - d_0^2)}{n}$ measure the diameter of the four rings, d_0, d_1, d_2, and d_3 and evaluate $(d_2^2 - d_0^2)$ and $(d_3^2 - d_1^2)$. The difference of the squares of successive rings is constant, as can be seen from the formula for $(d_n^2 - d_0^2)$, so that the mean of these two evaluations gives $\frac{2\,(d_n^2 - d_0^2)}{n}$.

This may be repeated for all exposures of the undisplaced line.

The difference $(d_m' - d_m)$ should be determined in each of these cases for the displaced rings on each side. A few remarks on the use of the tube may be helpful. It sometimes happens that the neon spectrum is too intense with the higher fields. In this case the tube should be taken down and the mercury shaken into the enlarged ends. The heating coils should be run rather hot with the tube horizontal until a slight distillation of mercury has occurred into the capillary. The appearance will be that of a dew of mercury. The tube will then run satisfactorily, and for steady running the heating coil should be just hot to the touch. Before beginning the experiment the tube should be tested in the highest field used in order to see if the neon spectrum is obtrusive.

The Lummer-Gehrcke Plate

The theory of this instrument is similar to that of the Fabry-Perot etalon. It produces spectra of a high order and has a high resolving power. In one form it is of quartz of refractive index 1·544, with a

length, 130 mm.; width, 15 mm.; and thickness, 4·5 mm. The high resolving power makes it suitable for the study of complex lines and for the measurements of the small changes in wave-length which occur

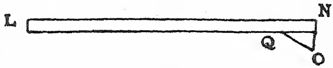


FIG. 239

in the Zeeman effect. In order to illustrate its use, an experiment will be described for the determination of the ratio $\frac{e}{m}$ from observations on this effect.

The plate is illustrated in fig. 239 and is shown mounted on its stand in position on the spectrometer in fig. 240 (at p. 1), where screws permitting adjustment in various directions can be seen.

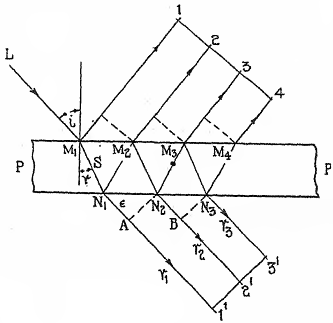


FIG. 241

The action of the plate is illustrated in fig. 241. Here LM_1 denotes an incident ray of monochromatic light making an angle, i, with the normal to the plate. After refraction the angle is r.

The figure shows the production of two beams emerging from the plate on opposite sides. The rays, $M_1 1$, $M_2 2$, $M_3 3$, etc., and $N_1 1'$, $N_2 2'$, $N_3 3'$, etc., on account of their different courses due to successive reflections and refractions, are in different phases when they reach the position denoted by 123 and $1'2'3'$ respectively. These two traces

bands. The upper beam consists of a system of bands with alternating intensities, the maxima having an intensity which may be measured by 2J, while the minima have intensity, J. Thus the effect is the same as is obtained by imparting to the whole field an intensity, J, and drawing bands across it of double the intensity. The transmitted system, however, consists of a set of maxima of intensity, J, with alternating minima of nearly zero intensity. For convenience in observation the second system is to be preferred, and, by using light at grazing incidence so that i is a right angle and r the critical angle for quartz, the sharpness of the lines is increased.

It is not possible to give here the reasoning which leads to these statements; the discussion is given in the original paper (*Ann. d. Phys.*, X, p. 457, 1903).

These points find application in the Hilger pattern.

A slot will be noticed on the right of the carrier of the plate, fig. 240. Just opposite this is a prism of such dimensions that the light along the directions M_1N_1, M_2N_2, etc., is in the critical direction.

The prism lies on the under-surface of the plate, so that the beam emerging from above is the transmitted beam.

The other beam is absorbed by the black lining of the stand on which the lower surface of the plate rests.

In fig. 239 the prism is represented by OQ.

The Theory of the Plate

Corresponding to every direction, i, of incidence there will be in the wave-front, $1'2'3'$ . . ., a variation of phase from point to point on account of the different courses taken by the rays. These rays are received by some optical instrument, for example, the telescope of the spectrometer, and focused in the focal plane. Corresponding to the waves drawn, we shall have a point image in the focal plane. These rays, however, are those lying in the plane represented by the paper. Above and below them lie rays coming from rays above and below LM_1, and parallel to LM_1, whose courses are exactly similar, so that in the focal plane above and below this point image lies a series of points forming a line. We shall work out the intensity corresponding to this line. If we consider a slightly different direction of incidence, $i + \delta i$, the emergent beam is also slightly different in direction and in its course through the plate. The phase differences in the corresponding wave-front will thus be different and the corresponding line in the focal plane will have a different intensity.

A wave travelling along a direction denoted by r will at a time, t, be represented by

$$a \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)$$

as in the theory of the Fabry-Perot etalon.

t is the time measured from some convenient instant, and r the distance from a convenient origin.

In our case the origin will be conveniently the point, N_1, and we shall consider the wave motion corresponding to the position, $1'2'3'$, at which the time is measured by t.

On refraction into the plate there is a diminution of intensity, and since the intensity is proportional to the square of the amplitude of the wave motion, we can express this by regarding the refraction as causing a diminution of amplitude, so that amplitude a in the air becomes sa in the quartz, where s is a fraction. On passing out at N_1, the amplitude is again reduced by a fraction, s'. The amplitude for the ray $N_1 1'$ is thus $ss'a$.

We may not suppose that s and s' are the same, since in one case the refraction is from air to quartz, and in the other from quartz to air.

There is also a change of intensity on reflection at the points M_2, M_3, M_4, etc., and N_1, N_2, N_3, etc. We shall suppose this to diminish the amplitude each time by a fraction, σ.

Thus the ray, $N_2 2'$, has undergone refraction at M_1 and N_2, and reflection at N_1 and M_2. The incident amplitude, a, is thus reduced to $\sigma^2 ss'a$; and similarly the amplitude of $N_3 3'$ is $\sigma^4 ss'a$.

The disturbance which gives rise to the ray, $N_1 1'$, starts out from N_1 with an amplitude, $ss'a$, and at $1'$, since it has travelled a distance, r_1, we may represent the displacement by

$$ss'\,a \cos 2\pi\left(\frac{t}{T} - \frac{r_1}{\lambda}\right).$$

Denote the foot of the perpendicular from N_2 on $N_1 1'$ by A, from N_3 on $N_2 2'$ by B, and denote the equal distances N_1A, N_2B, by ε. Let the distance, M_1N_1, be denoted by δ and the refractive index of the quartz by μ. Then the distance traversed in quartz is equivalent to the distance, $\mu\delta$ in air.

Thus the point $2'$ is at an equivalent distance $(r_2 + 2\mu\delta)$ from N_1, and since $r_2 = r_1 - \varepsilon$ we may denote the displacement at $2'$ by

$$\sigma^2 ss'a \cos 2\pi\left(\frac{t}{T} - \frac{r_1 - \varepsilon + 2\mu\delta}{\lambda}\right);$$

and similarly the displacement at $3'$ is

$$\sigma^4 ss'a \cos 2\pi\left(\frac{t}{T} - \frac{r_1 - 2\varepsilon + 4\mu\delta}{\lambda}\right).$$

The number of such terms depends upon the dimensions of the plate, and in practice this number is about 15. The terms beyond these are small on account of the successive reflections and refractions, and for the present purpose it is sufficient to assume that the series is infinite. A more detailed study of the action of the plate is necessary in order to appreciate the consequence of this approximation. In this case

falling upon a surface of separation from a rarer one, so that there is no change of phase on reflection to be considered.

As in the theory of the Fabry-Perot etalon, the summation may be made by the use of exponentials, the general term of the series being

$$ss'a\sigma^{2n} \exp 2\pi \left(\frac{t}{T} - \frac{r_1 - n\varepsilon + 2n\mu\delta}{\lambda}\right)$$

including $n = 0$.

The summation can be carried out as before, and it will be found that the resultant amplitude is $ss'a\ (1 + \sigma^4 - 2\sigma^2 \cos \beta)^{-\frac{1}{2}}$, where $\beta = \dfrac{2\pi\ (2\mu\delta - \varepsilon)}{\lambda}$.

The remarks on the location of the interference pattern at infinity and on the contributions made by the various parts of an extensive luminous source, such as the slit of a spectroscope, apply here.

The expression $(1 + \sigma^4 - 2\sigma^2 \cos \beta) = (1 - \sigma^2)\ 2 + 4\sigma^2 \sin \dfrac{\beta}{2}$ and has minima at $\beta = 0,\ 2\pi,\ 4\pi$, etc., and maxima at $\beta = \pi,\ 3\pi,\ 5\pi$, etc. Thus lines of maximum intensity correspond to the former and the formula for the plate is

$$2\mu\delta - \varepsilon = n\lambda.$$

If the thickness of the plate is d, $\delta = d \sec r$ and

$$\varepsilon = M_1M_2 \sin i = 2d \tan r \sin i = 2d \tan r\mu \sin r.$$

Thus $$2\mu\delta - \varepsilon = 2\mu d \cos r = n\lambda.$$

A short proof of this formula, which does not give any indication of the intensity of the lines, is obtained from the consideration that the path difference between consecutive rays, e.g. $N_1 1'$ and $N_2 2'$, is equal to $(2\mu\delta - \varepsilon)$. All rays emerging from the plate may be taken in pairs separated by the same distance, N_1N_2. These interfere to produce brightness if $(2\mu\delta - \varepsilon)$ is a whole number of wave-lengths, i.e. when the condition $2\mu d \cos r = n\lambda$ is satisfied.

The system of interference fringes is produced by passing a beam of light into the prism attached to the plate (fig. 242). This reflects the rays into the plate, so that the angle, r, is nearly the critical angle and the emergent rays are almost at grazing emergence.

The pattern which consists of nearly straight lines is formed in the focal plane of the objective of the telescope used to focus them.

If we refer to fig. 240 and note the position of the plate it is clear that the different orders come out from the plate above one another. As the path difference increases the angle of emergence decreases, so that the higher orders will lie higher in the field of view than the lower in the case of a telescope with an erecting eyepiece.

The different orders correspond to angles very slightly differing from grazing incidence.

By substituting the value, $\mu = 1{\cdot}544$, and the value for the critical angle, $r = 40°22'$, together with the value of d given above, we find that for $\lambda = 5890$ tenth metres (10^{-10} m.), the order is approximately 18,000.

In order to set the instrument in position, the spectrometer is first set up in the manner previously described, and the slit illuminated by means of some convenient monochromatic light. The plate is placed in position and adjusted by the screws until the brightest image is obtained in the eyepiece. Since the orders lie one above another, a

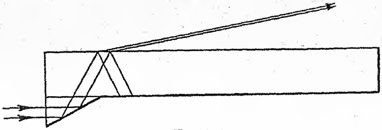


Fig. 242

vertical slit cannot be used, for the different orders will appear overlapping the image of the slit. Thus the crossed slits must be used with a small rectangular source. The length of the slit need not be very small, for on account of the dimensions of NO (fig. 239), only a fraction of the slit is effective in producing bands. The slit should be small enough to avoid overlapping, but wide enough to produce intense bands.

The images may then be viewed or photographed as desired.

The Determination of the Ratio $\frac{e}{m}$ for an Electron from the Normal Zeeman Effect by means of the Lummer Plate

In the similar experiment in which the Fabry-Perot interferometer was used the source was examined by rays of light emitted perpendicularly to the magnetic lines of force. Three lines appear in the field of view in that case. If the source is examined by rays travelling in the direction of the lines of force, two lines appear separated from the position of the original line by equal amounts to either side, the displacement being the same as in the former case. There is a difference in the state of polarization in the two cases, for in the first case the three lines are linearly polarized, the two displaced lines in planes at right angles to the plane of the undisplaced line, while in the second case the lines are circularly polarized in opposite directions. The magnitude of the change of wave-length has already been given in terms of $\frac{e}{m}$ in E.S.U., viz.

$$\delta\lambda = \frac{eH\lambda^2}{}.$$

In this experiment the pole pieces must be drilled centrally along the direction of the lines of force running between them, and the apparatus must be aligned so that a capillary discharge tube placed between the pole pieces emits rays through one pole piece to the slit of the collimator of the spectrometer. The alignment is assisted by placing a bright light, e.g. a sodium flame on the far side of the hole and obtaining the pattern from the shorter sodium line by setting the drum at 5890 Å. The source for which the effect is examined may again be a tube containing neon and mercury vapour which was described in the Fabry-Perot experiment, the yellow line of mercury (5790·66 Å.) being used again. Adjustments are made in order to obtain a bright and well-defined fringe system as described above.

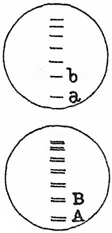


FIG. 243

The magnetic field must be calibrated by means of a fluxmeter, so that it is known from the current in the exciting coils.

On applying the magnetic field it will be observed that the lines broaden, and in some cases actual separation will occur, as in fig. 243.

The state of polarization may be tested by intercepting the light just before it enters the object glass of the telescope by a quarter-wave plate.

The plate must be suitable for the particular wave-length concerned —the usual plate met with in laboratories will be suitable to examine the yellow light. This part of the experiment is introduced so that the student may take the opportunity of verifying the character of the vibrations in the lines.

The quarter-wave plate reduces the circular vibrations to two linear vibrations at right angles to one another and on examining these with a Nicol it will be found that when one line is cut out the other is present in the field.

After carefully adjusting the apparatus so that the effect is seen by eye, attach the camera and photograph the spectrum without the

magnetic field. By means of the rack and pinion raise the plate, put on the field, and take a second photograph below the first.

The plates when developed will give lines as in fig. 243.

Separation of the lines is seen in the higher orders, e.g. at A and B. Let a denote the line when no field is applied to the source, and A the corresponding line with the field applied. Measure the mean displacement of the components of A by means of a microscope, and measure the separation between the two successive orders a and b.

Denote these distances by l and L respectively.

We proceed to work out a formula, showing how $\delta\lambda$ may be derived from the ratio $\frac{l}{\mathrm{L}}$.

Let the angle, i, which denotes the angle of emergence from the plate corresponding to the order, n, change to $(i + \Delta\mathrm{I})$ in the order $(n + 1)$. At the same time let r become $(r + \Delta\mathrm{R})$.

Since these angular changes are small, it follows from

$$2\mu d \cos r = n\lambda, \qquad 2\mu d \cos (r + \Delta\mathrm{R}) = (n + 1)\lambda,$$

$$\sin i = \mu \sin r, \qquad \sin (i + \Delta\mathrm{I}) = \mu \sin (r + \Delta\mathrm{R})$$

that

$$2d \tan r \cos i\ \Delta\mathrm{I} = -\lambda.$$

$\Delta\mathrm{I}$ is the angle between rays which produce two consecutive lines, a and b, on the photographic plate (fig. 239).

Let Δi denote the angular difference which corresponds to two lines in the order n of wave-lengths λ and $(\lambda + \Delta\lambda)$. This angle corresponds to the displacement between the components of the line A.

Since the plate is at a fixed position and the angles are small,

$$\frac{\Delta i}{\Delta\mathrm{I}} = \frac{l}{\mathrm{L}}.$$

From the equation for the plate

$$-2\mu d \sin r\,\Delta r + 2d \cos r \frac{d\mu}{d\lambda}\Delta\lambda = n\,\Delta\lambda.$$

The variation in the refractive index must be included because of the change of wave-length.

From the relation $\qquad \sin i = \mu \sin r,$

it follows that $\qquad \cos i\,\Delta i = \mu \cos r\,\Delta r + \sin r \frac{d\mu}{d\lambda}\Delta\lambda,$

and combining these results in order to eliminate Δr

$$2d \tan r \cos i\,\Delta i = \left(\frac{2d}{\cos r}\frac{d\mu}{d\lambda} - n\right)\Delta\lambda.$$

Substituting for n from the equation for the plate

$$\tan r \cos i\,\Delta i = -\left(\frac{\mu}{\lambda}\cos r - \frac{1}{\cos r}\frac{d\mu}{d\lambda}\right)\Delta\lambda.$$

Thus $$\frac{l}{L} = \frac{\Delta i}{\Delta I} = \frac{2d}{\lambda}\left(\frac{\mu}{\lambda}\cos r - \frac{1}{\cos r}\frac{d\mu}{d\lambda}\right)\Delta\lambda.$$

The angle r in this formula is the critical angle for quartz, since the emerging rays are all very nearly grazing the surface.

Thus $$\mu \cos r = \mu\sqrt{1 - \sin^2 r} = \sqrt{\mu^2 - 1}$$

and $$\frac{l}{L} = \frac{2d}{\lambda^2\sqrt{\mu^2 - 1}}\left(\mu^2 - 1 - \mu\lambda\frac{d\mu}{d\lambda}\right)\Delta\lambda$$

$$= \frac{eHd}{2\pi mc^2\sqrt{\mu^2 - 1}}\left(\mu^2 - 1 - \mu\lambda\frac{d\mu}{d\lambda}\right).$$

The table below which gives values of μ for different values of λ should be used to draw a graph from which $\frac{d\mu}{d\lambda}$ may be obtained by measuring the slope at the particular wave-length of the source.

Several orders usually show sufficient separation for the purpose, and as many as possible should be used in order to obtain an average value of the ratio $\frac{l}{L}$ for as many results as possible.

REFRACTIVE INDEX OF QUARTZ IN THE VISIBLE SPECTRUM AT 10°

Wave-lengths (in $\mu\mu$)	Refractive index
396	1·5581
410	1·5565
434	1·5540
486	1·5497
508	1·5482
533	1·5468
589	1·5442
643	1·5423
686	1·5410
760	1·5392
768	1·5390

Measurement of Wave-length by Diffraction at a Straight-edge

The theory of this experiment will be found in Schuster's 'Theory of Optics' in Chapter V.

In the first few sections of the chapter it is shown that on observing a straight-edge illuminated by monochromatic light from a narrow slit, a series of alternate bright and dark bands will be seen.

If S denotes the slit (fig. 244), EF the straight-edge fixed parallel to it, and PQ a plane perpendicular to SE, we shall have a series of bands along PQ. Let Q be the nth bright band from P, where P is the point of intersection of SE and the plane.

Denote PQ by x_n, then it can be shown that

$$x_n = \frac{1}{2}\sqrt{\frac{p\lambda\,(4n-1)\,(p+q)}{q}}.$$

In this formula p is the distance, EP, and q is equal to SE, the distance between edge and slit.

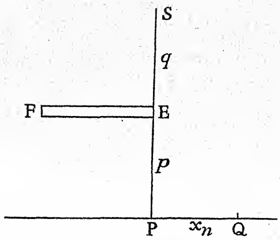


FIG. 244

The bands are not regularly spaced as are interference fringes.

To carry out the measurement, set up in one of the stands of the optical bench a straight-edge parallel to the slit which is illuminated by sodium light. First adjust the parallelism by eye, and finally make a slight rotation by the screw of the holder until the bands are most distinct; then make the slit as narrow as is compatible with sufficient illumination. Observe the fringes with the travelling microscope, and measure the distance between the first or second bright band and one of the most distant that can be seen plainly.

If this is the nth and the first observed is the mth, we have

$$x_n - x_m = \frac{1}{2}\sqrt{\frac{p\lambda\,(p+q)}{q}}\left(\sqrt{4n-1} - \sqrt{4m-1}\right).$$

The student should take the opportunity of observing the diffraction fringes arising from the light passing a narrow wire, needle-point and narrow slit.

In the case of the narrow wire, diffraction bands unequally spaced will be seen outside the geometrical shadow. Within it, a series of equally spaced bands will be observed. These may be described as interference fringes due to the two parts of the wave, one on either

side of the wire. The effects of these are equal to those of the two half-period zones which lie at the edges of the wire so that they act like two sources at a small distance apart.

Determination of the Wave-length of Light by means of a Plane Diffraction Grating

A diffraction grating is made by ruling a large number of equidistant parallel straight lines on glass. The lines are ruled by a diamond point moved by an automatic dividing engine containing a very fine micrometer screw which moves sideways between each stroke. A photographic replica of a plate made in this way is often used in its place.

In handling the grating do not touch the faces of the glass, hold it between thumb and finger by the edges.

Adjust the collimator and telescope for parallel rays in the usual way and observe the direct image of the slit, noting how it lies in the field of view.

Set up the grating with its face normal to one side, EF, of the triangle formed by the levelling screws (cf. fig. 168). Throw an image of the slit into the telescope by reflection from one face of the grating, and adjust the screws to bring it into the same part of the field as that occupied by the direct image. This makes the faces vertical.

In order to set the grating at right angles to the rays adjust the collimator and telescope at right angles to each other, and turn the table until the slit is reflected on the cross-wires of the telescope. Then turn the table a further 45°.

It now remains to tilt the grating so that the lines are parallel to the slit.

View the first diffracted image, making the slit as narrow as is convenient, and adjust the screw D until the best image is obtained. The lines and slit are then parallel.

Find the diffracted images on both sides of the line of direct vision. It will be easy to observe two orders, and if a bright light is used and an image of the brightest part of the flame thrown on the slit by a short focus lens the third may be seen also.

If $(a + b)$ is the width of the grating element the formula is

$$(a + b) \sin \theta = n\lambda$$

for normal incidence, θ being the angle of diffraction, λ the wave-length of the incident light, and n the order of the spectrum.

Obtain θ by taking half the angular distance between the corresponding images on each side.

$(a + b)$ is deduced from the number of rulings on the grating.

Use any source of monochromatic light or light giving well-marked lines as that from a discharge tube containing hydrogen which gives three well-marked lines, red, green, and violet, known as C, F, and H_γ.

The above formula is obtained by considering rays in pairs passing through adjacent clear spaces of the grating.

For example consider the two spaces AB and CD.

We may consider the rays passing through them in pairs, taking together those rays which are symmetrically situated, as for example, QLQ′ and TMT′.

These reach the grating in the same phase, and the line ABCDE represents a section in the plane of the figure of the wave-front incident on the grating.

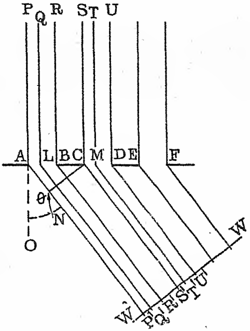


FIG. 245

Consider a wave-front, WW, at a later interval and suppose that it makes an angle, θ, with the plane of the grating.

Draw the rays AP′, BR′, etc., perpendicular to this wave-front and making the angle, θ, with the normal, AO, to the grating. These rays are all received by a telescope and united in the focal plane. Their paths from the grating to the instrument differ so that they reach the focal plane with different phases. This phase difference is due to the difference of path traversed after leaving the grating.

Take the case of the two rays, AP′ and CS′. Their path difference is AN, where N is the foot of the perpendicular from C on AP′. Any other pair of rays, e.g. LQ′ and MT′, situated symmetrically in AB and CD have the same path difference. This holds for all pairs of rays symmetrically situated. Thus the phase difference on arrival in the apparatus that receives them is the same for each pair.

We shall thus have reinforcement if AN is equal to a whole number of wave-lengths.

Now $$AN = AC \sin \theta.$$

If a is the width of a space and b that of a line,

$$AC = a + b,$$

$(a + b)$ is called a grating element. Thus a bright line occurs in a direction θ, provided that

$$(a + b) \sin \theta = n\lambda.$$

The Plane Reflection Grating

This reflection grating consists of a plain polished sheet of metal, across which parallel lines are drawn very closely together as in the transmission grating.

It is necessary to mount the polished face vertically, and this is done as in the previous case.

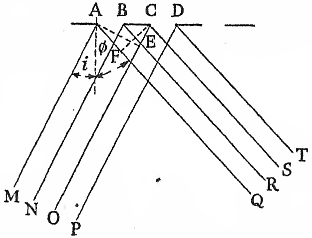


Fig. 246

The light does not fall from the collimator normally on the grating but at an angle, i.

In order to measure i, the grating is first set with the polished face normal to the rays from the collimator which, together with the telescope, has been adjusted for parallel rays.

The table carrying the grating is turned through a definite angle, say 10°, and is then fixed.

Suppose rays from any direction make θ with the normal to the grating and are received in the telescope.

Consider any three rays, MA, NB, and OC (fig. 246), incident on the grating element ABC. AB is polished while BC is the position of the line where the polish is scratched and where the incident rays are absorbed.

The path difference between the extreme rays MAQ and OCS is AF$\sim$CE, where the dotted lines, AE and CF, denote incident and

reflected wave-fronts respectively. If $AB = a$, $BC = b$, this difference is equal to

$$(a + b)(\sin \theta \sim \sin i).$$

It is possible to divide up the bundle of rays falling on AB and CD into pairs, one from each bundle, having this same difference of path. In order to do this it is only necessary to choose rays occupying the same relative positions in the two bundles.

If this path difference is a whole number of wave-lengths and the emergent parallel rays AQ, BR, CS, etc., are brought to a focus by a telescope, we shall get a bright image due to reinforcement of the rays. We thus have

$$(a + b)(\sin \theta \sim \sin i) = n\lambda,$$

where n may have the values 1, 2, 3, etc. The corresponding values of θ, say, θ_1, θ_2, θ_3, etc., give the directions in which the different orders of diffracted images are seen.

If the observation is made from a direction on the same side of the normal as i, the formula is

$$(a + b)(\sin i + \sin \theta) = n\lambda.$$

Obtain first the reading of the scale marking the position of the telescope when it is directed towards the slit with no grating intervening. As above, set the normal to the grating at some definite angle, i, to the incident rays. Move the telescope round to receive light in a direction corresponding to $\theta = i$ on the opposite side of the normal to i. This corresponds to order zero—it is the position where the ordinary reflected beam is received. Now move round to the next position where a distinct image can be found. This will correspond to $n = 1$. Carry out this process as long as it is possible to observe images at all. The higher orders get fainter but resolve the lines more than the lower.

If the number of lines per unit length is given for the particular grating, it is possible to deduce $(a+b)$, which must be expressed in centimetres. From the values of i, θ, and n, it is then possible to evaluate λ. It should be possible to obtain separation of the sodium lines in the second order, and the wave-length for each constituent should be calculated.

Use three or four different values of i and take the mean of the series of values of λ obtained.

Resolving Power of the Object Glass of a Telescope

Let a parallel beam of monochromatic light fall on a slit, AB, and be focused in the focal plane of a lens placed to receive it on the other side of the slit. This is illustrated in fig. 247, where the lens is placed at some distance from the slit for convenience of showing the effect

of rays received in different directions. In practice, the lens may be in contact with the plane of the slit.

According to Huyghen's principle each element of the slit, e.g. PQ, will act as a source and emit rays into the lens. In the figure the rays emitted in a direction inclined at θ to the normal to the slit are shown. The normal rays travel along paths of equal length and are

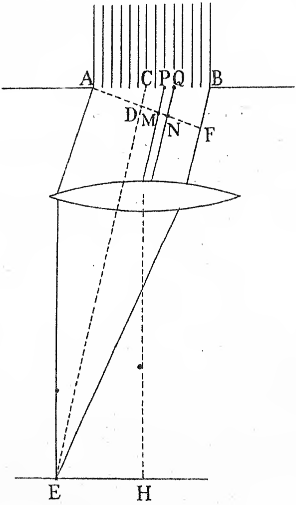


FIG. 247

brought to a focus at the central point, H, of the focal plane of the lens. The rays incident on AB are supposed to be in the same phase, so that there will be reinforcement at H. Rays received in the direction, BF, are also focused in the focal plane of the lens, but at a point, E, lying to one side of H, and these do not travel along equal paths from the slit. Consequently, in superposing the rays in order to find the total amplitude at E it is necessary to take account of the differing phases in the individual rays. The total amplitude may be calculated by observing that no change in phase occurs beyond the plane of AF drawn normal to the rays, and thus making an angle, BAF $= \theta$, with

the plane of the slit. Thus the total amplitude in the plane of AF is concentrated at E.

The slit will be considered to be of width a and very long, so that the effect may be expressed as the amplitude per unit length of the slit. Let the total amplitude for the width, AB, be A, and let it be distributed uniformly. This means that the disturbance arriving at AB may be denoted by $\frac{A}{a}\sin\frac{2\pi t}{T}$ per unit width. This disturbance is transmitted as a wave and at a distance, r, measured from a point in AB along a ray, the magnitude of the disturbance at time, t, is

$$y = A_0 \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right),$$

T denoting the period, λ the wave-length, and A_0 the amplitude at the point of origin in AB.

Thus, at time, t, the effect at MN arising from the element, PQ, of width, dx, is given by

$$dy = \frac{A}{a}\,dx \sin 2\pi\left(\frac{t}{T} - \frac{x\sin\theta}{\lambda}\right),$$

where AP $= x$ and consequently PM $= x\sin\theta$.

The total effect in the plane of AF is obtained by integrating this quantity from $x = 0$ to $x = a$. The magnitude is thus

$$y = A\frac{\sin\varphi}{\varphi}\sin\left(\frac{2\pi t}{T} - \varphi\right),$$

where $\varphi = \frac{\pi a\sin\theta}{\lambda}$ and denotes the phase difference in the central ray, CDE, for CD $= \frac{a}{2}\sin\theta$.

The effect can therefore be described as being due to a central ray of amplitude, $A\left(\frac{\sin\varphi}{\varphi}\right)$, with a phase retardation, $\frac{\pi a\sin\theta}{\lambda}$, appropriate to its inclination, θ.

The intensity in this direction is proportional to $\left(\frac{\sin\varphi}{\varphi}\right)^2$ and maxima and minima of illumination occur in certain directions which may be deduced from this expression.

The maxima occur for values of $\varphi = 0, 1{\cdot}43\pi, 2{\cdot}46\pi, 3{\cdot}47\pi$, etc., and minima for the values $\pi, 2\pi, 3\pi, 4\pi$, etc.

The graph drawn in fig. 248 shows the relation between intensity and values of φ.

When the light is examined at H, the intensity corresponds to the ordinate at I, and on moving round we come to a place, E, say, where there is darkness corresponding to the zero ordinate at π.

In this case $$\varphi = \pi = \pi \cdot \frac{BF}{\lambda},$$

i.e. $$BF = \lambda.$$

Thus a minimum occurs in the direction in which $BF = \lambda$.

Now the angular distance between CE and CH is the same as angle BAF, and since this is small it is measured by $\frac{\lambda}{AB}$ or $\lambda \div$ (breadth of incident beam).

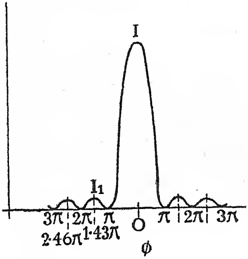


FIG. 248

If another beam were incident on AB in the direction along CE, there would be a maximum for this beam at E, and it would overlap the minimum of the former.

The intensity curves might then be represented on the same diagram as in fig. 249. The curve on the left represents the intensity curve

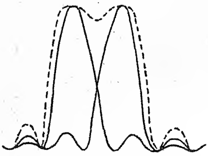


FIG. 249

for the second direction while the upper dotted curve shows the two compounded.

The resultant has two pronounced maxima with an appreciable dip between. It will thus be possible, by the aid of this falling-off in intensity between two bright regions, to distinguish the two beams or they will be 'resolved'. Moreover, the angle between the two directions is ECH

or $\lambda \div$ (width of beam). This is taken as a limiting case, and the resolving power is measured by this ratio. If the angle between the incident beams is less than this the two maxima cease to be distinguishable as two and blend into one.

In order to verify this theory, a telescope is fitted with an adjustable slit which is placed as close as possible to the object glass. The width of the slit is carefully measured by means of a micrometer microscope.

A suitable object for this experiment may be made by coating a sheet of plane glass with tinfoil and cutting two fine parallel lines in the foil with a razor blade at a distance of two or three millimetres apart. When these are illuminated with a sodium flame they provide two bright slit-sources. This object should be placed at different distances from the object glass of the telescope, the aperture of which may be varied by means of an adjustable slit placed immediately in front.

A certain minimum width of this slit will be found for which the two lines appear as separate lines. This width varies with the distance from the object glass to the two slits, and for smaller widths the lines appear as one.

A table is made of the minimum widths of the slit and the corresponding distances.

The angle subtended by the two fine lines at the object glass is measured by $\frac{d}{D}$, where d is their separation and D the distance.

The theory above described shows that the value of this angle is $\frac{\lambda}{a}$, where λ is the wave-length of the light, and a is the width of the aperture.

The object of the experiment is to compare the theoretical and practical resolving powers, the former determined by $\frac{\lambda}{a}$ and the latter by $\frac{d}{D}$.

An examination of the practical resolving powers measured in the Wheatstone Laboratory of King's College, London, during the past session has shown that the values obtained were about 20 per cent greater than the theoretical value.

Repeat the experiment with two point sources and circular apertures of different diameters.

In this case the field of view contains a bright small central circle with concentric alternate bright and dark rings. If two sources close together are such that the bright centre due to one falls on the first dark ring of the other, the two sources just cease to be distinguishable as separate.

Sir G. Airy has shown that if d is the diameter of the aperture, θ the angle subtended when separation ceases,

$$\theta = 1{\cdot}22\,\frac{\lambda}{d}.$$

The point sources should be two fine holes in a sheet of tin. They should be illuminated by monochromatic light and placed at such a distance from the object glass that they just cease to be seen as two holes.

Place the telescope at different distances from the holes and focus it on them. Adjust the aperture diameter until the holes just cease to be distinguishable as two separate sources of light. Measure the distance between the centres of the holes by means of a micrometer microscope, and the distance between the plane of the holes and the aperture by a metre rule. From these measurements deduce θ and compare it with the theoretical value, $\frac{1{\cdot}22\lambda}{d}$, in each case.

The Determination of the Diameter of Small Bodies by the Method of Haloes and by Direct Measurement

The result due to Airy which has just been quoted can be used to determine the diameter of small spherical bodies placed in the path of a parallel beam of light. This result was given for application to the diffraction pattern of a small circular aperture in a screen, but by Babinet's principle it may be applied to the case of a small, opaque body which replaces the aperture when the screen is removed.

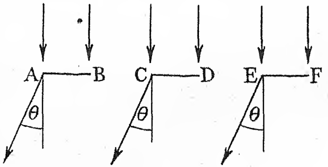


Fig. 250

If a number of such bodies be scattered over a glass surface and parallel rays allowed to fall on them the diffraction pattern observed consists of a series of concentric rings. The pattern is like that which would result from a single body, but is more intense. The reason for this is illustrated in fig. 250.

If AB, CD, and EF denote three bodies in the path of parallel rays of monochromatic light, rays of equal intensity will be diffracted in the direction, θ. These parallel rays will be focused at a point in the focal plane of the receiving instrument or on the retina of the eye.

The complete pattern results from a rotation of AF about its normal, so that a ring occurs in the focal plane corresponding to the angle, θ.

This is the principle of Young's eriometer designed to compare the diameters of small spherical particles, and it is upon Young's method that this experiment is based.

A small hole of 0·25 mm. to 0·5 mm. is made in a sheet of metal or cardboard and on a line passing through the centre of the hole drawn in the plane of the sheet small holes are made on either side of the centre spaced at distances 1 cm. apart. These holes should be as small as possible and are only for the purpose of marking the equal distances.

A piece of plane glass is lightly dusted with lycopodium spores and held in a stand between the observer and central hole. A series of bright and dark rings will be seen concentric with the hole. According to Airy's calculation the first dark ring is seen in a direction, θ_1, with the direction of the incident light, where $\theta_1 = \frac{1{\cdot}22\lambda}{d}$, λ and d denoting the wave-length of the light and the diameter of the particles respectively.

The second and third dark rings are seen in the directions $\theta_2 = \frac{2{\cdot}27\lambda}{d}$ and $\theta_3 = \frac{3{\cdot}23\lambda}{d}$.

The first, second, and third bright rings are seen in the directions

$$\varphi_1 = \frac{1{\cdot}63\lambda}{d}, \quad \varphi_2 = \frac{2{\cdot}68\lambda}{d}, \quad \text{and} \quad \varphi_3 = \frac{3{\cdot}70\lambda}{d}.$$

By moving the glass plate, the diameters of these rings may be made to vary, and in any position the diameter may be measured by means of the series of holes made along the line on the screen.

The mean diameter of the first dark ring should be made to coincide with a whole number of intervals marked by the small holes. Let the distance between the plate and the central hole be D_1 and the radius of the ring r_1, so that the direction in which the ring is seen is $\theta_1 = \frac{r_1}{D_1}$, the angle being small.

Then
$$\frac{r_1}{D_1} = \frac{1{\cdot}22\lambda}{d}.$$

If D_1 is varied, r_1 changes in proportion and a graph of r_1 against D_1 will have a slope from which d may be deduced, provided that λ is known. It is convenient to use a sodium flame as a source of light.

The process should be repeated for as many rings as possible, e.g. for the second dark ring the relation is

$$\frac{r_2}{D_2} = \frac{2{\cdot}27\lambda}{d}.$$

It will be found that the dark rings can be used with greater accuracy than the bright ones.

The light is not incident in the form of a parallel beam, but the pencil incident upon the spores producing the narrow pencil which falls into the eye, can be treated as a small parallel bundle of rays. The experiment should be repeated for holes of varying diameter in order to investigate the existence of an optimum value for the diameter.

Plot the results obtained for the diameter of the spores against the diameters used for the central hole in the screen. The best value should be that obtained by extrapolation to the zero value of the diameter of the holes.

This experiment offers an opportunity for a study of the law of errors. Lycopodium spores, which are the spores of lycopodium club moss, are chosen for this experiment because they are of extraordinary uniformity. There are, however, variations in size due to accidental influences which occur in the course of their growth. In addition, accidental errors occur in observation, and by recording the measurement of a large number of spores and by plotting the number against the diameter, the nature of the error curve can be investigated.

The diameters of the spores may be measured with a microscope provided with a scale. This scale may need calibration, and this can be done by examining a glass scale provided with 0·1 mm. divisions, in order to see how many microscope scale divisions correspond to 0·1 mm.

In a particular case it was found that one division was equivalent to 0·00263 mm. The greater the number of spores examined the better for the statistical part of this experiment, but in practice the number is necessarily limited. It has been found that 200 is a convenient number for attaining the object of this exercise. The results should be arranged in tabular form.

Range of group in scale divisions	Mean of group	No. of particles	Range of group in scale divisions	Mean of group	No. of particles
8·0– 8·5	8·25	1	11·5–12·0	11·75	50
8·5– 9·0	8·75	1	12·0–12·5	12·25	28
9·0– 9·5	9·25	1	12·5–13·0	12·75	23
9·5–10·0	9·75	5	13·0–13·5	13·5	13
10·0–10·5	10·25	9	13·5–14·0	13·75	6
10·5–11·0	10·75	26	14·0–14·5	14·25	1
11·0–11·5	11·25	36	—	—	—

The number of particles corresponding to the various mean values should be plotted on a graph and a smooth curve drawn through them.

By drawing a series of rectangles with bases equal to the range of the group and of heights equal to the number of observations in the group, a histogram is obtained, which is one of the conventional ways of exhibiting the variations. The curve obtained is usually symmetrical and the value of the diameter at the maximum frequency may be taken as the best value.

The Determination of the Width of a Slit by Diffraction

In the experiment on the resolving power of the object glass of a telescope it was shown that when parallel rays of light fall normally on a slit of width, AB, the amplitude of the total disturbance seen in the direction, BF (fig. 243), is given by the expression $\frac{\text{A} \sin \varphi}{\varphi}$, where $\varphi = \frac{\pi \text{BF}}{\lambda}$. If the width of the slit is denoted by a and the direction, BF, makes an angle, θ, with the normal to the slit,

$$\text{BF} = a \sin \theta,$$

and the amplitude is proportional to

$$\frac{\sin\left(\frac{\pi a \sin \theta}{\lambda}\right)}{\frac{\pi a \sin \theta}{\lambda}}.$$

This expression has minima corresponding to the directions given by

$$\frac{\pi a \sin \theta}{\lambda} = \pi,\ 2\pi,\ 3\pi, \text{ etc.}$$

When θ is small $\quad \theta = \frac{m\lambda}{a}$, where $m = 1, 2, 3$, etc.

Let the slit of the collimator of a spectroscope be strongly illuminated by light from a sodium flame either by direct illumination or by focusing an image of the source on the slit. A beam of parallel light is then produced by the collimator, G. It is focused in the usual way and may be allowed to fall on a slit plated vertically at the centre of the table of the spectrometer. The slit may be made conveniently by mounting a pair of safety-razor blades on a stand and making them adjustable. When approximate adjustments have been made, parallel bands will be seen in the focal plane on looking through the telescope. They should be made as sharp as possible by rotating the slit and as bright as possible by using an intense source of light and adjusting the slit of the collimator.

After setting the cross-wires of the eyepiece of the telescope on the centre of the diffraction pattern, measure the angles to as many successive dark lines as possible, on both sides of the centre. By assuming

the wave-length of the light used in the experiment, deduce the width of the slit from the formula. Great care must be taken in the reading of the angles on the spectrometer, since the angle, θ, is a small angle obtained from differences. The angle for each order should be obtained by swinging the telescope from the corresponding minimum on one side to that on the other, and by halving the difference. If the maxima are used, the formula for θ for the different bright bands is

$$\theta = 0, \quad \frac{1{\cdot}43\lambda}{a}, \quad \frac{2{\cdot}46\lambda}{a}, \quad \frac{3{\cdot}47\lambda}{a}, \text{ etc.}$$

After determining the width in this way, the results should be compared with direct readings made with a micrometer microscope, and the experiment should be repeated several times with slits of different widths.

Michelson's Method of Measuring Stellar Diameters

The experiment on the resolving power of the object glass of a telescope leads to the conclusion that rays of light travelling in directions inclined to one another cannot be resolved if their inclination is less than $\frac{1{\cdot}22\lambda}{d}$, where d is the diameter of the aperture or $\frac{\lambda}{a}$, if the aperture is in the form of a slit of width a. The magnitude of the angle which measures the resolution is determined by the separation of the central maximum in the intensity curve (fig. 248) from the first minimum. The closer these can be made the higher the degree of resolution. The fact that in astronomical observations angles well below this order of magnitude occur, even if the aperture is made very large, as in the case of a 100-inch telescope, led Michelson to develop a method of increasing the resolution by means of two slits placed in front of the lens. In his later work, Michelson was able to increase the resolution by the use of mirrors which had an effect equivalent to producing a wide separation of the slits, but for the present purpose the two slits will be placed immediately in front of the lens in a plane touching its surface. The character of the diffraction pattern obtained may be readily deduced by a continuation of the argument given in connexion with the experiment on the resolving power of an object glass. It was shown that the effect of a single slit of width, a, in a direction, θ, was equivalent to a disturbance,

$$y = \mathrm{A}\,\frac{\sin\varphi}{\varphi}\,\sin\left(\frac{2\pi t}{\mathrm{T}} - \varphi\right),$$

along the direction of the emergent rays.

If two such slits occur separated so that the distance between their middle points is b, the combined effect when the rays from both are received in a telescope can be obtained by superposition. The phase

difference in the two central rays is due to the difference in path (HF — GE) and is of magnitude $\frac{2\pi b \sin \theta}{\lambda}$. Thus, to the above expression another is added differing by the subtraction of this phase. The sum gives

$$Y = 2A \frac{\sin \varphi}{\varphi} \cos \left(\frac{\pi b \sin \theta}{\lambda}\right) \sin \left(2\pi \frac{t}{T} - \varphi - \frac{\pi b \sin \theta}{\lambda}\right).$$

$\varphi = \frac{\pi a \sin \theta}{\lambda}$, and when a is much smaller than b, as in the case of two narrow slits separated by a distance of 1 or 2 cm., the cosine term

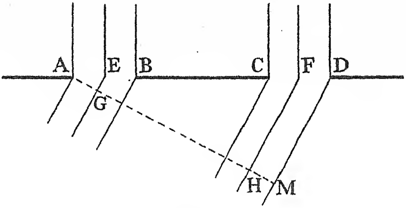


FIG. 251

varies much more rapidly than $\frac{\sin \varphi}{\varphi}$. It has been seen that maxima for the latter occur at separations of the order $\theta = \frac{\lambda}{a}$, but the cosine term has maxima at values of $\theta = 0, \frac{\lambda}{b}, \frac{2\lambda}{b}$, etc., showing a much more rapid variation.

A variation of intensity proportional to the square of the product of these terms is observed, and this variation is represented in fig. 252.

The outer curve represents the variation of $\frac{\sin^2 \varphi}{\varphi^2}$ alone, and the inner that of the square of the cosine term.

If rays from a direction slightly inclined to the normal fall on the two slits a similar pattern is superposed and, if they are widely enough displaced, the effect of one will not completely obliterate that of the other. If, however, the inclination is so close that the maximum of the second does not lie beyond the minimum, B, of the first, the rapid variations will not be seen and the effect will be that of uniform brightness. The minima of the cosine term occur midway between the

maxima, and the first minimum occurs in the direction inclined at θ to the normal, where $\sin \theta = \frac{\lambda}{2b}$, but since θ is a very small angle it is possible to write $\theta = \frac{\lambda}{2b}$.

Thus rays making this angle, θ, with the normal to the plane of the slits, will produce a pattern with the central maximum lying over the

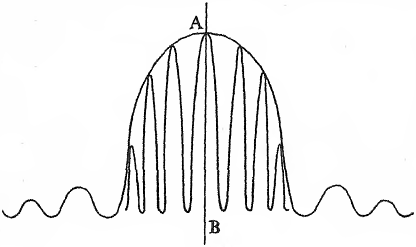


FIG. 252

first minimum in the normal ray pattern. The smallest inclination which can be resolved is thus $\theta = \frac{\lambda}{2b}$.

In the measurement of stellar diameters the angle subtended by the star's disk at the earth's surface is required. It is thus necessary to resolve the directions drawn from the point of observation to the opposite ends of the star's diameter. The case differs from that just considered, in that the source is continuous between the two points.

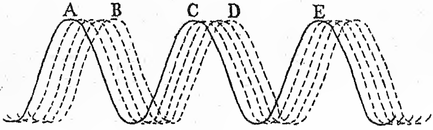


FIG. 253

The result obtained applies to the case where the two sources are point sources. In the case of a continuous source a continuous sequence of patterns occurs, and the effect of this can be appreciated best from a diagram.

In fig. 253 three maxima corresponding to the maxima of the cosine term are shown. They correspond to the inner curves of fig. 252, and

are now shown on a large scale laterally. The maxima of the curves are not strictly equal, but the variation, following that of the factor $\frac{\sin^2 \varphi}{\varphi^2}$, is slow in comparison with the variations of the cosine factor. The continuous curve of the figure corresponds to light from one side of the star and the extreme dotted curve on the right corresponds to that from the opposite side. Between these the sequence of dotted curves is drawn to represent the effect of light from intermediate points of the star's disk. So long as the space between the successive crests is not filled, a drop in intensity will occur corresponding to the space between B and C in the figure, and the appearance will be that of rather diffuse interference bands. When the extreme dotted curve falls on the adjacent wave of the continuous curve no such diminution occurs, and the field of view is uniformly bright. Thus the limit occurs when the ray from one side of the star's diameter produces a pattern with a central maximum overlying the first maximum from the centre of the pattern produced by the ray from the other side. The angle between the rays is thus equal to the angle made by rays at θ to the normal to the slits, where $\theta = \frac{\lambda}{b}$.

The extreme rays may, however, be sufficiently separated to cause a pattern to extend out as far as D, i.e. the position, D, may be at the central maximum due to rays from the other end of the diameter. In this case uniformity will appear when D falls at E, and in that case $\theta = \frac{2\lambda}{b}$. A similar argument shows that other conditions for uniformity are $\theta = \frac{3\lambda}{b}, \frac{4\lambda}{b}$, etc.

Thus, if it be possible to vary the distance, b, the fringe pattern produced by a star which subtends an angle, θ, at the instrument will become uniform for the smallest separation, b, given by $b = \frac{\lambda}{\theta}$, and as b is increased the fringes will reappear, disappearing for values of $b = \frac{2\lambda}{\theta}, \frac{3\lambda}{\theta}$, etc.

No account has been taken of the fact that the star's disk is circular, the above results are for sources in the form of a slit. In the case of a circle, the central strips contribute more light than the outer ones and the first disappearance occurs when $\theta = \frac{1{\cdot}22\lambda}{b}$.

In order to illustrate these points experimentally, two parallel slits about 1 mm. wide may be cut with a razor edge in a piece of cardboard separated at a distance rather more than the length of the radius of

the object glass of the telescope used. Another way to obtain suitable slits is to paste a sheet of tinfoil on a plane sheet of glass and make parallel slits in the foil by means of the razor edge. It has been found that a telescope with a three-inch objective is a suitable one for this purpose. The slits should be mounted vertically to lie symmetrically across the lens. At some distance, as large as is convenient in the laboratory, an adjustable slit should be set up and illuminated by light from a strong sodium lamp, an intense source being essential. The slit should be viewed varying the width, if necessary, until the fringes appear and then observing the width of the slit when they disappear. The smallest separation of the slits for which this occurs corresponds to the case $\theta = \frac{\lambda}{b}$. Each time the fringes disappear the width of the slit should be measured as accurately as possible with a travelling microscope. If this width be d_1 at the first disappearance and D the distance between the source and the two slits, the angle, θ, is $\frac{d_1}{D}$. This should be equal to $\frac{\lambda}{b}$, and for large openings the ratios should be an integral multiple of this amount. The purpose of the experiment is to verify these relations.

Subsidiary Maxima of a Plane Diffraction Grating

In the ordinary use of the diffraction grating as in the experiment described earlier in this chapter, only the positions of the principal maxima are considered, and only these are required. The elementary theory of the grating refers only to these maxima and makes no suggestion of the fact that a sequence of lines of smaller intensity occurs between the different orders of the diffraction spectra, which consist of lines situated at the principal maxima corresponding to particular wave-lengths.

The more exact theory shows that $(N - 2)$ subsidiary maxima occur between the principal ones in the case where the total number of rulings is N. The intensity in any direction, θ, can be obtained by a continuation of the procedure of the last experiment. Instead of two slits separated by a distance, b, the number is now N and the total amplitude is obtained by adding the effects of all the spaces of the grating. These effects lag behind one another by the phase

$$\frac{2\pi (a + b) \sin \theta}{\lambda},$$

where a = width of the grating space and b the width of the opaque element between the spaces. Thus the terms to be added are of the form

$$\frac{\sin \varphi}{\varphi} \sin \left\{\frac{2\pi t}{T} - \varphi - \frac{2\pi p\,(a + b) \sin \theta}{\lambda}\right\},$$

p having the values 0, 1, 2, . . . N. This summation has the amplitude

$$\frac{\sin \varphi}{\varphi} \cdot \frac{\sin N\delta}{\sin \delta},$$

where
$$\varphi = \frac{\pi a \sin \theta}{\lambda}$$

and
$$\delta = \frac{\pi (a + b) \sin \theta}{\lambda}.$$

On investigating the variation of the square of the expression it is found to have principal maxima in the directions given by the elementary theory, but there are $(N - 1)$ minima and $(N - 2)$ subsidiary maxima between consecutive principal maxima.

The magnitude of $\left(\frac{\sin \varphi}{\varphi}\right)$ varies slowly with θ in comparison with the variations of the other factor, so that the variations observed are due to the second factor. The purpose of the experiment is to determine at what values of δ the subsidiary maxima occur.

In the case of the usual diffraction grating with a large number of lines, these maxima cannot be observed on account of their close packing, but gratings of a few elements, i.e. of the order of 10, can be used to demonstrate their existence.

A convenient grating is made by ruling clear strips on a surface silvered mirror and using it as a transmission grating. It is mounted in a cell clamped to a stand on an optical bench, and the number of elements exposed is varied from two upwards by arranging an adjustable aperture, F, in front of it.

An intense source of monochromatic light should be used, and it may conveniently consist of a slit illuminated by a lens condensing light from a sodium lamp. A second lens is used as a collimator to produce parallel rays of light from the slit. The grating is placed in the parallel beam and the diffraction pattern is observed by a telescope formed of an objective of about 30 cm. focal length and an eyepiece mounted to move by means of a micrometer screw perpendicular to the axis of the telescope. By this means it is possible to measure the small angles of diffraction in this experiment; the ordinary spectrometer method would not be suitable.

It is important to get the axes of the collimator and telescope coincident in order to avoid aberrations as much as possible. It is sufficient to arrange the axes parallel to that of the optical bench. The collimator can be focused for parallel light by means of a telescope already focused on infinity.

The apparatus is arranged as illustrated in fig. 254, the eyepiece, E, being focused on the image of the slit after setting the collimating lens, C, to produce a parallel beam. The mountings on the bench may

be used to carry the apparatus, and there is usually a micrometer mounting carrying the eyepiece.

An examination of the spectra should first be made beginning with the exposure of two slits in the grating, G. The appearance will be that of the last experiment consisting of rather diffuse interference fringes, but on exposing three or more slits sharper spectra appear with subsidiary maxima between the bright successive orders. The spectra may be made sharp by rotating the grating to and fro in order to bring its lines parallel to the slit, or, if it is more convenient, the slit may be rotated. It should be noted that the number of subsidiary maxima is two less than the number of elements exposed.

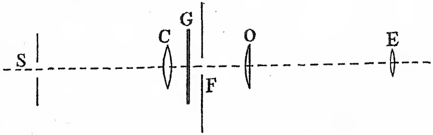


FIG. 254

In order to make a quantitative examination, a convenient number of grating elements to use is six. This part of the work consists in verifying that the subsidiary maxima correspond with those of the expression

$$\frac{\sin^2 6\delta}{\sin^2 \delta}.$$

Let the position of the central image occurring in the focal plane of the telescope objective be taken as the origin and let the lateral distance of any point from this origin be denoted by x. The angle, θ, which measures the inclination of the diffracted ray to this point to the central ray, is small, so that $\theta = \frac{x}{f}$, where f is the focal length of the lens, and $\sin\theta \simeq \frac{x}{f}$.

Thus $$\delta = \frac{\pi(a+b)\sin\theta}{\lambda} = \frac{\pi(a+b)x}{\lambda f} = \mathrm{A}x,$$

where A is a constant. This constant may be determined from the position of the principal maxima for in these cases $\delta = \pi, 2\pi$, etc.

If the positions of as many of these maxima as possible be located on either side, δ can be determined for each and an average value for A obtained. The positions of the subsidiary maxima should now be located on either side. If this is done for the range extending to two orders, sixteen readings will be obtained. By arranging these in two

columns side by side, one for the readings on one side and the other for readings on the other, the differences taken horizontally will give twice the corresponding values of x.

A curve $y = \dfrac{\sin^2 6\delta}{\sin^2 \delta}$ should then be plotted over the corresponding range, i.e. from $\delta = 2\pi$ on one side of the origin to $\delta = 2\pi$ on the other. Locate the subsidiary maxima and compare their positions with those determined in the experiment.

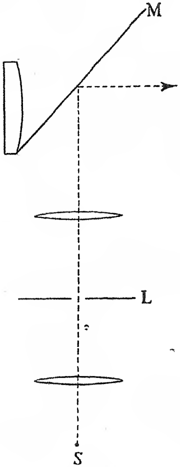


FIG. 255

In order to avoid magnification in the direction of the lines without destroying the lateral magnification, a cylindrical lens should be used as the eyepiece. The most suitable form of pointer in this case is a virtual one formed by the image of a fine slit thrown centrally in the field of view of the eyepiece. A diagram of the arrangement is shown in fig. 255.

M is a piece of cover glass attached by soft wax or Plasticine to the cylindrical lens, and light from a fine slit in L is partially reflected by it into the eye at E. As the lens moves across the field of view the image of the slit maintains the same position relative to the lens, and thus serves as a fine pointer without any distortion or magnification

such as would occur with a cross-wire placed to coincide with the image. Care must be taken to avoid parallax between the virtual pointer and the image of the diffraction pattern.

Distortion and spherical aberration must be avoided, and stops should be provided to limit the lens surfaces exposed.

Polarization by Reflection. Verification of Brewster's Law

When light is reflected from surfaces the reflected beam is partially polarized, that is to say, that the transverse vibrations constituting the light have, on the whole, a greater component in a particular direction than in any other.

Ordinary light is supposed to consist of a transverse vibration which changes its direction in space, though of course, always in the wave-

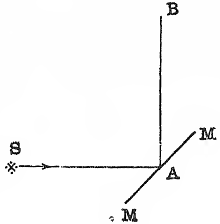


FIG. 256

front, so rapidly that on the average in any appreciable interval of time the component in one direction is the same as that in any other.

The reflected light has lost this property and is polarized so that it has a greater component in one direction. This direction is normal to the plane of incidence.

The transmitted light has a greater component in the plane of incidence.

In the diagram (fig. 256), a ray, SA, is represented as being reflected at a glass sheet, MM, so that AB is polarized. In order to test the polarization the ray is received in an analyser, which in some instruments consists of a Nicol prism. This is a prism of Iceland spar which is cut into two along a diagonal plane and cemented together with Canada balsam. Iceland spar has the property of dividing a ray of light into two rays refracted in different directions and one polarized perpendicularly to the other. The layer of Canada balsam serves to reflect one of these rays to the side of the prism where it is absorbed by the blackened walls of the case, and the other ray is transmitted.

In this way the emergent beam is made to consist of light vibrations all in one direction. Incident light with its vibrations in this direction passes through the Nicol, while if the vibrations are perpendicular to this direction the light is unable to get through. Light with vibrations in any intermediate direction has only the components parallel to the direction of transmission passed on.

The vibrations transmitted are parallel to the shorter diagonal of the end of the prism.

On examining AB with this prism, it will be found that as the prism is rotated there is a change of intensity in the transmitted light. This

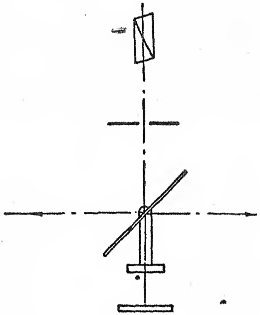


FIG. 257

means that AB consists of vibrations, with the components in one direction greater than in another, or it is partially polarized.

On altering the inclination of MM the alternations can be varied in extent, and in one position the change from brightness to darkness is a maximum. The angle of incidence when this occurs should be noted by the help of the scale of angles attached to MM. Theoretically, the light should be completely polarized in one position, so that for one particular setting of the Nicol darkness should be complete. In practice, it will be found that the light is not quite all cut out, though with care, a position will be found when this is very nearly true. The apparatus is represented diagrammatically in fig. 257.

Brewster's law is that for complete polarization

$$\tan i = \mu,$$

where i is the angle of incidence and μ is the refractive index of the reflecting material.

This law should be verified. It will be found best to allow light from a window on the opposite side of the laboratory to fall on MM, and to adjust for the maximum effect, and afterwards to set up a sodium flame as at S, and make the exact adjustment.

In another form of apparatus the Nicol is replaced by a second glass plate which can be rotated about the direction, AB, as well as inclined at different angles to the horizontal.

When the mirrors are parallel the polarized light in AB is readily reflected by the second mirror, while on rotating it through a right angle from this position about AB this polarized light will not be reflected.

We shall thus obtain alternations in intensity on rotating the mirror about the vertical axis, and there will be a maximum effect for a particular inclination of MM, in which case $\tan i = \mu$.

Rotation of the Plane of Polarization. Laurent's Saccharimeter

The essential parts of the saccharimeter are two Nicol prisms, N_1 and N_2, illustrated in fig. 258, one of which serves to polarize a beam

FIG. 258

of light passing through it while the other analyses the transmitted beam and detects its plane of polarization. These Nicols are spoken of, respectively, as the polarizer and analyser.

When N_1 has reduced the light vibrations to a particular direction, viz. parallel to the short diagonal at the end of the prism, all the light transmitted by N_1 can pass through N_2 if N_2 is oriented exactly in the same way as N_1, i.e. if its shorter diagonal lies parallel to that of N_1, and its length lies parallel to that of N_1. We are here neglecting the diminution of intensity due to absorption, which always goes on, since actual bodies are not perfectly transparent. We mean that no light is cut out in this case on account of polarizing effects of N_2.

In this position the Nicols are said to be parallel. If, however, N_2 is turned from this position through a right angle, no light from N_1 can get through N_2, since N_2 is now so oriented that the light vibrations falling on it are in a direction perpendicular to its short diagonal, and such vibrations are not transmitted.

In this position the Nicols are said to be crossed.

Certain substances like quartz, and solutions like that of sugar, possess the property of rotating the plane of vibration of light as it passes through them, so that if N_1 and N_2 are crossed when the active

substance is not placed between them (in which case no light will get through N_2), on inserting the active material, on account of the change in direction of the vibration, some light will pass through N_2.

It is found that a rotation of N_2 in one direction or the other will bring it into a position when the light is once more stopped. This shows that the light is still polarized, but its vibrations have changed direction in traversing the medium. We ought, therefore, to be able to measure the amount of this rotation by measuring the angle through which N_2 is turned; but, unfortunately, N_2 can be turned through an appreciable angle when the light is cut out without any apparent return

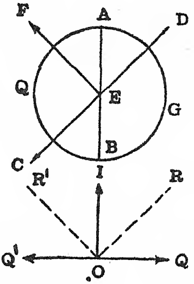


FIG. 259

of the light. This lack of sensitivity is overcome in the saccharimeter by a special device.

Just in front of the polarizing Nicol on the side towards the analyser, is placed a semicircular sheet of quartz cut parallel to the optic axis. The complete circle is made up by a semicircle of glass of such thickness that it absorbs the same amount of light as the quartz. The position of this circle is at H, and it covers the open end of N_1 completely.

When the light falls on the quartz it is separated into two components, polarized normally to one another, which travel through the quartz with different velocities. Let one component be represented by OI and the other by OQ just as the light reaches the quartz plate, these are the components of a vibration along OR. It is here supposed that the vibration at O takes place in the direction, OR, so that its component directions are OQ and OI. As the disturbance passes through the plate there will be a gradual change of phase between these components on account of the differing velocities of transmission. After a time the disturbance will reach a point in the plate where one

component displacement is along OI while the other component is along OQ′. These combine to give a displacement OR′.

The quartz plate is cut so that, as the disturbance just leaves the plate on the other side, this difference of phase exists between the components. The difference is one-half of a period, and the plate is called a half-wave plate or half shade.

Of course the light which traverses the glass side proceeds undisturbed, and its oscillations are still along the direction, ED, shown in the upper part of the diagram, parallel to OR.

If N_2 receives this light when its short diagonal is at right angles to OR′ this component is not transmitted, and no light from the quartz side gets into the eye, while part of the component OR gets through and the glass side appears illuminated. Generally, light passes in from both sides of the plate, but both sides are not equally illuminated. When both sides present the same illumination the principal plane of the Nicol is either along AB or normal to it, for it is clear that in either of these positions the components transmitted are the same for both sides.

It happens that the eye can readily detect a change from the equality of illumination in both halves of the field, particularly if both halves are equally dark, i.e. when the Nicol is so placed that the smaller components are transmitted.

If the Nicol is set for equal illumination on both sides, and an active substance is interposed, it will be necessary to rotate the Nicol to find once more the position of equal intensities. The amount of rotation measures the angle of rotation of the plane of polarization.

Another common method of bringing about an increase in sensitivity is to use the biquartz. This consists of two semicircular disks of quartz fitted together to form a complete circle. One of these rotates the plane of polarization of the incident light in a clockwise, and the other in a counter-clockwise direction. The amount of rotation per unit thickness varies with the colour of the light. For a particular thickness the rays from a sodium flame will be turned in opposite directions through a right angle, so that if the short diagonal of the Nicol lies parallel to this direction the yellow rays get through, and if the diagonal is perpendicular to this direction these rays are cut out.

When white light is used it is robbed of the yellow constituent when the Nicol lies in this latter position, and the colour observed is greyish and is called the tint of passage.

It is easy to detect a slight change from this uniform colour, for an appreciable change takes place to a partly blue and partly red field, one colour belonging to each side.

Let LMNS denote one end of the analysing Nicol, and let UV denote the direction of vibration of the light.

This light will be cut out if MS—the shorter diagonal—lies normally

to UV. A small rotation of MS counter-clockwise will bring it into a position to cut out the light and so will a larger rotation in the opposite direction, the sum of these rotations being 180°.

It is thus not easy to decide which way the plane has been turned. But if two lengths of the rotating substance be used, one slightly longer than the other, the rotation for the longer must be greater than for the shorter.

The direction of rotation of MS which shows a larger angle in the case of the longer is the direction in which the rotation has taken place.

Tubes of glass with carefully worked end-pieces are used to carry the solution to be examined. The ends are held in position by metal caps screwed against them. It is necessary to have rubber washers

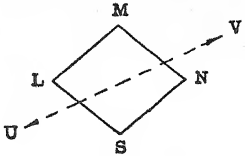


FIG. 260

between the glass ends and the tube to avoid strain when screwing up, for a strained end will produce rotation.

Find the amount of rotation for a solution of sugar in water and deduce its specific rotation. This quantity is defined to be the amount of rotation produced by one decimetre of solution divided by the weight of dissolved substance in unit volume.

Let w gm. be dissolved in 100 c.c. and suppose a length, l cm., produces rotation, θ. The specific rotation is

$$10\frac{\theta}{l} \div \frac{w}{100} = 1000\frac{\theta}{lw}.$$

Repeat for various strengths of solution, and for different lengths of tubes.

When several tubes are obtainable, it is interesting to observe the effect of causing transmission through different lengths of a solution of a particular strength.

By this means it may be verified that the amount of rotation is proportional to the distance traversed by the light in the solution.

Another instructive experiment is to make solutions of different known concentrations, which may be measured by the number of grammes of substance dissolved in 100 c.c. of solvent, and to measure

the amount of rotation in traversing a particular distance through the solution. A curve showing the relation between the rotation and concentration should be plotted.

The Lippich Polarizing System

In modern polarimeters the half shade is replaced by a more convenient method of dividing the field. In one type of polarimeter the field is divided into three parts (fig. 261), the two outer similarly illuminated for all positions of the analysing Nicol and the central portion, which may be differently illuminated from the neighbouring regions and which has to be matched with the outer parts of the field.

The mode of producing the divided field is illustrated in fig. 262. N is the polarizing Nicol and LL are two small Nicols fixed in position

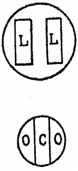


FIG. 261

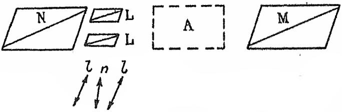


FIG. 262

and mounted in a brass cylinder in front of N. The directions of vibration of the light emerging from the cylinder and falling on the optically active substance are represented by the arrows, l, n, l. The central portion passes through the Nicol, N, only, while the outer parts pass through N and L.

When the analyser, M, is turned so that it transmits vibrations along a direction bisecting an angle between n and l, the whole field is uniformly illuminated. For all other positions of M the field is not uniform.

Thus, to measure the amount of rotation of any substance placed at A, the analyser is first put into the position corresponding to uniformity of field. The substance to be examined is then put into position and M again rotated until the field is once more uniform. The angle of rotation measures the rotation due to the active substance.

The optical system by which the field is examined is not shown in the diagram. It lies to the right of M and is focused on the plane through the right-hand ends of the Nicols, LL. This system has the advantage that it is suitable for the examination of all wave-lengths, whereas the half shade has to be constructed for one wave-length only. Fig. 261 shows at OCO how the field is divided into three parts.

The Half Shadow Angle

The various devices employed to enable accurate observations to be made in polarimetry, which have been described, produce two beams of polarized light with vibrations in directions inclined to one another.

In the Lippich system we have denoted the two directions by l and n.

In fig. 263 these directions are denoted by Bl and Bn respectively, and the angle between them is 2θ. This angle is called the 'half shadow angle', and the magnitude of this angle has an important bearing on the question of sensitivity.

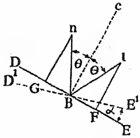


FIG. 263

Suppose that BC bisects the half shadow angle, and that DBE denotes the direction of vibration of the light which traverses the analyser.

When this lies at right angles to BC the intensities of the two beams are equal as seen through the analyser, for the components of the displacements transmitted, viz. BF and BG, are equal, and the intensities are in the ratio

$$BF^2 : BG^2.$$

Suppose the analyser is turned through an angle, α.

Then the transmitted components are

$$Bl \cos lBE' \quad \text{and} \quad Bn \cos nBD',$$

or

$$Bl \sin (\theta - \alpha) \quad \text{and} \quad Bn \sin (\theta + \alpha).$$

Thus the ratio of the intensities is

$$\sin^2 (\theta - \alpha) : \sin^2 (\theta + \alpha).$$

In photometric work it is assumed that the eye can detect a difference of intensity of 1 per cent.

Thus, if we regard θ as a given angle, we may say that the change in setting of the analyser of the amount, α, will just be detected when

$$\sin^2 (\theta - \alpha) = 0{\cdot}99 \sin^2 (\theta + \alpha).$$

From this equation it follows that when α is a small angle, so that we may write

$$\sin \alpha = \alpha \quad \text{and} \quad \cos \alpha = 1,$$

$$\alpha = 0{\cdot}0025 \tan \theta.$$

If α has a small value the apparatus is sensitive, and it would appear that the sensitivity is improved by making θ as small as possible.

But as θ gets small, difficulties arise on account of the fact that the light is never plane polarized in practice, but is always elliptically polarized, and θ may not be indefinitely diminished.

Polarimeters are usually fitted with a small movable arm projecting from the tube which carries the polarizer. This arm carries an index mark which moves over a scale. By means of it the polarizing Nicol can be rotated so that the half shadow angle can be adjusted within limits. One form of the apparatus is illustrated in fig. 264 (at p. 1).

The sensitivity is thus to some extent under the control of the observer, who will discover, as he becomes familiar with the instrument, the best adjustment for sensitivity which suits him.

The reader may be referred for a more detailed and complete account of all these questions to the article on 'Polarimetry' in the *Dictionary of Applied Physics*.

Soleil's Compensator

Sometimes the saccharimeter is fitted with a piece of apparatus consisting of two quartz wedges. This is known as Soleil's compensator. Fig. 265 illustrates the apparatus. The wedges are ABC and DEF,

Fig. 265

and these are mounted in metal holders which can be moved by means of a rack and pinion, so that the wedges are carried in either direction parallel to AC or DE.

Thus, if a ray of light is passed perpendicularly to AC, it is possible to place varying thicknesses of quartz in its path. The quartz wedges are cut so that the optic axis lies perpendicular to AC and DE, and polarized light passing through them suffers rotation of its plane of polarization.

The amount of rotation can be varied by moving the wedges by means of the rack and pinion.

The quartz wedge is placed just in front of the analyser.

An index mark moves along a scale as the wedges are displaced, so that a record can be made corresponding to each thickness of quartz interposed.

The analyser is first rotated until, with the quartz wedges occupying a convenient zero position, the field is equally dark on both sides, supposing that the polarimeter is fitted with a half shade. The quartz wedges are then moved a small amount by the rack and pinion, and the amount of rotation of N_2 necessary to restore the uniformly dark field of view is recorded.

By making a number of observations a curve can be plotted which shows the rotation corresponding to the various dispositions of the wedges.

When light has undergone a rotation before passing through the wedges, this rotation may be counteracted by interposing the correct thickness of quartz.

Thus, suppose the Nicols and quartz occupy the zero position described above, and that an optically active material is interposed, the rotation resulting may be counteracted by displacing the quartz wedges either so as to increase the thickness traversed or to diminish it. When the appearance in the analyser is the same as that of the zero position, we know that the wedges have caused a rotation equal in magnitude, but opposite in direction to that of the active substance.

Thus by observing the record opposite the index mark we can deduce the amount of rotation due to the substance.

In plotting the graph, rotations in one direction will lie on one side of the origin up to 180°, while those in the other direction will lie on the other side up to 180°.

CHAPTER XIV

PHOTOMETRY

Introduction

The light emitted from a small source is absorbed very little by the air through which it passes, so that we may say that any surface, surrounding the source completely, will receive the same total amount of light.

Let this total amount be denoted by M.

Imagine a cone with its apex at a small source of light and let its solid angle be ω. All surfaces receive the same amount of light on the parts lying within this cone.

If ω is small, say $\delta\omega$, the cone may be regarded as defining a particular direction, and the intensity will be regarded as the same for all rays within this cone. If δL denote the total amount of light emitted within this cone per second, we write: $\delta L = K\,\delta\omega$.

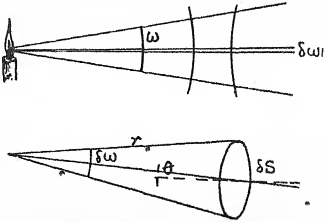


Fig. 266

This equation defines K, which is sometimes called the 'candle power' of the source.

In general, K is dependent upon the direction, but when K is the same for all directions

$$M = 4\pi K,$$

where M is the total amount of light emitted per second by the source.

Let the small cone cut a surface in the element, δS, and let the mean direction of this cone make an angle, θ, with the normal to δS.

Then $$\delta S = \frac{r^2 \delta\omega}{\cos\theta}.$$

and $$\delta L = \frac{K\cos\theta}{r^2}\,\delta S.$$

The intensity of illumination is defined to be the amount of light falling on unit area per second,

i.e. $$I = \frac{\delta L}{\delta S} = \frac{K \cos \theta}{r^2}.$$

The intensity, I, varies inversely as r^2 and directly as $\cos \theta$. When I is the same for two similar surfaces they appear to the eye to be equally bright, and on this principle the use of photometers depends.

The efficiency of a source of light may be defined as the ratio of K to the material or energy consumed per second.

The average candle-power, in the case when K is not constant, divided by the amount consumed per unit time is called the mean spherical efficiency. This term is employed because the average value of K is defined as $\frac{M}{4\pi}$, and this denotes the average amount of light falling on 1 sq. cm. of a unit sphere placed with its centre at the source.

The efficiency of a candle is

$$K \div \text{weight of wax consumed per second,}$$

of a gas flame

$$K \div \text{cubic feet of gas consumed per minute,}$$

and of an electric lamp

$$K \div \text{watts supplied to it.}$$

The watts are measured by the product, volts $\times$ amperes, one watt denoting the rate of working when a current of one ampere falls through a potential difference of one volt.

The unit in which K is measured is the candle-power of a standard candle in a horizontal direction when the flame is 50 mm. high.

If another source is used and compared with the standard, the two are adjusted to the same height and placed at such distances, r_1 and r_2, from a conveniently placed screen that each makes it appear equally bright. In this case by what has been said above concerning intensity we have

$$I_1 = \frac{K_1 \cos \theta}{r_1^2} = I_2 = \frac{K_2 \cos \theta}{r_2^2},$$

since the angles are the same. If K_1 is the standard candle-power its value is unity, and,

$$\frac{K_2}{r_2^2} = \frac{1}{r_1^2}.$$

It is assumed that the student is familiar with the simple forms of photometer such as Rumford's and Bunsen's. It is difficult to make accurate comparisons of the illuminating powers of sources with these instruments.

We shall be concerned chiefly with the more accurate types of instruments in this chapter. Even with these, care and practice are required, but a skilled observer can obtain accurate results.

The Efficiency of Sources of Light

Take a gas-burner provided with an indicator registering the quantity of gas supplied, and adjust the flame until it stands at the same level as a candle flame and the grease spot of a Bunsen photometer.

The candle may be taken as the standard with the value of K unity, and it must be shielded from draughts and must burn steadily.

Determine K for the flame corresponding to different rates of supply of gas, by adjusting the distances between the sources until the photometer screen appears alike on both sides.

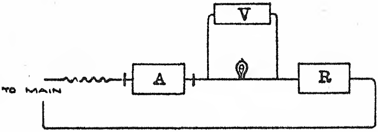


FIG. 267

Plot a curve showing the relation between the efficiency and the supply per hour.

It will be found that the efficiency increases with the supply up to a maximum and then diminishes.

It is most economical to adjust the supply to the value appropriate to the maximum.

We may similarly measure the efficiency of an electric lamp. Arrange the lamp, L, on the same level as the grease spot and candle flame as before, and measure its candle-power when the current is supplied at different voltages.

Fig. 267 illustrates the arrangement of apparatus.

V is the voltmeter joined to the terminals of L, and A measures the current in amperes. The resistance, R, is adjustable and is used to vary the current supplied to the lamp. The power is obtained by connexion to the main through a plug. A convenient voltage is 100 volts.

Plot a curve showing the variation of candle-power with rate of supply of energy as measured in watts.

The Flicker Photometer

One form of this instrument is illustrated in figs. 268 and 269. The essential part consists of a white wheel, W, of which the edge is about 1 cm. wide, and is cut to the shape of a ridge running spirally. This

is shown at RR, in fig. 269. The wheel is mounted in a box, black on the inside, provided with a rod, so that it may be supported in the carrier of an optical bench.

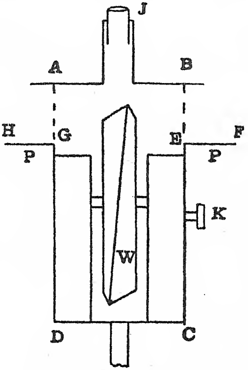


FIG. 268

The wheel is provided with a central axis and is rotated by a spring within the box, which is wound up by the key, K.

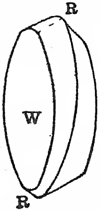


FIG. 269

The two sources to be compared are placed one on each side of the apparatus. These are carried in stands on the bench, so that the distances from the wheel are measured accurately and their heights are adjusted with the aid of two lenses fixed on the box.

The small doors, EF and GH, are closed, and the lenses used to throw an image of the sources on marked points, P and P, on the doors. When the images fall on these points the heights of the sources are correctly adjusted.

The edge of the wheel is observed by means of an eyepiece, J, and usually unequal parts of the edge are seen on the two sides. When the wheel rotates the widths of these vary, and unless the illumination is equal on both sides a flickering effect is observed. If the illumination is the same on each side the ridge character disappears.

The sources are moved to or from the apparatus until the flicker effect ceases, when the illuminations are equal, and if I_1 and I_2 represent the illuminating powers of the sources, and r_1 and r_2, their distances from the wheel, we have

$$\frac{I_1}{{r_1}^2} = \frac{I_2}{{r_2}^2}.$$

The distances should be measured from the middle of the wheel. It is not easy to decide when the flicker ceases, and the degree of accuracy obtainable is not very high.

Compare in this way a standard candle and a lamp.

Guild's Flicker Photometer

The great advantage of the flicker photometer is that it makes it possible to compare sources of different colours with a considerable degree of accuracy. But it has been found that certain conditions must be satisfied, and standard conditions are recommended by specialists in this branch of physical measurements. The conditions necessary to ensure that the flicker photometer measures the true brightness are that the illumination should be high and the field of view small. Further, this small field of view should be surrounded by a larger field which is maintained at a steady brightness approximately equal to that of the small flicker field.

Thus the standard conditions recommended are that the angular diameter of the flicker field should be 2° and the field brightness that of a surface of magnesium oxide under an illumination of 25 metre-candles. The surrounding field increases the comfort and ease of the operation. Guild has described an instrument (*Jour. Sc. Instruments*, I, p. 182, 1923–4) designed for use with sources of different colours which fulfils the conditions required for accuracy in its use. A modification of some parts of the original apparatus is described here.

A rotating disk, AB (fig. 270), made of two sectors cut accurately at 90° is mounted on pivot bearings and is driven by a belt from a shaft.

One face of the disk receives light through an opening, O_1, on the right of the box, situated so that its centre is at the level of the axis of the disk, and as the disk rotates an alternating field is provided.

There is a similar aperture, O_2, on the left, which lies above the driving shaft and pulleys and admits light which is reflected by a 45° prism, P, on to a screen, S, coated with magnesium oxide, which scatters the rays falling upon it. The screen is viewed alternately with the rotating sectors as these cross the line of sight, ES. The holes, H, are just large enough to allow the essential rays through to the sighting instrument. Scattered light must be prevented from reaching the comparison surfaces which should appear black when the sources are cut off. The sighting tube provides a large steady field round the

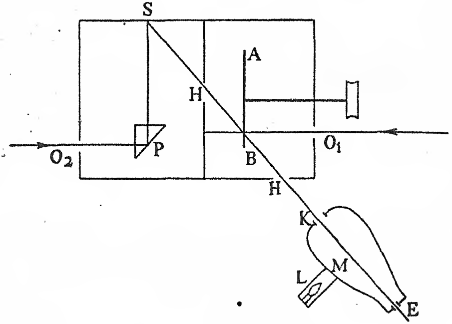


FIG. 270

central flicker field, consisting of a hemisphere of 9 cm. diameter, with a conical tube carrying an eyepiece, E. A small hole, K, 8·7 mm. in diameter, bounds the flicker field, and with a total length of sighting tube of 24 cm. the angle subtended at E by the flicker field is 2°. A side tube, L, carries a 4-volt lamp behind an opening, M, covered with a diffusing medium. The interior of the sighting tube in front of M is coated with magnesium oxide.

Method of Use

In this type of photometry a substitution method must be followed. This means that a lamp is kept fixed on one side for comparison with the two lamps to be compared which are placed in turn on the other side. If the distances when matching occurs are d_1 and d_2, measured from the two lamps to the sector plane, the ratio $\frac{d_1^2}{d_2^2}$ gives the intensity ratio.

Since sensitivity diminishes at high speeds of rotation, the speed of alternation should be reduced to the minimum at which a position of no flicker can be found. The best speed has to be found for each measurement, since the critical speed is higher the greater the difference in colour of the alternating fields.

Settings should be made by passing one of the sources to and fro through the position of no flicker reducing gradually the amplitude of the movement. This is more sensitive than the method of making a gradual approach to the final position. It is convenient to keep the comparison lamp and photometer fixed and to move the lamps to be compared in the way described.

In order to make the initial adjustment of brightness, a lamp of known candle-power should be placed to give the standard illumination of 25 metre-candles on the sector. The comparison lamp is then put into the position where it produces no flicker. The surrounding field is adjusted to equality with the small field as nearly as possible by means of the 4-volt lamp.

If these points are attended to, an accuracy of 1 per cent can be obtained even though wide differences of colour in the sources occur.

The Lummer-Brodhun Photometer

This is one of the most accurate photometers. It is illustrated in fig. 271, and consists of a box, LMNO, containing the prisms, P_1, P_2, A, and B, by means of which light is reflected and transmitted into the telescope, T, placed at 45° to the sides of the box, with its object glass in one corner.

The two sources to be compared are at S_1 and S_2, from which light falls on the slab, DD, which consists of magnesium carbonate. From the diffuse sides of the slab, rays are scattered and absorbed by the sides of the box, except those that cut the sides of P_1 and P_2 normally. These are reflected by the hypotenuse faces into the two right-angled prisms, A and B.

These prisms are the principal part of the apparatus.

The hypotenuse of A is rounded off, except for a circular central portion which is placed in optical contact with the large face of B. The reason for this is that rays of light may pass from A to B at this junction just as if the prisms formed one solid medium. Rays falling on other parts of the hypotenuse faces are totally reflected.

In this way rays from P_1 pass on through B, forming the central bundle of rays in the beam emerging from the right and entering T. The rays outside this circle are reflected and are absorbed by the sides of the box. In the same way, rays from P_2 are reflected outside the circle, while those falling on the circle are transmitted. Thus the field of view of the telescope is illuminated by a central circle of rays, originating at S_2, while the outer rays come from S_1. Generally, these

two parts will be of different brightness, and on moving S_1 or S_2, the two parts may be made equally bright. The eye can judge this easily and readily appreciate a slight deviation from equality. It is on this fact that the sensitiveness depends.

From DD, both sets of rays follow similar paths and light is absorbed equally. When the field of T is uniform, the slab is illuminated equally

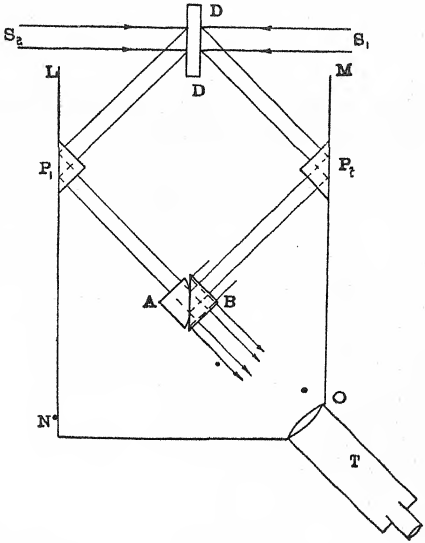


FIG. 271

on both sides. If L_1 and L_2 denote the illuminating powers respectively, and d_1 and d_2 denote the respective distances from DD, we have

$$\frac{L_1}{d_1^2} = \frac{L_2}{d_2^2}.$$

Thus any two sources may be compared with accuracy.

Compare two sources as in the last experiment.

In order to eliminate errors arising from inequality of the reflecting power of the two surfaces of the slab and a possible difference in optical

paths, DP_1A and DP_2B, it is usual to mount the box, LNOM, on a central horizontal axis lying perpendicularly to S_1S_2. Readings are first taken with the telescope as in the figure, and then with the box turned through two right angles so that the telescope lies on the left.

The Nutting Photometer and its use for the Determination of the Absorption of a Solution

This photometer is an accurate instrument of a high degree of accuracy. It is based upon principles described by P. G. Nutting, but has been modified by Messrs. Hilger and Co., whose apparatus is here described.

It may be used in combination with the constant deviation spectrometer (pp. 289 to 293), and the apparatus is illustrated in fig. 272 (at p. 1) in the position ready for use.

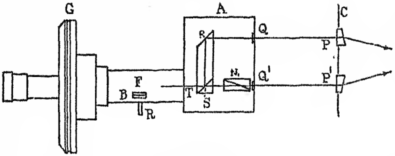


FIG. 273

The essential features of the photometer are illustrated in fig. 273.

The box, A, is of aluminium, blackened inside, provided with two small windows, Q and Q′, by means of which light is admitted along the two directions, QR and Q′S.

Light from a suitable source is deviated by two prisms, P and P′, carried in the plate, C. On the plate a number is inscribed, indicating the distance that the source must be placed from it, in order that the necessary deviation may be produced. This distance is usually 19 cm.

Behind the window, Q′, lies a Nicol prism, N_1, which polarizes the light entering at Q′; but behind Q in the form of apparatus shown no Nicol prism is placed.

In another form a Nicol lies behind Q also, and its purpose is to counteract as much as possible the elliptic polarization that occurs on reflection at the surfaces marked R and S. It is found that this is reduced to a minimum by a particular orientation of the Nicol.

The prism, RS, is composed of three slabs, the two outer are alike and are cut at the ends, R and S, at an inclination of 45° to the length and to the incident light.

Thus the beam, QRST, is totally reflected at R and S, by these two end-faces. Between these slabs lies a central one of the same

composition and thickness cut at R at 45° like the others, but cut square at the other end, and projecting as the diagram indicates. Thus the central portion of the beam, QR, is totally reflected at R, and transmitted out at the other end, and is absorbed by the blackened walls of the box. The beam, Q'S, is totally reflected at the end, S, by the upper and lower slabs, and the light absorbed by the walls of the box, but the central portion is transmitted in the direction, ST, by the central slab.

Thus a tripartite field is produced and may be observed from the end of the tube, F. The outer portions are illuminated by light which has entered at Q, and the central portion by light from Q'.

The instrument is constructed and the prism and Nicol, N, chosen to cause the light within the box to suffer approximately the same absorption along the two paths, QRS and Q'S.

The tube, F, carries a second Nicol which is not shown, which acts as an analyser and which can be rotated by means of the divided metal circle, G. This circle carries two scales, one marked in degrees and the other giving 'densities', a term which will be explained below.

When the instrument has no zero error the degree scale reads zero when both Nicols are parallel. In this case the light already polarized by N_1 is transmitted by N_2, while the unpolarized outer portions of the field are polarized by N_2 in the same way as the central portion by N_1.

Thus when there is no absorbing medium between the source and one of the windows, the field appears under these circumstances of the same intensity in the outer and inner portions.

F also carries a condensing lens which may be adjusted by the rod, R, projecting downwards from the tube.

The purpose of this lens is to converge the light so that on exit all the light may enter the pupil of a normal eye. This is an important condition with which accurate photometric apparatus must comply in order that intensity comparisons may be of any use.

The rod carries a scale past an index mark in the slot, B, on which are engraved two sets of numbers which have to be associated in pairs. The upper scale records the distance of the source from the front of A in cm., and the number below this record on the lower scale denotes the maximum breadth of the source which is permissible if the photometric condition is to be satisfied.

Beyond the circle, G, the tube carries a lens system which focuses the light on to the slit of the spectrometer, and if it is desired to examine the field of view directly by eye an additional eyepiece is fitted into the tube.

The apparatus requires careful adjustment which may be carried out as follows:

Remove the prism from the spectrometer, illuminate the collimator slit, and place on the prism table a piece of plane mirror or a right-angled

totally reflecting prism, and adjust it until an image of the slit lies on the cross-wire of the telescope. Now place the source to be used with the photometer at the distance from the collimator slit at which it is to be situated during the experiment, having regard to the photometric condition which limits the distance to some extent on account of the size of the source. Adjust the source until an image of it lies in the centre of the field of view of the telescope, the collimator slit being now wide open and the eyepiece removed from the telescope, so that the source appears to lie at the centre of the object glass. Slide the photometer into position with the end of the tube, F, as nearly as possible 1·4 cm. from the slit, and with the window, Q′, directly between the collimator slit and source. Cover up the window, Q.

Adjust the photometer by means of the three screws on which it stands, and by rotation about a vertical axis until the image of the source lies once more in the centre of the objective of the telescope.

Place the plate, C, into position at a distance of 19 cm. from the source. The distance between P and P′ is 3·8 cm., so that by moving the source a distance 1·9 cm., it can be brought to lie opposite the middle point of PP^1 and it should then be in the correct position.

This may be judged first by observing if bright circular patches of light lie symmetrically round the windows, Q and Q′.

Make sure that, with the rod, R, adjusted so that the index lies opposite the mark denoting the distance between the front of A and the source, the width denoted by the lower reading is greater than that of the source. If this is not the case the source has either to be displaced farther from A or diminished in size. Place the constant deviation prism and eyepiece of the telescope in position and adjust the spectrometer correctly for sodium light as indicated on p. 292. In doing this, a strongly coloured Bunsen flame may be placed just closer to C than the light source.

Make the line as sharp as possible by rotating slightly the milled head at the end of F.

Now remove the Bunsen flame and open the slit to let in a convenient quantity of light. It may happen that the central part of the field of view is slightly displaced with respect to the outer parts. This is exaggerated in the upper part of fig. 274. This may be corrected by a further small rotation of the photometer about a vertical axis. The three parts of the field should be separated by fine dark lines. If these are too wide they may be made narrow by adjusting the base screws of the apparatus.

Correction of Zero Error

In practice, it is usually found that when the apparatus is set at zero the three parts of the field are not uniformly bright, and that the error is not the same for different wave-lengths. In order to correct for this,

the readings of the apparatus are recorded when the field is uniform for different wave-lengths. A shutter eyepiece is fixed to the telescope to cut down the light except over a narrow central strip, the wave-length for which is recorded on the drum. When the central portion is the brighter, rotation of the analyser cuts it down, and the readings on both sides of the zero are observed and the mean taken for a series of different wave-lengths and a curve plotted, showing the relation between error and wave-length.

These readings are observed on the degree scale or density scale as may be required.

If the central part of the field is the darker it is not possible by merely turning the analyser to bring about uniformity. In this case

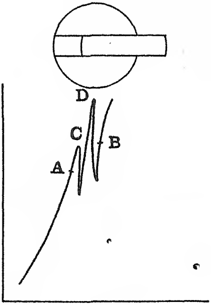


FIG. 274

a weak absorber, e.g. a plate of glass, is put into the path of the beam just before it enters Q. The thickness is chosen so that in the zero position the central field is just stronger, when the correction may be made as before. This should not often occur because the instrument is made so that the central portion is brighter than the outer, so that as the apparatus deteriorates with time, the central portion may still be the brighter and the interposition of the weak absorber may be avoided.

The substance of which the absorption is required is placed in the path of the light before it enters Q.

In the case of a liquid it is necessary first to measure the absorption of the vessel containing it.

The observations consist of noting the readings on the scale, G, when the field is uniformly bright for both directions of rotation of G.

Let I_0 denote the intensity of the light entering at Q′, and let I

denote the intensity of that which enters at Q. The density scale records the values $\log \frac{I_0}{I}$, i.e. the logarithm of the ratio of the intensity of the light entering the medium to that transmitted by it.

If this number be divided by the thickness of the material traversed, a quantity known as the 'extinction coefficient' is obtained.

Let a denote the amplitude of the polarized light transmitted by N_1 and suppose that the analyser is turned through an angle, θ, from

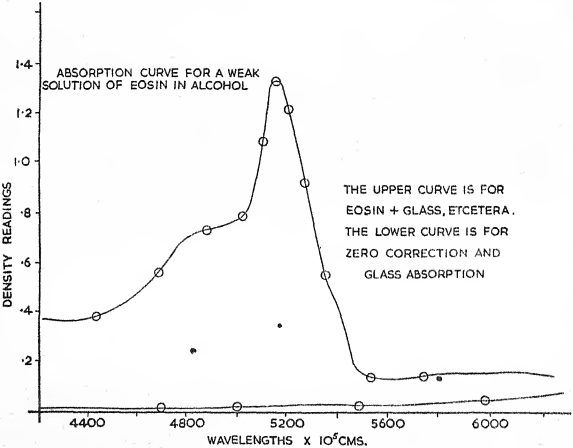


FIG. 275

the position in which it is parallel to N_1. The amplitude of the vibrations transmitted by the analyser is therefore $a \cos \theta$.

When the field is uniformly matched the intensity is I, and hence

$$\frac{I_0}{I} = \frac{a^2}{a^2 \cos^2 \theta},$$

$$\therefore \log \frac{I_0}{I} = \log \sec^2 \theta.$$

This shows the relation between the density scale and scale of degrees.

A solution which gives a characteristic absorption curve is one of eosin in alcohol. Eosin may be obtained from a bacteriological

laboratory and the solution must be very dilute, or so much absorption takes place that the transmitted light is very feeble.

Carry out the determination described above and plot on the same curve the values of the 'density', $\log \frac{I_0}{I}$, and the corresponding wave-lengths, for the zero error, for the vessel alone and for the vessel and solution.

From these it is possible to obtain the value, $\log \frac{I_0}{I}$ for the solution alone.

Fig. 275 shows the result of an experiment with eosin. Sometimes the variations occur very rapidly and a curve is obtained like that drawn in diagram 274. Unless frequent observations are made in the neighbourhood of AB the variations at C and D may be missed. Whenever the curve shows any sign of change the neighbourhood where this occurs must be examined with care.

It may not be necessary to determine exactly the value of $\log \frac{I_0}{I}$ for the solution, and in this case it is sufficient to make the zero correction and that for the containing vessel together, the whole appearing as a combined correction curve.

SOUND

To Find the Frequency of a Note by means of the Siren

In this instrument a musical note is produced by puffs of air following one another in rapid and regular succession. The series of puffs is produced by blowing air through a number of holes in a rapidly rotating plate.

The diagram (fig. 276) illustrates the instrument. It consists of a cylindrical metal chamber provided with a tap through which air can

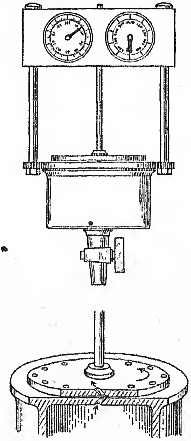


Fig. 276

be blown. In the upper end is cut a series of holes lying regularly spaced on a circle with its centre on the axis of the cylinder. Above this lies a circular metal disk provided with a similar series of holes which fit above the former. The disk is mounted so that it can rotate about the axis of the cylinder and so alternately cover and expose the lower series of holes.

The two series of holes slant in opposite directions as the figure shows, and when a current of air is blown into the chamber and the disk given a slight rotation, the puffs of air on escaping produce a pressure which drives the disk.

By adjusting the influx of air the regularity and speed can be controlled so that notes of varying frequencies can be produced. If N is the number of revolutions made by the disk per second, and the number of holes is n, the frequency is Nn. In order to measure the number of revolutions the disk is provided with a metal bar provided with a screw at one end, which works two dials, one registering units and tens, and the other hundreds of revolutions.

This form of apparatus is due to Cagniard de la Tour, but it has the disadvantage that the speed can only be increased with greater air pressure and a consequently louder note. It is also difficult to keep the speed uniform.

It is preferable to drive the disk with an electric motor, of which the speed may be regulated by including a resistance in the supply circuit, and the holes should be cut normally to the disk in order to avoid air pressure in the direction of the rotation.

The siren gives a large number of harmonics, and it is necessary carefully to single out the fundamental note.

Let it be required to find the pitch of a given note, as for example, that produced by an open organ pipe.

Carefully adjust the speed of the siren until beats are heard between the note it gives and that of the organ pipe by blowing at a particular pressure from a bellows connected to the chamber. The blower should then endeavour by a slight change of pressure to produce from the siren a note giving no beats. To some extent the frequency may be controlled by the tap, but it is important to keep a steady pressure on the bellows.

At the same time a second observer should measure the speed of revolutions by observing the number of revolutions recorded on the dials in a definite time (30 or 60 seconds).

Verify the result of the determination of frequency by measuring the length of the pipe and its diameter. For an open organ pipe emitting the fundamental, the wave-length is approximately twice the length.

The correction necessary to obtain a more accurate result is to add 0·6 radius for the open end and 2·8 radius for the flute mouthpiece. If the length of the pipe be l, and the radius of the pipe, r, the half wave-length is given by

$$\frac{\lambda}{2} = l + 3{\cdot}4r,$$

or

$$\lambda = 2\,(l + 3{\cdot}4r)$$

The velocity of sound in air for the temperature at which the experiment is carried out may be obtained from formula (A), p. 450.

Thus the frequency is $\dfrac{V}{\lambda}$.

Determine the frequencies of several organ pipes theoretically and experimentally. Draw up a table recording the speed of rotation of the siren disk and the observed and calculated frequencies in each case.

The Tonometer

In Scheibler's tonometer a number of tuning-forks are arranged in ascending order of frequency, each of which gives the same number of beats with its neighbour. The forks thus form a series in which the frequency increases by equal steps and they are arranged so that the highest frequency is twice that of the lowest.

In Appunn's tonometer the forks are replaced by reeds set in vibration by a blast of air from bellows of large capacity, and the apparatus has the appearance of a small harmonium provided with a series of stops by means of which any note may be sustained.

This form of apparatus is not so accurate as the original one, for Lord Rayleigh has shown that the frequency of a vibrating reed is to some extent affected by the vibrations of its neighbours. As it is necessary to vibrate two successive notes in the experiment we have no longer a constant register of frequency as in Scheibler's instrument.

It is first necessary to find the absolute frequency for each note on the instrument. Suppose there are $(k + 1)$ notes, and consequently k intervals between them, and that the frequencies are

$$N_1, N_2, \ldots, N_{k+1},$$

beginning from the lowest.

If the number of beats be observed between all the successive notes and be denoted by

$$n_1, n_2, \ldots n_k,$$

respectively, we have the following relations:

$$\begin{aligned} N_2 - N_1 &= n_1, \\ N_3 - N_2 &= n_2 \\ &\cdots\cdots \\ N_{k+1} - N_k &= n_k. \end{aligned}$$

Then adding both sides

$$N_{k+1} - N_1 = n_1 + n_2 + \ldots + n_k.$$

But $$N_{k+1} = 2N_1;$$

$$\therefore \quad N_1 = n_1 + n_2 + \ldots + n_k.$$

We thus find N_1 by counting the successive numbers of beats, and we can then deduce from the equation the frequencies of all the other

notes of the series. In practice, of course, the number of the beats will vary between the successive notes in different parts of the scale to a slight extent.

In order to determine the frequency of the lowest note, count the number of beats at five or six different parts of the range and deduce the average difference of frequency.

Let this be denoted by n. Then the frequency of the lowest note is $k \times n$.

After this determination has been made, the frequency of any given note coming within the range may be determined. For example, suppose the frequency of a fork is required. Find by trial the note nearest to it in pitch and count the number of beats when the notes are sounded together.

If this number be denoted by x and N_l is the frequency of the note nearest the unknown, the frequency of the latter is $N_l \pm x$.

The frequency of the note next above N_l is $N_l + n$, and on sounding this and the unknown note together the frequency of the beats will be less than n if its pitch is higher than that of N_l, and greater than n if its pitch is lower. This enables a distinction to be made between $\pm x$.

In making this last part of the experiment it is advisable not to rely on the accuracy of the average number of beats for each interval, but to measure separately the frequency of the beats of the notes immediately above and below N_l.

It will then be easy to decide on the exact position of the unknown pitch above or below N_l.

The Determination of the Frequency of a Tuning-fork by the Method of the Falling Plate

A smoked glass plate, P, is suspended vertically by a piece of thin string or thread over two nails, QQ, the thread being attached to the upper edge of the plate by means of sealing-wax or by any other convenient method.

The fork is held in a clamp, H, and carries a light style of bristle or thin aluminium wire attached to one prong by as little wax as possible. The style is just in contact with the plate so that when the plate falls it removes some of the soot and leaves a trace.

In order to prevent breakage, a padded wooden stand, AB, is placed just below the plate.

The fork is stroked gently by means of a violin bow and the plate is allowed to fall straight down by burning the thread between the nails, QQ.

A wavy line is traced by the style similar to that shown in fig. 278, but with more waves, and usually of small amplitude.

A point, O, is chosen just clear of the indistinct portion drawn when the plate was moving down in the first stage of the motion, and

consequently before its velocity had sufficiently increased to open out the waves. From it is counted a number, n, of complete waves to the point S. Again, n waves are counted to the point S_1. Let the spaces, OS and SS_1, be of lengths, s and s_1, respectively. Then the time taken to fall over these two lengths is the same, let us say t.

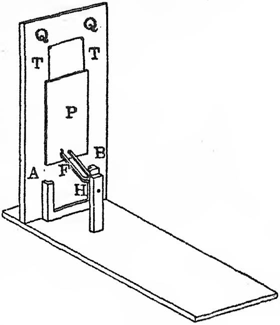


FIG. 277

If N denotes the frequency of the vibrations

$$n = Nt.$$

Let u denote the velocity of the plate at the instant corresponding to the mark, O.

Then by the equation for space described under acceleration, g,

$$s = ut + \tfrac{1}{2}gt^2,$$

and $$s + s_1 = 2ut + 2gt^2,$$

for the time of description of $s + s_1$ is $2t$.

Hence $$s_1 - s = gt^2,$$

or $$t = \sqrt{\frac{s_1 - s}{g}};$$

$$\therefore \quad N = n \cdot \sqrt{\frac{g}{s_1 - s}}.$$

The distances, s and s_1, are measured carefully by means of a travelling microscope and the value of N obtained is that for the fork vibrating with the load, consisting of the style and wax, and affected by the friction of the style against the plate.

This should be allowed for by taking a second fork of nearly the same frequency as that under examination, before attaching the style, but of slightly higher pitch. Carefully load this fork by adding wax until no beats are heard when both sound together.

Then, when the first fork is loaded and has the style touching the plate, as in the experiment, again sound the two together and count the beats. The number of these per second gives the number of periods lost per second on account of the loading and friction.

This number added to the value of N, determined in the experiment, gives the corrected frequency.

In marking off the points O, S, and S_1, be careful to choose them at corresponding points of the waves.

If O is at the summit of a crest, S and S_1 must lie in a similar position n and $2n$ waves later, respectively.

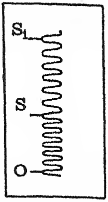


FIG. 278

The plate may be conveniently blacked by holding it just over a turpentine flame which gives a good deposit of soot; the flame of a paraffin lamp also gives a satisfactory deposit.

Chronographic Methods of Determining the Frequency of a Fork

In both the methods to be described under this title a fork is set in vibration and electrically maintained, with a style attached to one of the prongs lightly touching the blackened surface of a cylinder which can be rotated about its axis.

On rotating the cylinder, the track made by the style appears as a wavy line which can be opened out, so that each vibration is distinctly separated from its neighbours, by adjusting the speed of rotation.

The drum may be rotated by the handle, H, or by a falling weight not shown in the diagram, which rotates the drum at a convenient speed.

In this case the rotation is regulated by a governor and by releasing a catch the handle, H, can be employed for rapid movement of the

drum. This also is not shown in the figure. The drum moves along an axis cut with a screw thread, so that the track drawn by the style forms a wavy helix round it.

It is then necessary to have some record of time with which to compare the vibrations. The two methods differ in the way the time is obtained.

In the first a stand is mounted conveniently near to the fork, as illustrated in fig. 279, which carries a pointer actuated by an electromagnet, M. The pointer, P, is carried on a lever one end of which consists of a strip of iron or steel, which is attracted by the electromagnet core when a current flows through the exciting coils. When

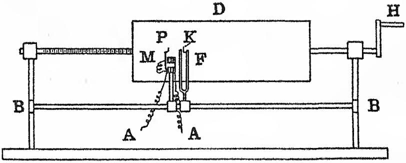


FIG. 279

the current is off, the lever is held in a position with the strip a short distance from the magnet by means of a spring.

Thus, as the cylinder rotates, a line is drawn round the surface. On passing the current in the coils the pointer moves and makes a kink in the line. If the current is put on at regular and known intervals, a time record along the side of the waves is produced on the drum. Thus by counting the number of waves between consecutive kinks the frequency of the fork may be deduced.

The regular intervals are obtained by connecting the wires, AA, to a battery, and completing the circuit by means of the mercury cup, M, and the pendulum, P, as shown in fig. 280.

Mercury is poured into a hole cut in wax, W, so that it stands just above the wax surface, and the wax so placed that a strip of wire hanging down from the pendulum just touches the surface as it passes its lowest position.

The circuit is completed, twice in each complete period and at each instant the pointer makes a record on the drum.

It is improbable that the interval between successive records will be one-half a period, since this would require exact coincidence of the point of contact with the mercury and the lowest point of the swing. The time between alternate records will, however, measure the time of a complete period.

Thus, in counting up the vibrations, find the mean number between alternate records.

Obtain a long helix, begin at the first stroke of the pointer, and count the number of vibrations up to some later odd-numbered stroke. Find the mean number per complete period. Repeat this, beginning with the second stroke and ending at some later even-numbered stroke, and find the mean number again. The two values should agree, but if there is a slight variation take the average value of the two results.

To obtain a blackened surface, take a sheet of smooth white paper and wrap it round the drum, one layer thick, holding it in position by

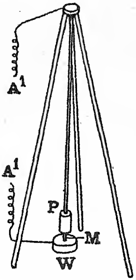


Fig. 280

means of gummed paper. To blacken it, rotate it over a turpentine flame, or coat it with camphor smoke.

The coating of soot should not be very thick, the style will then remove the soot and leave a white wavy trace.

The paper may be smoked again when once used and the track covered up.

The time of a complete period is measured in the usual way, by timing the pendulum.

The frequency obtained is, of course, subject to a correction similar to that of the last experiment and this should be determined and applied in the manner described.

Fig. 281 represents a convenient method of carrying out the determination in an alternative way.

The cylinder is held vertically, and is rotated by a handle or string round a drum as shown.

The fork is maintained electrically as before, but the axle of the drum is connected to one secondary terminal of a small induction coil, while the second terminal is connected to the fork. The primary circuit

is completed through a pendulum and mercury cup as before. Each time the primary circuit is completed, the induction coil is excited and a spark passes to the drum from the style, knocking off a little soot and leaving a white dot to record the instant of closing of the primary circuit. The same procedure and precautions are adopted as before. A thin, flexible copper wire will be found suitable for suspension of the pendulum, and with a small induction coil there is no inconvenience on account of shocks obtained when the apparatus is touched.

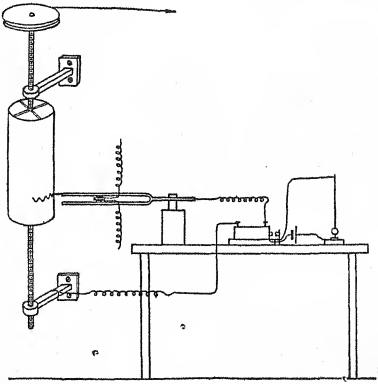


FIG. 281

This method was employed by A. M. Meyer. It should be noted that uniformity of motion of the drum is not necessary. The methods are both inferior to those in which an optical method is employed for determining frequency as in Rayleigh's method.

The Frequency of a Tuning-fork by the Stroboscopic Method

The fork is fitted with two very light plates fastened at the extremity of the prongs, one to each, and so that one may vibrate freely past the other. These plates may be of thin cardboard or aluminium, so that the loading affects the fork to the smallest possible extent, and they are stuck on to the fork by means of a little wax. A disk is taken provided with a number of dots placed at equal intervals round circles concentric with the disk. Each circle has its own interval length.

The fork is maintained electrically (p. 123), and the disk placed

behind it with its dotted surface brightly illuminated. Each plate attached to the fork is provided with a slot, and when the fork is at rest the slots lie directly behind one another. Thus the disk can be seen through them.

The slots and one of the circles of dots are so placed that the dots can be seen by looking through the slot, as the disk slowly rotates. If the fork is vibrating, it is possible to see through the slots twice in each complete period, and thus $2n$ times per second, where n denotes the frequency.

The disk is caused to rotate uniformly by means of an electric motor provided with a resistance in circuit to vary the speed.

The speed is gradually increased by adjusting the resistance until when the fork is vibrating, the dots in one of the circles appear to be at rest.

In this case a dot moves up as the disk rotates, so that each time the slots are in line a dot is just in line with them. The eye sees apparently one stationary dot, and the effect of rotation is lost by looking through the slot.

In order to count the number of rotations per second made by the disk, a counting arrangement is attached and the time taken over a definite number of rotations, when the dots remain apparently steady, by means of a stop-watch.

Let there be N rotations per second in the case when the dots belong to a circle containing p of them.

In this case the time taken by a dot to take the place of the one preceding it is $\frac{1}{\mathrm{N}p}$ second. This is equal to one-half the period of the fork.

Hence
$$\frac{1}{\mathrm{N}p} = \frac{1}{2} \cdot \frac{1}{n},$$
or
$$n = \tfrac{1}{2}\mathrm{N}p.$$

Now further increase the speed of rotation until once more the dots appear steady. In this case the speed of the disk is such as to cause a dot to take the place of the dot two intervals in front. The time taken to do this is $\frac{2}{\mathrm{N}'p}$, where N' is the number of rotations per second.

From this we may calculate the value of n once more.

On further increasing the speed until a dot takes the place of another three intervals in front, we can obtain a third calculation. Repeat this for several series of dots.

With the fork loaded it is necessary always to correct for the loss of pitch due to loading as before.

We may, however, avoid this by making one prong of the fork bright over a small area, and by rotating a disk provided with several ...

of concentric holes in front of it. If the fork is well illuminated and the disk carefully mounted, it will happen that for some particular speeds the fork appears stationary when seen through the holes.

The experimental details and the mode of deduction of the frequency of the fork from the observations are similar to the former.

The observations in stroboscopic experiments can nowadays be made much more conveniently by the use of a neon lamp than by the method described above.

This lamp has the property that it lights up immediately a voltage is supplied to it without any appreciable lag, and is extinguished immediately when the voltage is taken away.

The lamp consists of a flat piece of aluminium with a rod of the same metal lying a short distance from it and parallel to it in a bulb containing neon at low pressure.

When included in the secondary circuit of an induction coil, it lights up each time the current in the primary is broken. There is no effect at 'make', because in the construction of the usual type of induction coil the effect at make is suppressed and that at break of the primary circuit intensified.

If the primary coil is put into the circuit which actuates the fork, the primary current is made and broken once per vibration of the fork and the lamp flashes out once per complete vibration.

Of course, the make and break attached to the primary must not be allowed to work, the fork takes its place. All that is necessary is to make the connexions in the usual way and to screw the platinum-iridium point close up to the clapper to prevent separation and to place some object, e.g. a small block of wood, between the soft iron on the clapper and the end of the armature to prevent oscillation as the current fluctuates.

The lamp is used to illuminate the rotating disk provided with a series of dots on a white background and the speed adjusted until one row of dots appears stationary.

When this is the case one dot just moves up to take the place of a dot somewhere in front of it during the interval of darkness between the flashes of the lamp, i.e. during the period of vibration of the fork.

The calculation of frequency is made as before.

It is a great advantage to be able to avoid loading or marking the fork and to have the disk in any convenient position where it may easily be observed.

Instead of a series of concentric circles with regularly spaced dots, it is more convenient to draw on a circular disk a series of concentric regular polygons. A small triangle is drawn just about the centre of the disk, about this a square, then a pentagon, and so on. The triangle may be coloured white, the space between it and the square blackened, the space between the square and pentagon coloured white, and so on

alternately. When the speed is adjusted exactly, one of these figures appears stationary and the frequency is easily calculated. By varying the speed the figures may be made to appear stationary in turn and several determinations of frequency made.

The Composition of Two Simple Harmonic Vibrations in the Same Direction (Beats)

An apparatus which will combine graphically two simple harmonic vibrations in the same direction has been invented by Koenig. It consists of a large fork mounted on a stand and provided with a clamp by means of which a strip of glass can be held horizontally and fastened to one of the prongs (fig. 282).

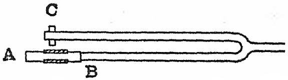


FIG. 282

AB denotes the strip, and C a weight attached to the other prong for the purpose of balancing. The glass is coated with a thin layer of lamp-black, by means of a smoky flame.

A second fork is mounted above the first and carries a light style adjusted so that it just touches the glass plate.

This fork is fixed to a sliding base by means of which the style can be drawn along the smoked plate.

If both style and plate are vibrating, a curve can thus be traced which represents the motion of the upper fork relative to the lower.

Thus, if at any instant the lower fork is displaced a distance, y, from the standard position, and the upper is displaced a distance, y', the displacement of the style over the plate is $(y' - y)$. The forks are made to vibrate with nearly equal amplitudes. This may be done by bowing or by using a strip of metal which is wide enough to open out the forks to a convenient extent; the metal is then quickly removed. The slider is drawn along, not too quickly, and the trace obtained examined.

When the forks have nearly the same frequency this will consist of a wavy line with waves of varying amplitude. In this case, with the amplitudes of the forks nearly equal, the smallest waves will have almost zero amplitude, and the fluctuation expresses graphically what the ear recognizes as beats.

The time between consecutive minima measures the interval between the beats.

If the glass plate is long enough, three or four intervals extending

the slider must be adjusted so that the individual vibrations are drawn out to an extent which enables them to be counted easily.

It is best to count the number of intervals on the plate and the total number of vibrations between the first and last minimum points. This enables the mean number of vibrations between consecutive beats to be obtained.

Let this number be x.

We shall suppose that the two vibrations have frequencies n and $(n+m)$ per second, so that they may be represented by

$$y' = a \sin 2\pi (n+m) t$$

and
$$y = a \sin (2\pi nt + \alpha),$$

where α is included to take account of any difference in phase that may exist when the vibrations begin.

The resultant displacement recorded on the plate is

$$\begin{aligned} Y = (y' - y) &= a \{\sin 2\pi (n+m) t - \sin (2\pi nt + \alpha)\} \\ &= 2a \cos \left\{2\pi \left(n + \frac{m}{2}\right) t + \frac{\alpha}{2}\right\} \sin \left(2\pi \frac{m}{2} t - \frac{\alpha}{2}\right). \end{aligned}$$

In the cases when beats are heard, m is much smaller than n or $\left(n + \frac{m}{2}\right)$, so that the simplest way of regarding this expression is to consider it as a S.H.M. of amplitude:

$$2a \sin (2\pi \,.\, \tfrac{1}{2}mt - \tfrac{1}{2}\alpha) = A \text{ (say)},$$

and then
$$Y = A \cos \left\{2\pi \left(n + \frac{m}{2}\right) t + \frac{1}{2}\alpha\right\}.$$

represents a S.H.M. of amplitude, A, and frequency, $\left(n + \frac{m}{2}\right)$.

The amplitude, A, attains a maximum value, $2a$, and sinks to zero alternately. It is zero for the values of t given by

$$\pi mt - \tfrac{1}{2}\alpha = 0, \pi, 2\pi, \text{etc.},$$

i.e. for values of $t = \dfrac{\alpha}{2\pi m}, \quad \dfrac{\alpha}{2\pi m} + \dfrac{1}{m}, \quad \dfrac{\alpha}{2\pi m} + \dfrac{2}{m}$, etc.,

i.e. at intervals of $\dfrac{1}{m}$ seconds, or m times per second.

Thus the number of beats is m per second, the same number as the difference between the frequencies of the two notes. The frequency of the note heard is $\left(n + \frac{m}{2}\right)$.

In the curve traced on the plate we shall find $\left(n + \frac{m}{2}\right)$ vibrations and m beats per second. The number of vibrations between two beats is

$$\frac{1}{m}\left(n + \frac{m}{2}\right).$$

But this is counted on the plate and found to be x.

Thus $$\frac{1}{m}\left(n + \frac{m}{2}\right) = x,$$

or $$n = m\,(x - \tfrac{1}{2}).$$

The beats can be timed by means of a stop-watch.

When the notes are sounding, count them for as long as possible, and take the time of the interval. If the number of beats counted is N and the stop-watch is started on the first and stopped at the Nth, the interval between the beats is

$$\frac{\mathrm{T}}{\mathrm{N} - 1},$$

where T is the time interval recorded on the stop-watch.

Thus $\frac{1}{m}$ is known, and the above equation gives n. The frequencies of the notes are thus n and $(n + m)$.

The method must be regarded as an illustration of the phenomenon of beats—it is not an accurate method for the determination of frequency.

The Composition of Two Simple Harmonic Vibrations Perpendicular to one another. (Lissajou's Figures)

Let the co-ordinates of a point, P (fig. 283), be (x, y) and let P move so that $$x = a \sin pt$$
$$y = b \sin (p't + \alpha).$$

The motion of P then consists of two simple harmonic motions along two perpendicular directions.

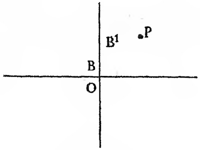


Fig. 283

When t is zero, x is also zero; but y has the value $b \sin \alpha$.

Thus the two S.H.M.s are in different phases.

From the equations it follows that:

$$t = \frac{1}{p} \text{ arc sin } \frac{x}{a}$$

and $$t = \frac{1}{p'} \text{ arc sin } \frac{y}{b} - \frac{\alpha}{p'}.$$

Hence, between x and y there exists for all values of t the relation:

$$\frac{1}{p} \text{ arc sin } \frac{x}{a} = \frac{1}{p'} \text{ arc sin } \frac{y}{b} - \frac{\alpha}{p'}.$$

This represents the curve on which P lies.

When the relation between p and p' is simple, as for example:

$$p = p', \quad p = 2p', \quad 2p = 3p', \text{ etc.},$$

the point, P, describes the curve completely in a short time, and afterwards retraces it.

The case, $p = p'$, is very simple, for we then have

$$x = a \sin pt,$$
$$y = b \sin (pt + \alpha).$$

In the general case this represents an ellipse with its centre at the origin.

If $\alpha = 0$, then

$$\frac{x}{a} = \frac{y}{b},$$

and P describes a straight line.

If $\alpha = \frac{\pi}{2}$, P describes the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is an ellipse whose principal axes lie along OX and OY.

In the case when the amplitudes of vibration are equal along both directions, $a = b$ and the locus of P is the circle

$$x^2 + y^2 = a^2.$$

Thus when the periodic times are equal, in general P describes an ellipse; special cases of this general case are the straight line and circle. All these curves are therefore appropriate to the case of equal frequencies.

If OB is equal to $b \sin \alpha$, B represents the position of P at the time $t = 0$, or at instants later by a complete period than this initial time.

If the frequencies are not quite equal, let us say that the frequency along OY is a little the greater, then when another period is complete the displacement along OY will be a little different from that in the first case, i.e. P will lie at some point, B′.

If $$OB' = b \sin \alpha',$$

α' is slightly different from α, and there has been a slight change in phase on the part of the OY motion. Thus α will continually vary, and will pass through all the values from 0 to 2π. This will cause a continual change in the shape of the curve described by P. It will sometimes be a straight line and sometimes an ellipse. If it happens that the two amplitudes are equal, we shall have a circle sometimes.

The closer p' is to p, the more slowly this change takes place, so that by watching the movement of P we can test the closeness of the two frequencies. If the figure is maintained steady, without change, the frequencies are equal.

The same argument can be applied to the cases where one of the other simple relations exists between p and p'.

When the shape of the curves corresponding to these relations is known, the approximate ratio of the frequencies can be recognized and the exactness of the ratio tested by observing the rate of change of shape throughout the series.

We can deduce the ratio of the frequencies by examining one of the curves, e.g. fig. 284. It cuts the Y axis in four points and the X axis

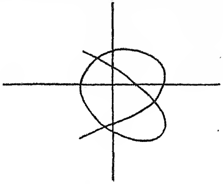


FIG. 284

in three, so that the frequencies are in the ratio, 4 : 3, for the vibrating point makes four vibrations parallel to OX in the same time that it makes three parallel to OY.

This principle may be employed to investigate the way the period of vibration of a rod, fixed at one end, varies with the length of the rod. The apparatus consists of a vertical flat rod or spring with a lens fixed at its upper end and a horizontal spring carrying a screen with a small aperture (fig. 285).

Light from a source, S, is focused by a lens, L_1, on to the aperture, A, while the lens, L_2, carried by the vertical spring, V, focuses an image of the aperture on the screen so that we have a bright point, P.

Motion of the horizontal spring alone causes P to trace a vertical line, and represents the S.H.M. of A.

Motion of the vertical spring alone produces a horizontal S.H.M. on the screen. When both springs vibrate together, the path of P represents the combination of the two motions.

The vertical spring is of fixed length, but the horizontal spring can be clamped at various points, so that the vibrating length can be adjusted.

Make the first adjustment so that P describes an ellipse, straight line, or circle without change of form.

In this case the periods are the same.

Now change the length until another steady curve is obtained without change of form. Draw it carefully and deduce the ratio of the periods of vibration.

Make several determinations of the ratio of frequencies and corresponding lengths, and draw a curve showing the relation between the length and frequency, taking the vertical spring as a standard.

Fig. 286 shows the curves described by P for a few frequency ratios which will serve for reference.

The Vibration Microscope

The essential features of this apparatus are the same as those described in the last experiment. The two vibrating springs are

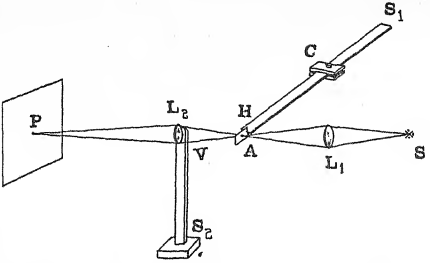


Fig. 285

replaced, one by a fork and the other by a vibrator, the frequency of which is to be compared with that of the fork.

The second vibrator is sometimes another fork or violin string. A bright source of light such as a speck of chalk is attached to the second vibrator, while the fork carries, attached to one prong, a lens which forms the object glass of a small microscope.

If the lens alone vibrates, on looking through the eyepiece the motion of the chalk is simple harmonic, on account of the vibrations of the lens. If the other vibrator is in motion and the lens is at rest, the motion observed is, of course, that of the vibrator alone. These two motions are arranged to take place in two perpendicular directions so that a figure of the type described in the last experiment is observed. It is steady if the frequencies are exactly adjusted, but goes through the appropriate series if the frequencies are not identical. The rate of progress through the series may be observed and determined by means of a stop-watch. After one completion of the cycle there has been a gain of a whole vibration by one vibrator over the other. If the

frequencies be N and N′, and the time for completion of the series is t seconds, then
$$N \sim N' = \frac{1}{t},$$
for Nt and N′t are the numbers of vibrations made respectively, and these differ by one.

If the fork be slightly loaded we can find which is the greater of the frequencies by again observing the rate of progress through the

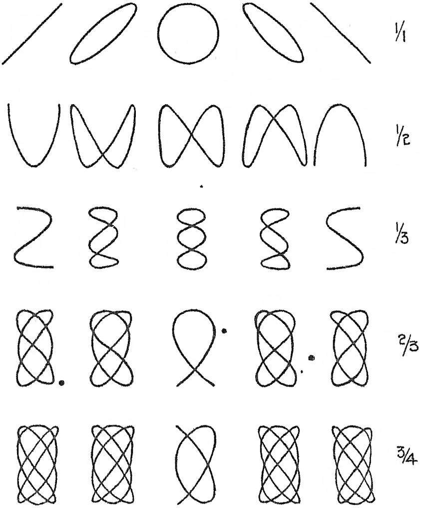


Fig. 286

cycle. If this time is shorter than before, the time of gaining a period is less than before and the frequency of the fork is the smaller.

We shall describe how the apparatus may be used to find out the character of the vibrations performed by a stretched string in the manner in which Helmholtz used the apparatus to examine the vibrations of a violin string.

The string, SS, is stretched below the prongs of the fork, PP, to one of which the lens, L, is attached. The part of the string to be

examined is slightly blacked by ink, rubbed with wax when dry, and powdered with starch or chalk. A few white particles will remain sticking to the string, and one of them is illuminated by a lamp and focused by the microscope and its movements observed. The tension of the string is adjusted until the figure apparently described by the chalk, as seen in the microscope, remains steady. The vibrations of the fork are electrically maintained while the string may be bowed or plucked.

The frequency of the fork is known, so that, as its motion is simple harmonic, we can find the displacement due to it at any of the instants

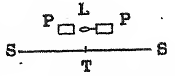


FIG. 287

during the vibration. The string vibrates at right angles to the direction of vibration of the fork, so that from the curve obtained we can subtract the vibrations of the lens and draw a curve showing the displacement of the string at different times.

The white speck is first obtained in the centre of the field, and its mean position represented by the origin, O (fig. 288).

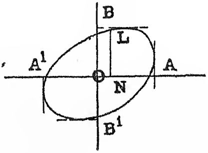


FIG. 288

The curve is drawn to a convenient scale accurately from measurements observed by the scale in the eyepiece. If the microscope is not furnished with a scale, throw an image of an illuminated scale to lie coincident with the string and view the white speck and image together.

In the figure, OA denotes the amplitude of vibration of the fork, its motion being assumed to take place along AOA^1.

If its frequency is n, the motion is given by

$$x = OA \sin 2\pi nt.$$

Thus for any time, t, we can find the displacement, ON, along the x-axis. At this instant the displacement along the other direction, i.e. due to the motion of the string, is NL. Thus by observing several

values of the ordinates and the times corresponding, we can plot a second curve showing the time-displacement for the string. Its shape will indicate the character or quality of the note emitted.

Fig. 289 shows the observed curve and the time-displacement diagram for a stretched string when bowed. The first curve was observed while the string was being bowed. The bow is drawn slowly and regularly across the string. Slight fluctuations are liable to occur during this process, but they appear as slight variations of a figure remaining, on the whole, permanent. The dimensions of this figure were obtained.

The deduction of the diagram for the string is made in the following way:

Draw on squared paper the figure observed (fig. 289), and draw the extreme vertical tangents, AP and A^1P^1. We are assuming that the vibrations of the fork are executed along AA^1.

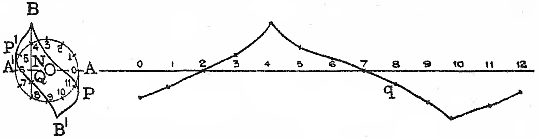


FIG. 289

With the middle point of AA^1 as origin, describe a circle on AA^1 as diameter, and divide the circumference into a convenient number of equal parts. In the diagram the number is twelve.

If a perpendicular be drawn from any point on the circumference of this circle on to AA^1, the displacement of the foot of this perpendicular from O will represent the displacement of the prong of the fork from its mean position.

Along the line, A^1A, produced, beginning at 0, twelve equal intervals are marked off, as shown, to represent intervals of time corresponding to the points on the circle. Beginning at A, draw ordinates to the curve passing through the points marked on the circle. For convenience in drawing, only one of these, that through the point, 8, is drawn. Let this cut the curve at the point, Q, and the line, AA^1, in N. Then NQ denotes the displacement of the string from its central position in magnitude and direction.

This displacement is plotted at the point, 8, on AA^1 produced, and we thus obtain a point, q, on the displacement diagram for the string. This process is carried out for all the points on the circle, and the diagram plotted.

It will be noted that the ordinate, NQ, cuts the curve in a second point, B. There is no doubt as to which point is to be taken when

actually drawing the curve, for we begin at A and pass along one branch of the curve and back along the other.

In this discussion we have associated the upper part of the curve from P to B and then to P^1, with the times corresponding to the points from 0 to 6.

Transverse Vibrations of Strings. (Melde's Experiment)

The object of the experiment is to verify the laws of vibration o a string under tension. In such a case a disturbance travels along the string with a velocity, v, given by

$$v = \sqrt{\frac{T}{m}}.$$

T denotes the tension expressed in absolute units, i.e. poundals or dynes, and m is the mass per unit length.

When the string is fixed at both ends, there is a node at each end in its fundamental mode of vibration, with a loop or antinode in the middle. The corresponding wave-length is twice that of the string. If this wave-length is denoted by λ_1, we have

$$n_1\lambda_1 = v = \sqrt{\frac{T}{m}},$$

$$n_1 = \frac{1}{\lambda_1}\sqrt{\frac{T}{m}},$$

n_1 denoting the frequency of this note.

This mode is illustrated in fig. 290. The string may also vibrate to produce the overtones or harmonics as illustrated in figs. 291 and 292.

FIG. 290

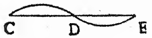


FIG. 291

FIG. 292

In these cases, if the frequencies are n_2 and n_3, and the corresponding wave-lengths λ_2 and λ_3, we have

$$n_2 = \frac{1}{\lambda_2}\sqrt{\frac{T}{m}}, \quad \text{and} \quad n_3 = \frac{1}{\lambda_3}\sqrt{\frac{T}{m}};$$

and if l denotes the length of the string:

$$\lambda_1 = 2l, \quad \lambda_2 = l, \quad \lambda_3 = \tfrac{2}{3}l, \quad \text{and so on.}$$

Thus $$n_1 : n_2 : n_3 : \ldots = 1 : 2 : 3 : \ldots.$$

In the experiment the string is set in resonant vibration by impulses having the same frequency as one of its modes of vibration. For this purpose ordinary string is unsuitable. It is not uniform and does not divide into equal segments; but it will be found that a length of fishing line is satisfactory, as a rule it is usually sufficiently uniform.

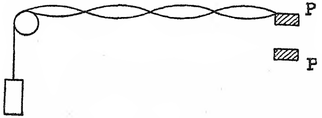


FIG. 293

One end of the cord is attached to the prong of a tuning-fork, by tying it to a small wire hook soldered on to the prong, or to a small screw which is held in a hole bored in the prong.

The other end passes over a small pulley and carries a weight which produces and measures the tension.

The vibrations of the fork are electrically maintained (see p. 123), and by properly adjusting the length and tension the string can be made to break into stationary undulation with well-defined nodes.

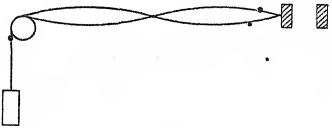


FIG. 294

The fork may be placed so that the motion of the prongs is in the direction perpendicular to the string (fig. 293), or along it (fig. 294).

In the former case the frequency is the same as that of the fork, in the latter it is half as great.

For when in the second case the prong is in the extreme position on the left, the string is slack in the first vibration, and when in the extreme position on the right, it is horizontal and tight. The inertia of the string carries it onward so that when the prong returns to the extreme left position, and thus completes one vibration, the string completes a half-vibration.

The student should examine the interesting effects produced when the prong moves in a direction between these two and thus produces in the string a combination of the two nodes of vibration.

We shall consider the former case only in this description of the experiment.

The frequency of the fork is denoted by N, and when the string is in resonant vibration, this is also the frequency of the mode of vibration of the string, corresponding, let us say, to a wave-length, λ.

Then
$$N = \frac{1}{\lambda}\sqrt{\frac{T}{m}},$$

or
$$\frac{\lambda^2}{T} = \frac{1}{mN^2} = \text{constant}.$$

Thus by varying the tension and consequently the wave-length, we should find $\frac{\lambda^2}{T}$ constant.

To determine λ, measure the distance between the first well-defined node on the right of the string and the last on the left. Let this be d and suppose there are k loops between, then,

$$\lambda = \frac{2d}{k}.$$

Note that the ends of the string at the fork and the pulley should not be taken. There is a certain amount of movement at these two points.

Draw up a table showing values of λ, T, and $\frac{\lambda^2}{T}$.

KUNDT'S TUBE

(*a*) The Determination of the Velocity of Longitudinal Waves along Rods

Kundt's apparatus consists of a glass tube about a metre long, and of diameter about 3 cm., provided with an adjustable piston near one end. The tube is supported horizontally on a table by resting it on two wooden V-shaped stands.

Near the other end of the tube is a second piston, Q, attached to the end of a metal rod, DQ (fig. 295). This rod is clamped at its middle point, C.

For the purpose of the experiment the tube must be quite dry, and a light powder, such as lycopodium powder, is placed in a line at the bottom of the tube extending along its length between the two pistons. A convenient way of inserting the powder is to spread it along a metre rule, place the rule in the tube, and turn it upside down.

If the tube is not dry it must be warmed above a Bunsen flame, and a current of air blown through it.

The metal bar, which may be of brass with a diameter of about 0·5 cm., can be set into longitudinal vibration by stroking it along CD with a piece of wash-leather and powdered resin.

In the fundamental mode of vibration the ends, D and Q, are antinodes, and the fixed point, C, is a node. The wave-length is twice the length of the rod.

$$\lambda = 2l.$$

If the frequency of the note is n, and the velocity of the waves v,

$$n\lambda = v.$$

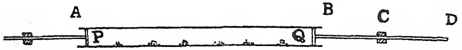


FIG. 295

If the distance between the pistons is L, the air between them will have a fundamental wave-length, 2L, and overtones with corresponding wave-lengths

$$\mathrm{L},\ \tfrac{2}{3}\mathrm{L},\ \tfrac{1}{2}\mathrm{L},\ \text{etc.}$$

The corresponding frequencies are

$$\frac{\mathrm{V}}{2\mathrm{L}},\ \frac{\mathrm{V}}{\mathrm{L}},\ \frac{\mathrm{V}}{\frac{2}{3}\mathrm{L}},\ \text{etc.},$$

where V denotes the velocity of sound in air.

If one of these frequencies is the same as that of the rod, the air will be set into strong resonant vibration, and will move the light powder. This will settle down at and near the nodes where the air is least in motion. As many as possible of these should be used to find the average distance between two nodes. Choose as carefully as possible the position of a node at one end of the tube and locate the node nearest the other end. Measure the length between these two points, and divide by the number of spaces, such as NM (fig. 296).

FIG. 296

The pattern will be somewhat similar to that in this figure, and the longest line of each set marks the nodal position.

It will probably happen that there is not exact resonance at first, but by slowly moving the piston, P, forward or backward, the length between the pistons may be adjusted so that resonance occurs.

Twice the distance between the nodes is the wave-length of the

sound in air. The velocity of sound at 0° in air is 33,060 cm. per second, and at a temperature, t

$$V_t = 33060\left(1 + \frac{t}{273}\right)^{\frac{1}{2}}. \qquad \ldots(A)$$

Thus the frequency, $n' = \dfrac{V_t}{\lambda'}$, where λ' is the wave-length in air. Since resonance occurs,

$$n = n';$$

$$\therefore \quad \frac{v}{2l} = \frac{V_t}{\lambda'},$$

or

$$v = \frac{2l}{\lambda'} V_t.$$

In carrying out the experiment it is a good plan for one observer to continue stroking the rod, while the other carefully adjusts the piston until the powder moves violently and settles down into the pattern of fig. 296.

(*b*) The Velocity of Torsional Vibrations in a Rod

If instead of stroking the rod longitudinally with the resined cloth, it is held near the end, D, and the cloth turned so that it slips over the surface in a direction that would cause the rod to rotate round its axis, a note is emitted of different frequency from that given when the rod is in longitudinal vibration. This note corresponds to torsional vibrations and will set the air in resonant motion as before.

Find in this way the velocity of these waves.

It is not easy to obtain a loud note by this method—the force applied should not be great, but with a little practice it should be possible to produce the note.

(*c*) The Determination of Young's Modulus and the Modulus of Rigidity

These constants may be determined from a knowledge of the velocity of longitudinal and torsional waves in the bar.

The formula for the former is

$$v_l = \sqrt{\frac{E}{\rho}},$$

and for the latter

$$v_t = \sqrt{\frac{n}{\rho}},$$

where E is Young's modulus, n the modulus of rigidity, and ρ the density of the material of which the bar is made. Its value may be taken from a table of physical constants. When expressed in C.G.S. units, E and n are expressed in dynes per sq. cm.

Find the values of E and n from the determination of the velocities in the previous experiments.

(d) The Velocity of Sound in Carbonic Acid Gas

Kundt's tube may also be used to determine the velocity of sound in gases. Suppose the tube to be filled with a gas in which the velocity is V_g and the wave-length, λ_g.

Then the frequency is

$$\frac{V_g}{\lambda_g} = \frac{V}{\lambda'};$$

$$\therefore \quad \frac{V_g}{V} = \frac{\lambda_g}{\lambda'} = \frac{\text{distance between nodes in the gas}}{\text{distance between nodes in air}}.$$

The procedure is, therefore, first to obtain resonance between the rod and air column and to find the mean distance between the nodes, then to drive out the air and fill the tube with the gas, and again obtain resonance and measure the distance between the nodes in this case.

V_g is then found from the last equation.

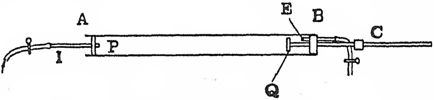


Fig. 297

The gas must be quite dry or the powder will stick to the glass and fail to respond to the motion of the gas when resounding. It may be necessary to pass it through drying tubes before filling the tube.

A slight modification of the apparatus is necessary for this purpose (fig. 297).

In the former case the adjustable piston may fit loosely, but for the present purpose it must be both adjustable and gas-tight.

It may consist of a cork round the outside of which a rubber band or piece of cloth is stretched.

The cork carries a tube to admit the gas and the piston may be adjusted by means of this tube.

The metal rod passes through a tightly fitting cork, also provided with a tube which can be opened and closed by means of a stop-cock.

We shall suppose that the velocity of sound in carbon dioxide is to be measured.

Connect the source of gas to the inlet tube, I, and open the exit tube, E, to the air. Allow the gas to flow in steadily so that it will fall to the bottom of the tube and the air will flow over at E, also, so as not to disturb too much the powder which we assume has been

The experiment should be performed close to an open window or in a draught cupboard to prevent escape of the gas into the room.

Continue the passage of gas long enough to ensure that the air is driven out and then close the inlet and outlet tubes by means of stop-cocks, and proceed as in the last experiment.

Care must be taken that the temperature of the gas is the same as that of the air, unless the temperature of the gas is measured in some way. If the gas is delivered from a cylinder it will be colder than the air, and it must be allowed to acquire the air temperature before closing the stop-cocks. Otherwise we shall be measuring the velocity in the air at one temperature, and that in the gas at another. Reduce the velocity to that at zero.

To do this, note that $\frac{V_g{}^t}{V^t} = \frac{V_g{}^0}{V^0}$ = ratio of nodal distances, where the affixes t and 0 denote temperatures, assuming the same coefficient of expansion for each gas.

In order to determine when the air is all driven out from the tube, collect a little of the gas issuing from E in an inverted glass cylinder over mercury, and introduce on to top of the mercury column a little of a solution of caustic soda or potash. By noting how much of the gas is absorbed it can be seen if any air is left. The CO_2 is all absorbed by the solution and no gas should be left.

(e) To Calculate the Ratio of the Specific Heats of a Gas

From the result of the last experiment we may determine the constant, γ, for carbon dioxide, i.e. the ratio of the specific heat at constant pressure to that at constant volume.

For the velocity of sound in a gas at 0°, is given by the formula:

$$V_g{}^0 = \sqrt{\frac{\gamma p}{\rho_0}},$$

where p is the pressure and ρ_0 the density of the gas at 0° C. The value of p may be determined by the barometer since the tube has been filled at atmospheric pressure.

p must be expressed in dynes per sq. cm. To make the calculation take the density of mercury as 13·60 gm. per c.c., and $\rho_0 = 0{\cdot}001974$ gm. per c.c.

The measurement of the distances between nodes may be performed simply by the ordinary use of a metre scale; but a slight addition to the apparatus will add to the accuracy.

A metre rule is fixed just below the tube and parallel to it, and sliding over the rule or along one of its edges is a wooden base, B, carrying a metal disk, D, with a hole, H, at the centre, and a frame, F, with cross-wires (fig. 298).

H serves as an eyepiece and HC is aligned on the nodes marked out by the powder.

An index on the base indicates the position of the stand.

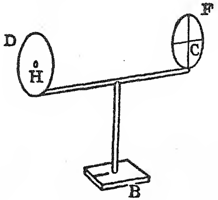


FIG. 298

If the apparatus is aligned consecutively on the nodes, and the positions of the index recorded, a table may be made out as follows:

No. of node	Reading at index	No. of node	Reading at index	Length of 5 half-waves
1		6		
2		7		
3		8		
4		9		
5		10		
	Average length of 5 half-waves ...			
	Mean half-wave-length			
	Mean wave-length			

Chladni's Figures

The nodes of a vibrating stretched string are points of its length where there is no motion. On either side of the node the string moves simultaneously in opposite directions. In the case of a vibrating plate there exist nodal lines, i.e. lines in the plate where there is no motion. They divide the plate into segments so that the parts on either side of the nodal line at any instant are moving in opposite directions.

The point of support of a plate is necessarily on a nodal line. Round and square plates are usually supported centrally in the experiment, but interesting results arise when they are supported elsewhere.

By touching any point on the edge or surface of the plate with the

if it lies on a nodal line the corresponding mode of vibration may be excited.

In the case of a circular plate clamped at the centre the nodal lines are radial, and the fundamental vibration gives two perpendicular

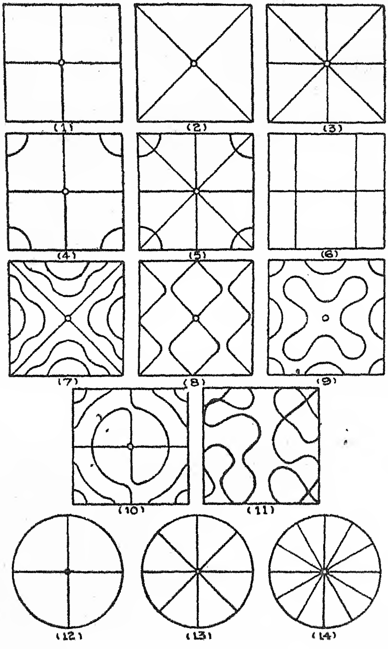


Fig. 299

diameters. These may be obtained by sprinkling white sand over the surface of the plate, touching two points on the edge separated by one-quarter of the circumference, and bowing vertically across the edge at a point one-eighth of the circumference from one of the fingers. A square plate clamped at the centre gives a large variety of

Some of these are shown in the figure in order to assist in producing them.

Some of the nodal points on the edge and surface of the plate should be touched, and the plate bowed along the edge at one of the points midway between the nodes. The fundamental vibration gives two perpendicular lines through the centre parallel to the edges of the plate.

The simpler figures are easy to obtain, but skilful manipulation of the bow is required to produce the more complicated ones.

The student should obtain as many as possible, and clamp the plate at other points than the centre to investigate other modes of vibration.

The Relation between Pitch and Volume in the case of a Narrow-necked Resonator

Consider the case of a bottle containing air closed by a piston, P, without friction in the neck of the bottle.

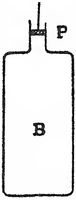


FIG. 300

Let v denote the volume of the bottle below the piston, m the mass of the piston, and A its area of cross-section. Let the piston be originally in the position of equilibrium and let the pressure outside be p_0, while that inside is p. Then we have

$$pA = p_0A + mg.$$

If now the piston be displaced downwards a distance, x, so quickly that the change may be regarded as adiabatic, a new pressure, p', will be generated, such that

$$p'(v - Ax)^\gamma = pv^\gamma.$$

We shall suppose that x is small, so that we may write

$$p' = p\left(1 - \frac{Ax}{v}\right)^{-\gamma}$$

$$= p + \frac{p\gamma A}{v}x.$$

The total force downward is now

$$mg + p_0A - p'A = (p - p')A$$

$$= -\frac{p\gamma A^2}{v}x.$$

Thus the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{p\gamma A^2}{v}x.$$

This is a simple harmonic motion and the time of vibration about the position of equilibrium is

$$2\pi\sqrt{\frac{mv}{p\gamma A^2}}.$$

In the calculation we have assumed that the pressure is the same throughout the gas during the oscillations.

This is not true, since time is required for the transmission of the pressure. The other assumption concerning the adiabatic character of the compression is very approximately true, especially as the neck is narrow and heat will not easily escape from the bottle or be transmitted to it. The piston will thus behave only roughly according to the formula, and will have an approximate period of oscillation given by the above value, i.e. it will have a frequency

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{p\gamma A^2}{mv}}.$$

Thus we have approximately

$$n^2v = \text{constant}.$$

It is the object of this experiment to verify to what extent this formula applies.

The piston, in practice, is the layer of air in the neck of the bottle and by pouring water into it, resonance is produced between it and a series of forks of known frequency. A medicine bottle will be found convenient and must first be calibrated by pouring in water from a measuring flask to various depths and measuring the height of the water surface above the bottom of the bottle. This should be done for a series of intervals up to the base of the neck.

Make a table thus:

Height of water	Volume poured in	Volume of air above height in column 1
h_1	v_1	$v_n - v_1$
h_2	v_2	$v_n - v_2$
h_3	v_3	$v_n - v_3$
.	.	.
.	.	.
.	.	.
h_n	v_n	0

v_n denotes the capacity of the bottle.

Draw a curve showing the relation between the heights of the water and the volumes of air above it, i.e. draw a curve with the values h as abscissae and the volumes in column 3 as ordinates. By means of this curve we can deduce the volume of air in the bottle corresponding to various heights of water.

Forks of pitch varying from 256 to 512 should be used and a curve plotted with volumes as ordinates and values of $\frac{1}{n^2}$ as abscissae. The resulting curve should be a straight line through the origin if the theory is correct, but in practice it will be found not to pass through the origin although the relation is very nearly linear. The curve will obey closely the law

$$n^2 (v + c) = \text{constant}.$$

The value of c is a correction to be applied to v.

This may be regarded as a neck correction, and the ratio of c to the volume of the neck should be recorded.

The agreement of pitch between the fork and bottle should be tested by blowing across the neck and noting beats between the note obtained and that of the fork.

Determination of the Frequency of a highly pitched Note by means of a Sensitive Flame

When a continuous note is produced in front of a smooth wall the reflected and incident trains of waves produce stationary undulation in front of the wall, and consequently there exist nodes and antinodes in the air. The experiment consists in locating the positions of consecutive nodes from the wall outwards. Thus the wave-length is determined, for the distance between two consecutive nodes is half a wave-length. By noting the air temperature and deducing the velocity of sound in air appropriate to it (eq. (A)), we may thus deduce the frequency of the note by the relation

$$V = n\lambda.$$

In order to locate the nodes a sensitive flame is used. This flame is produced by supplying the gas under pressure to a pin-hole burner. The requisite pressure can be obtained by leading the gas from the main into a large gas-bag and placing a board on the top of the inflated bag to carry a weight.

From the bag a pipe leads the compressed gas to the burner. When the bag is full, turn off the main tap and put weights on the board so that a tall flame from one to two feet high is produced and there is no flaring. It will be found that in the sensitive state, on making slight noises as, for example, by jingling keys in the neighbourhood of the jet, the character of the flame changes. It flares and shortens

recovering its former state when the sound ceases. This is caused by the motion of the air as the sound-wave passes.

Start with the flame close to the wall and sound a note from a highly pitched whistle, or other suitable source, and move the flame slowly outward from the wall.

It will be found that at certain points the flaring ceases and the flame increases in length. At these points the air is still, and the points are at nodes of the stationary wave motion.

If a smooth wall is not conveniently situated, set up a large sheet of glass or smooooth board at the end of a table, and move the flame outward from the surface along the table. The flame will probably need adjusting before it will respond readily. To do this use the tap leading from the bag to the burner, and also vary the weights producing the pressure. It appears to be necessary to use a rather long gas tube to convey the gas from the bag to the jet.

In obtaining the most sensitive flame it is to be noted that the orientation of the flame is important and the burner should be turned about a vertical axis, so that different sides of the flame are presented towards the direction of the sound. The flame appears to have different degrees of sensitiveness on different sides.

It is possible to produce a good sensitive flame by leading the gas from the mains to a glass tube with a finely drawn-out end. The pressure of the gas supply to the glass tube is controlled by a clamp on the rubber tube which connects it to the main supply. This method is simple and often saves the trouble of the gas-bag method.

CHAPTER XVI

MISCELLANEOUS MAGNETIC EXPERIMENTS

Measurement of the Pole Strength of a Bar Magnet, using a Grassot Fluxmeter

THE search coil of the fluxmeter (see p. 521) is placed on the bar magnet as shown in the figure, so that it encircles the mid-point of the magnet. At this stage the reading of the fluxmeter is noted. Then if the bar magnet is uniformly magnetized, the coil, when withdrawn, cuts all the lines due to the pole past which it moves.

The fluxmeter indicates the flux change in units which are specified; in the case of the instrument described on p. 521, each division corresponds to a change of flux equal to 10,000 maxwells.

If the change is x divisions during the withdrawal of the search coil, and there are n turns of wire in the search coil, then since from a pole of strength, m, there are $4\pi m$ lines, we have

$$4\pi m = \frac{10000x}{n},$$

or

$$m = \frac{10000x}{4\pi n}.$$

The experiment should be repeated, using search coils having different values for n.

The following are examples of experimental results.

Coil A: $n = 100$.

Initial reading of fluxmeter	3
Final reading of fluxmeter	46
Deflection, first experiment	49
Deflection, second experiment	50
Mean deflection	49·5

$$m = \frac{49{\cdot}5 \times 10000}{4\pi \times 100} = 394{\cdot}1.$$

Coil B: $n = 8$.

Deflection, 4 divisions.

$$m = \frac{10000 \times 4}{4\pi \times 8} = 394{\cdot}4.$$

Mean value of pole strength, 394·5.

The magnet was 1·58 cm. by 0·75 cm., i.e. 1·23 sq. cm. in cross-section. Thus the intensity of magnetization, assumed uniform, is

$$\frac{394 \cdot 25}{1 \cdot 23} = 320 \cdot 5.$$

Distribution of Magnetism along a Bar Magnet

This may be determined by using the fluxmeter in a manner very similar to that of the last experiment. The magnet is marked off in centimetres along its length, and the search coil is placed at the mid-point, around the magnet. The coil is then advanced in centimetre steps and the deflection of the instrument noted, i.e. for 0 to 1 cm., 1 to 2 cm., 2 to 3 cm., etc. The deflection in each case is proportional to the magnetization in the space moved over by the coil.

The variation of magnetization along the length is shown by plotting deflection against distance from the centre.

Gauss's Proof of the Law of Force

The most satisfactory proof that the force between two magnetic poles varies inversely as the square of the distance between them, was first given by Gauss.

The method consists of a comparison of the magnetic force at a point on the axis of a magnet with the force at a point on a line drawn at right angles to it, at its mid-point.

Let us first calculate the value of the force at two such points, assuming that the force between poles varies inversely as the nth power, so that the force between two poles of strengths, m and m', r cm. apart is

$$F = \frac{mm'}{r^n}.$$

End-on Position (A Position of Gauss)

Let NS be the magnet (fig. 301) and P a point along the axis produced, such that the distance from P to O, the centre of the magnet, is r cm. If the length of the magnet is $2l$ cm., the magnetic field at P, which is the force on unit pole placed at P, is

$$\frac{m}{(r-l)^n} - \frac{m}{(r+l)^n} = F_A.$$

This is the net repulsion due to N and S on the unit pole at P, and is equal to:

$$F_A = m\left\{\frac{1}{r^n\left(1-\frac{l}{r}\right)^n} - \frac{1}{r^n\left(1+\frac{l}{r}\right)^n}\right\},$$

$$F_A = \frac{m}{r^n} \frac{\left(1+\frac{l}{r}\right)^n - \left(1-\frac{l}{r}\right)^n}{\left(1-\frac{l}{r}\right)^n\left(1+\frac{l}{r}\right)^n}.$$

On expanding in powers of $\left(\frac{l}{r}\right)$ we have

$$F_A = \frac{m\left\{1+\frac{nl}{r}+\frac{n(n-1)}{2!}\left(\frac{l}{r}\right)^2 \ldots -1-\frac{nl}{r}+\frac{n(n-1)}{2!}\left(\frac{l}{r}\right)^2\ldots\right\}}{r^n\left\{1-\left(\frac{l}{r}\right)^2\right\}^n},$$

expanding the two expressions.

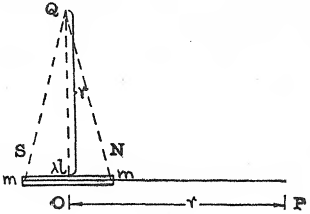


FIG. 301

Neglecting the powers of $\frac{l}{r}$ higher than the first, this expression becomes

$$F_A = \frac{2mnl}{r^{n+1}} = \frac{Mn}{r^{n+1}}, \qquad \ldots(1)$$

where M is the magnetic moment of the magnet.

Broadside-on Position (*B Position of Gauss*)

Again in fig. 301 let Q be r cm. from O, measured along OQ, which is drawn at right angles to the magnet at the mid-point.

Due to the pole at N, there is a repulsion at Q along NQ equal to

$$\frac{m}{\{\sqrt{(r^2+l^2)}\}^n},$$

and a similar attraction due to S, along QS.

The resultant of these forces may be obtained by considering the isosceles triangle, QNS, as a triangle of forces. The sides QS and QN

each representing a force $\dfrac{m}{(r^2+l^2)^{\frac{n}{2}}}$ the resultant NS is

$$\frac{2ml}{(r+l^2)^{n}\,(r^2+l^2)^{\frac{1}{2}}},$$

i.e.
$$F_B = \frac{M}{\left(r^2+l^2\right)^{\frac{n+1}{2}}} = \frac{M}{r^{n+1}\left\{1+\left(\frac{l}{r}\right)^2\right\}^{\frac{n+1}{2}}},$$

or when l is small compared with r.

$$F_B = \frac{M}{r^{n+1}}. \qquad \ldots(2)$$

Comparing equations (1) and (2), we see that for such a short magnet the magnetic forces at the two points, P and Q, are as $n : 1$.

The numerical value of n is obtained by comparing the magnetic forces at two such positions.

A magnetometer is employed. This consists of a small magnet fastened in a light frame which carries a small mirror, the whole being

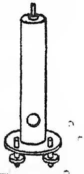


Fig. 302 (a)

Fig. 302 (b)

suspended by a thin silk fibre in a cylindrical brass case, as shown in fig. 302 (a). Fig. 302 (b) shows the usual form of small suspended magnet.

The magnetometer magnet is allowed to come to rest, care being taken that all the torsion is removed from the silk suspension.

Two boxwood scales, C and D, are arranged as shown in fig. 303, one in the magnetic meridian, and one at right angles to it.

A lamp, L, and scale, S, are arranged in the usual way, as shown, so that an image of the lamp is reflected and focused on the scale, S. The distance of S from the magnetometer mirror should be about one metre.

When the magnetometer needle is arranged to swing freely in the centre of the case by means of the levelling screws shown (fig. 302 (a)), the position of the needle with respect to the ends of the boxwood

scales can be very readily obtained, if the radius of the cylindrical case is measured.

If now a small bar magnet is placed at any point, such as B, fig. 303, in an east and west position, a magnetic field will be set up at M, corresponding to the broadside-on position. The distance, r, between the magnet centres being obtainable by direct measurement on the boxwood scale. The deflection of the magnetometer is found by observation of the movement of the reflected beam of light.

The magnet is then placed again east and west, at an equal distance from M, on the scale D (A, fig. 303), and the deflection again obtained.

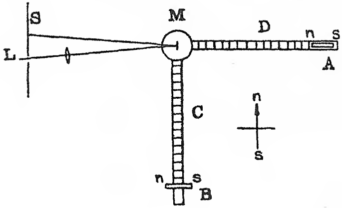


Fig. 303

It is desirable to eliminate the zero reading of the magnetometer; the double reflection, left and right, may easily be obtained by reversing the magnet at the two positions A and B.

The experiment is repeated for various values of r. For example, with a magnet 4 cm. long the values of the deflection at 50, 60, 70, 80 cm. may be obtained and tabulated as below.

Distance of magnet from centre magnetometer in cm.	End-on (A) position			Broadside-on (B) position			Value $\frac{A}{B} = n$
	Scale reading magnet		Double deflection A	Scale reading magnet		Double reading (B)	
	Direct	Reversed		Direct	Reversed		
50 cm.	17·6	−16·7	34·3	8·2	−9·0	17·2	2·0
60	10·3	−10·1	20·4	4·6	−5·1	9·7	2·1
70	6·5	− 6·2	12·7	2·8	−3·3	6·1	2·08
80	4·4	− 4·1	8·2	1·8	−2·3	4·1	2·07
90	3·3	− 2·9	6·2	1·0	−2·2	3·2	1·94
						Mean	2·04

The deflected position of the magnetometer needle depends on the relative strengths of the horizontal component of the earth's field, H, and the field due to the deflecting magnet. If θ is the angular deflection produced, we have

$$\mathrm{H}\sin\theta = \mathrm{F}\cos\theta,$$

or

$$\mathrm{F} = \mathrm{H}\tan\theta.$$

So that for the same value for r

$$\frac{\mathrm{F_A}}{\mathrm{F_B}} = \frac{n}{1} = \frac{\tan\theta_1}{\tan\theta_2},$$

where θ_1 is the deflection when the magnet is at A, and θ_2 when the magnet is at B, in the broadside-on position.

If the distance from the scale S is fixed and the corresponding deflections are d_1 and d_2,

$$\frac{\tan 2\theta_1}{\tan 2\theta_2} = \frac{d_1}{d_2},$$

i.e.

$$n = \frac{d_1}{d_2},\ \theta_1 \text{ and } \theta_2 \text{ being small.}$$

This ratio is tabulated in the last column of the table, and a mean value obtained.

The Variation of Residual Magnetism with Temperature

The object of this experiment is to investigate the behaviour of a magnetized carbon steel rod when subjected to temperature changes.

A rod of carbon steel of about 8 cm. in length and 0·5 cm. in diameter is magnetized between the poles of an electromagnet. The magnet so formed is set up inside a copper or brass tube, which is of slightly larger diameter, as seen in the upper portion of fig. 304.

The magnet is drilled with a small hole which receives a thermo-junction, as shown. This thermo-junction is calibrated as described on p. 580, and serves to register the temperature of the magnet. Some simple heating device is arranged to vary the temperature of the magnet. In the diagram a second brass tube provided on the upper surface with a series of holes is shown, fitted on the end of a brass Bunsen burner. If a brass Bunsen burner is not available a simple heater may be made by taking a second tube such as the one shown, perforated with a series of holes, having one end closed and at the other a device for admitting an air-gas mixture similar to that of the common form of gas ring.

The magnet is set up at right angles to the meridian, and at a distance d cm. from a mirror magnetometer, of the form shown in fig. 302; the distance, d, should be such that at room temperatures the magnet produces a full-scale deflection. The magnitude of the deflection

is noted, and the temperature is obtained from the thermo-junction balance point on the potentiometer.

The temperature of the rod is gradually raised by means of the heater, and the magnetometer deflection and the thermo-junction balance point are noted for every 20°C. rise until the temperature is about 180°C. At this point the observations are made much more frequently, as the critical part of the variation is being observed. At about 250°C. the readings may be again taken at about every 10°C. or 20°C. intervals, and observations continued until the temperature is at the maximum obtainable value for the heater employed.

The results are tabulated and the value of the temperature obtained for each point; the tangents of the corresponding angular deflections of the magnetometer are also tabulated.

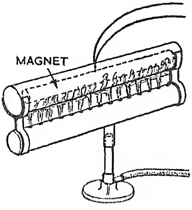


FIG. 304

The magnet is then allowed to cool slowly, and the observations are continued, until finally the specimen is once more at room temperature.

The magnetometer is at a constant distance from the magnet, and in a fixed control field, H, hence the moment of the magnet is proportional to the tangent of the angle of deflection.

This quantity is plotted against temperature, and a curve somewhat similar to that of fig. 305 is obtained. The firm line shows the relation between tan α and t°C. as the temperature rises, and the broken line the relationship as the temperature decreases.

For carbon steel in general, it will be found that at about 200°C. the magnetic moment is reduced to zero. An increase in temperature causes a reversal of the polarity as shown in the curve. This negative moment is a maximum at about 210°C. and then decreases to zero at about 800°C. In general such a high temperature will not be available with the heater described, but the most interesting part of the curve may be obtained. On cooling, the negative moment is a maximum at D, at a temperature about 10°C. or 20°C. above the original position, and on regaining room temperature will have a small positive moment, corresponding to E in fig. 305.

The above experiment which demonstrates an interesting feature of the magnetization of such an annealed carbon steel rod is due to Prof. S. W. J. Smith, and was described by him in the *Proceedings of the Physical Society of London*, No. xxiv, 15 Aug. 1912. Reference should be made to this paper, which gives an explanation of the observed results in terms of the magnetization of the iron and iron carbide molecules which compose the bar.

Briefly, the iron carbide molecules are set in line on magnetization and exert a demagnetizing effect on the iron molecules which are reversed in the internal field due to the carbide. At about 210° the carbide is totally ineffective, but the reversed iron molecules being

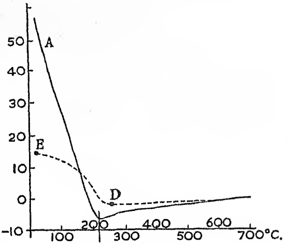


Fig. 305

more retentive have a maximum external effect. As the temperature rises, the iron molecules approach their neutral temperature, and finally at about 800° C. entirely lose their magnetic effect.

The Increase in Length of a Bar on Magnetization

To investigate the change in length produced in a rod of iron when magnetized, the following apparatus is set up. Fig. 306 shows a section of a brass case wrapped with two solenoids and provided with a water jacket, WW, through which water at a steady temperature may be circulated via tubes, T: the two coils, C, are wound together so that their magnetic effects are identical at the axis of the cylinder. The free ends of each coil are connected to separate terminals, P, Q, R, and S. The number of turns and mean diameter of the two coils are as nearly as possible the same, so that if a current is sent through from P to Q, the magnetic effect is the same as when the same current is sent from R to S through the second coil, or if the two coils are connected in opposition there is no magnetic field along the axis.

The direction of winding and the resulting magnetic field should be

tested at the outset by sending a current through the coils, and noting the magnetic effect on a compass needle held near the end of the solenoid.

At the top of the brass case is a circular brass end-plate, to which are soldered rigidly three pins, S, which terminate in points on which a sheet of plane glass, G, may rest.

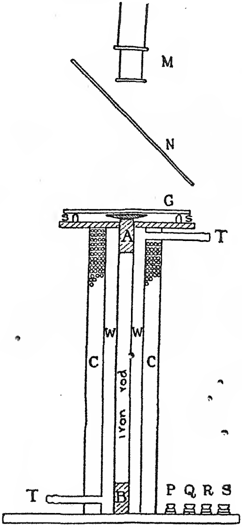


Fig. 306

The iron specimen to be investigated is first thoroughly demagnetized by heating, or by reversals of a diminishing current through a solenoid surrounding it (as described on p. 483). The specimen should be a cylindrical rod with plane ends made to fit the cavity provided along the axis of the solenoids. The end effects of the magnetizing solenoid are overcome by first placing inside the apparatus a cylindrical length of brass, B, with plane ends, the lower end resting firmly on the thick brass base of the instrument. Above the iron rod is placed a second

brass cylinder, A, also with plane ends; the upper end projects slightly above the level of the upper surface of the end-plate.

A long focus lens is fastened firmly to this brass cylinder by means of soft wax or Plasticine. The length of the support, S, is such that a small gap is left between the upper face of the lens and the lower surface of the glass sheet, G.

Above G is placed a second sheet, inclined at 45° to the normal. This reflects light from a sodium flame towards the lens and plane surface.

The reflected beams are viewed by a microscope, M, and the Newton rings formed are observed in this way. The microscope contains a cross-hair which is used as a point of reference.

When a current is sent through the solenoids arranged in series, with their magnetic fields in the same direction, the rod may be gradually magnetized by increasing the current strength. If any change in length occurs, there will be a movement of the Newton ring system, which may be observed in the microscope.

It will be seen (by reference to p. 322) that if the rod changes in length by $\frac{\lambda}{4}$ where λ is the wave-length of sodium light, a bright ring replaces the neighbouring dark ring. The direction of movement of the fringes will show whether the rod increases or decreases in length.

Now, due to the current circulating in the coils, a certain amount of heat will be developed. Thus, for a rod of 50 cm. length a rise in temperature of about 0·03° will cause an increase in length more than sufficient to move fringes corresponding to a change in size of the gap equal to $\frac{\lambda}{4}$. It therefore becomes necessary to ensure that the rod is maintained constant in temperature throughout. Water at constant temperature is circulated through the jacket, WW, for some time before commencing observations and continuously during the experiment. Under these circumstances tap water from a supply removed from any hot-water pipes will be satisfactory.

As a further check, to ensure that the effect to the heating is avoided, each observation is preceded by one in which the current to be used is sent through the two coils arranged to have their magnetic fields in opposition, i.e. the heating which may affect the dimensions of the bar will be present, and any movement of the fringes is due to that cause, since there is no net magnetic effect.

The connexions to the coils are rapidly changed so that their magnetic fields are in the same direction, and the same current causes a magnetization of the bar. The true effect on the length of the bar due to magnetization may therefore be obtained.

To facilitate the rapid change in the direction of the current, a Pohl commutator may be employed: the ends of one coil, say, R and S, are connected to one pair of the terminals of the commutator; the other

coil is connected in series with a battery and variable resistance to the middle terminals. Thus, by throwing over the movable arm, the fields of the coils may be made to act in opposition or together.

Carry out observations using currents up to a maximum value which is determined by the current-carrying capacity of the coils, CC.

Plot the results showing change in length in terms of the wave-length of sodium light, against the field strength measured in oersteds.

CHAPTER XVII

TERRESTRIAL MAGNETISM

Determination of the Horizontal Components of the Earth's Magnetic Field (H)

THE method due to Gauss, described below, is usually employed to determine the horizontal component of the earth's magnetic field; it may also be used to measure any magnetic field which is uniform over a sufficiently large volume.

The method involves two experiments. In the first, a magnet of known moment of inertia is suspended freely in the earth's field at the place where H is to be found, and from observation of the time of swing, the product MH where M is the moment of the magnet, may be calculated. In the second experiment, the field due to the same magnet is compared with the earth's field by means of a magnetometer.

Determination of MH

Let the magnet be suspended in a light stirrup and perform oscillations whose periodic time is T seconds, it being supposed that the suspension has no initial twist. Then if I is the moment of inertia of the magnet about the axis of suspension, i the moment of inertia of the suspending frame about the same axis, and τ the restoring couple per unit angular displacement due to torsion of the suspending fibre, we have

$$T = 2\pi\sqrt{\frac{(I + i)}{(MH + \tau)}}, \qquad \ldots(1)$$

for $(I + i)$ is the moment of inertia of the system, and $(MH + \tau)$ is the total restoring couple per unit angular displacement, when small displacements are considered, whence MH may be calculated.

Let $$MH = A. \qquad \ldots(1a)$$

To find $\frac{M}{H}$

A magnetometer is set up in the place at which MH was found, and the magnet placed with its centre d cm. from the centre of the magnetometer needle, east or west of it and lying east and west. If θ be the deflection of the magnetometer, and F the value of the field due to the magnet, we have

$$F = H \tan\theta,$$

and $$F = \frac{2Md}{(d^2 - l^2)^2},$$

where $2l$ is the distance between the poles of the magnet.

Alternatively, if the magnet is placed so as to be in the broadside-on position, i.e. east and west with its centre in the meridian through the centre of the magnetometer needle, the field produced at the magnetometer is given by

$$F_1 = \frac{M}{(d_1^2 + l^2)^{\frac{3}{2}}},$$

where d_1 is the distance between the centre of the magnet and the magnetometer in this case, and l is the distance between the poles of the magnet.

Whence for the end-on position

$$H \tan \theta = \frac{2Md}{(d^2 - l^2)^2}, \qquad \ldots(2)$$

and for the broadside-on position

$$H \tan \theta_1 = \frac{M}{(d_1^2 + l^2)^{\frac{3}{2}}}. \qquad \ldots(3)$$

In either case the value of $\frac{M}{H}$ may be found.

Let this value be B.

Whence from the two experiments

$$H = \sqrt{\frac{A}{B}}.$$

In performing these experiments all iron is removed from the neighbourhood at the outset, and care taken to maintain the magnetic conditions the same throughout the two experiments.

The magnet used may be a cylindrical bar magnet a few centimetres long.

Let the length of the magnet be $2l_1$ cm., the radius r cm., and the mass m gm. I, its moment of inertia about an axis through the centre of gravity and normal to its length, is

$$I = m\left(\frac{l_1^2}{3} + \frac{r^2}{4}\right).$$

To find T, use is made of a light stirrup suspended by a fibre from a torsion head. A brass cylinder of the same mass as the magnet is first placed in the stirrup, and all the twist is then removed from the fibre. The torsion head is then turned so that the brass rod is in the magnetic meridian. The rod is then replaced by the magnet.

The frame is provided with a small mirror, and a lamp and scale is arranged so that a beam of light reflected from the mirror on to the scale enables the movement of the suspended system to be observed.

Set the magnet swinging through a very small angle, and obtain the time taken for 100 complete swings. Repeat this and take a mean value, from which T may be obtained.

In order to determine i the method described on p. 49 may be applied. In the present case the application consists in finding the period of oscillation of the stirrup and mirror alone. Let this be T_1. Take a cylindrical rod of which the moment of inertia, i_0, is known and

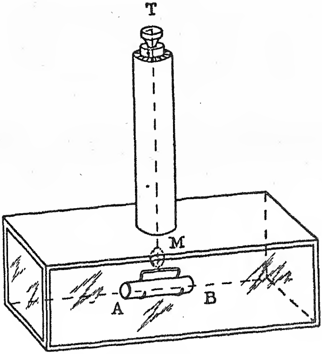


FIG. 307

of magnitude comparable with that of the stirrup. If it is placed in the stirrup with its centre of gravity on the axis of rotation

$$i_0 = m\left(\frac{l_1^2}{3} + \frac{r^2}{4}\right),$$

where m is the mass, $2l_1$ the length, and r the radius of the rod. The latter will usually be small enough to make $\frac{r^2}{4}$ very small compared with $\frac{l_1^2}{3}$ so that the second term in the formula may be neglected. Let T_2 denote the period when i_0 is in the stirrup, then

$$i = \frac{T^2 i_0}{T_2^2 - T_1^2}.$$

The term i in the formula appears as a correction to I and is usually small, so that it is not required to a high degree of accuracy.

The restoring couple due to the torsion, τ, may be obtained by rotating the torsion head through, say, a complete turn. Note the resulting angular deflection of the magnet, ψ radians, whence

$$(2\pi - \psi)\,\tau = \mathrm{MH}\psi,$$

or

$$\tau = \frac{\mathrm{MH}\psi}{(2\pi - \psi)}.$$

It will be found that, if the suspension fibre is thin, τ is negligible compared with MH. Including these corrections, we have

$$T = 2\pi \sqrt{\frac{I + i}{MH\left(1 + \frac{\psi}{2\pi - \psi}\right)}},$$

whence

$$MH = \frac{4\pi^2}{T^2} \frac{(I + i)}{\frac{2\pi}{2\pi - \psi}}.$$

The magnetometer experiment is performed, using a special form of magnetometer (Kew type). This consists of several small magnets rigidly fastened in a metal frame, which also carries a small mirror. The whole is mounted in a brass case, provided with suitably arranged glass windows so that a beam of light may be reflected from the mirror on to a scale S cm. away from it.

The magnetometer frame carries a long bar which has four pegs, P_1, P_2, P_3, P_4, so arranged that when a small carriage is placed on any

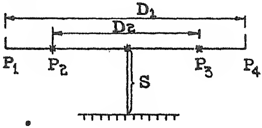


FIG. 308

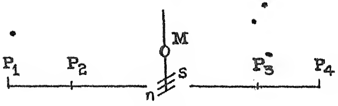


FIG. 309

of them, the cylindrical magnet, which the carriage supports, lies with its axis in the same plane as the magnetometer needle. Figs. 308 and 309 show diagrammatically the arrangement of the magnetometer.

The magnetometer and bar are first levelled (using a spirit-level) by means of the levelling screws on the base of the instrument.

P_1 and P_4, P_2 and P_3 are fixed, in the construction of the instrument, equidistant from the needle when the system is level. Measure $P_1P_4 = D_1$ cm., say; measure $P_2P_3 = D_2$ cm.

The scale is then adjusted so that the reflected spot of light is at its centre when the magnetometer is under the influence of the earth's field alone. The arm which carries the pegs is set first of all in the

direction of the magnetic meridian, and the small carriage in which the cylindrical magnet is carried is placed on P_1 and turned round so that the magnet is at right angles to the meridian. The true position for the magnet at right angles to the meridian is obtained by noting the deflection of the magnetometer. This will be a maximum when the normal position is acquired.

In this manner the deflection in cm. on the scale may be measured. The magnet is then rotated through 180° until a maximum deflection in the opposite direction is obtained in the magnetometer. Let these deflections be δ_1 and δ_1' cm., and the corresponding deflections when the magnet is supported on the other pegs be δ_2, δ_2', etc.

Obtain the mean value of $\delta_1, \delta_1', \delta_4, \delta_4' = \delta$, say, for distance $\frac{1}{2}D_1 = d_1$; also take the mean of $\delta_2, \delta_2', \delta_3, \delta_3' = \delta'$, say, for the distance $\frac{1}{2}D_2 = d_2$.

If φ and φ' are the corresponding angular deflections of the magnetometer, we have

$$\tan 2\varphi = \frac{\delta}{S} \quad \text{and} \quad \tan 2\varphi' = \frac{\delta'}{S}.$$

Hence we may calculate φ and φ' from the observed deflections.

The cylindrical magnet was placed as described above so that the field at the magnetometer was that in the broadside-on position; thus from equation (3) above

$$H \tan \varphi = \frac{M}{(d^2 + l^2)^{\frac{3}{2}}},$$

i.e.
$$\frac{M}{H} = (d_1^2 + l^2)^{\frac{3}{2}} . \tan \varphi = (d_2^2 + l^2)^{\frac{3}{2}} \tan \varphi'.$$

Now
$$(d^2 + l^2)^{\frac{3}{2}} = d^3 \left(1 + \frac{3}{2}\frac{l^2}{d^2} + \ldots\right),$$

neglecting the fourth and higher powers of $\dfrac{l}{d}$,

i.e.
$$\frac{M}{H} = d_1^3 \left\{1 + \frac{3}{2}\left(\frac{l}{d_1}\right)^2\right\} \tan \varphi,$$
$$\frac{M}{H} = d_2^3 \left\{1 + \frac{3}{2}\left(\frac{l}{d_2}\right)^2\right\} \tan \varphi'.$$

Hence l may be eliminated, giving

$$\frac{M}{H}\left(\frac{1}{d_1 \tan \varphi} - \frac{1}{d_2 \tan \varphi'}\right) = d_1^2 - d_2^2.$$

Hence $\dfrac{M}{H}$ is determined, and the value of H may be calculated using this and the value of MH previously obtained.

The bar carrying the four pegs may be alternatively arranged at right angles to the meridian, and the carriage containing the magnet

again placed on the pegs with the magnet at right angles to the meridian, producing a field of force at the magnetometer corresponding to the end-on position. The process described above is repeated, and the average deflections obtained enable the calculation of the mean deflections, θ_1 and θ_2, for the distances, d_1 and d_2.

Hence again

$$H \tan \theta_1 = \frac{2M}{D_1^3} \times \frac{1}{\left(1 - \frac{2l^2}{d_1^2} \dots\right)},$$

$$H \tan \theta_2 = \frac{2M}{D_2^3} \times \frac{1}{\left(1 - \frac{2l^2}{d_2^2} \dots\right)},$$

to the same order as before.

Once more eliminating l, half the distance between the poles of the magnet, we have

$$2 \cdot \frac{M}{H}\left(\frac{1}{d_1 \tan \theta_1} - \frac{1}{d_2 \tan \theta_2}\right) = d_1^2 - d_2^2,$$

whence $\frac{M}{H}$ may be again calculated. This second method gives larger deflections and is therefore preferable to the first.

The Dip Circle—Measurement of the Angle of Dip

The dip circle consists of a long magnet supported on a horizontal axis which passes approximately through its centre of gravity and is at right angles to its length. The magnet is supported on wheel bearings or on agate edges, so that it may turn freely and with a minimum of friction. On the same base as the support for the knife-edges, is a vertical graduated circle which enables the position of the ends of the needle to be obtained. The whole structure is supported on a circular table, which is capable of rotation around a vertical axis. This rotation may be measured to at least one minute of arc (see fig. 310) on a horizontal circular scale using a suitable vernier.

The position of the ends of the magnet may be read on the vertical scale by means of microscopes carrying verniers, which move round the scale. If the plane of the vertical scale be turned in the direction of the meridian, so that the horizontal axis of support of the magnet is at right angles to it, the magnet sets in the direction of the earth's lines of magnetic force, and the angle included between the horizontal and the position of the needle is the angle of dip.

However, there are many sources of error. The axis of rotation of the magnet may not coincide with the centre of the scale. One end, therefore, of the magnet would read too small, and the other too large, a deflection.

The axis of rotation may not be truly through the centre of gravity, in which case a couple is exerted, tending to turn the magnet so that the centre of gravity comes vertically under the axis of support.

The axis of the magnet does not usually coincide with the geometric axis of the needle, so that the centres of the ends of the magnet do not give a correct reading for the angle of dip.

A consideration of the observations taken and described below will show how these errors may be eliminated.

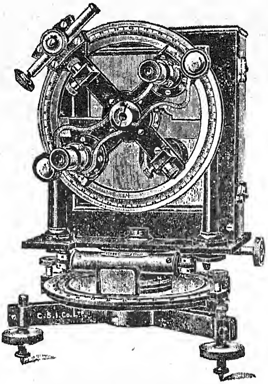


FIG. 310

To determine the angle of dip, the dip circle is set up and by means of the levelling screws on the base of the instrument, using a spirit-level, the 'horizontal' scale is made truly horizontal.

The plane of the meridian must then be found, so that the magnet may set freely. To do this, the instrument is first turned so that the upper south-seeking pole is at the 90° mark, and the reading of the vernier on the horizontal scale is noted.

It will, in general, be found that the lower, north-seeking pole of the magnet is not quite at the lower 90° scale reading. A slight turn of the screw which rotates the instrument about the vertical axis is then made until the North Pole is at the 90° reading, and the horizontal vernier scale is once more read. The circle is next turned through 180° and the two observations are repeated. The needle is now reversed in its bearings and the four observations repeated. The mean value of

the horizontal scale readings for these eight positions is taken. If the plane of rotation is set in this mean position it is perpendicular to the magnetic meridian. The case is then moved through an angle of 90° from this position so that the plane of the needle now coincides with the magnetic meridian.

The procedure is as follows:

(1) Read the position of each end of the magnet.
(2) Rotate the instrument through 180°, and once more read both ends of the magnet.
(3) Reverse the needle in its bearings and read both ends on the graduated circle.
(4) Turn the instrument through 180°, and again read both ends.
(5) Remagnetize the needle and repeat the above process.

The needle is remagnetized so that its poles are interchanged.

The North Pole for the observations, 1 to 4, above now becomes a South Pole. Under these new conditions eight more values are obtained. The mean value of the 16 readings gives the true value of the inclination. The observations may be conveniently tabulated as below.

Marked side of magnet pole	Marked pole dipping (needle readings)			Magnet remagnetized (needle readings)		
	Upper pole	Lower pole	Mean	Upper pole	Lower pole	Mean
East			α			α'
West			β			β'
Needle reversed in bearings						
East			γ			γ'
West			δ			δ'

$$\text{Mean } \frac{\alpha + \beta + \gamma + \delta}{4} = \text{A} \qquad \text{Mean } \frac{\delta' + \beta' + \gamma' + \delta'}{4} = \text{B}$$

$$\text{Angle of Dip} = \frac{\text{A} + \text{B}}{2}.$$

CHAPTER XVIII

PERMEABILITY AND SUSCEPTIBILITY

Measurement of Permeability by the Magnetometer Method

THE relation between the magnetizing intensity, H, the induction, B, and the intensity of magnetization, I, is given by the equation

$$B = H + 4\pi I.$$

If an external force, H_0, is applied, H has the value

$$H = H_0 - aI.$$

This means that the effect of the external force is diminished by the magnetization induced, the constant, a, depending upon the shape of the body.

The permeability, μ, is defined by the ratio $\frac{B}{H}$ and the susceptibility, k, by $\frac{I}{H}$, thus

$$\mu = 1 + 4\pi k.$$

If the specimen of magnetizable material is in the form of a long thin rod the effect of the induced magnetization in comparison with the external field can be neglected. This effect will cause no appreciable error if the length is about 200 times the diameter of the cross-section.

If the specimen is arranged horizontally or vertically near a small magnet, suspended by an unspun silk fibre, the magnet will be deflected by the induced magnetism in the specimen. From a knowledge of the deflection and of H, the value of B, μ, I, k, may be calculated.

To find the above quantities for a given wire specimen, the type of reflecting magnetometer described on p. 462 is used.

The inducing magnetic field is obtained from a solenoid, through which a current of known magnitude may be passed.

Iron Wire Specimen arranged Vertically

The arrangement of the apparatus is seen in fig. 311. L is a lamp and scale, M the magnetometer which is clamped to the wooden base shown, at a convenient distance from the specimen which is placed in a vertical solenoid, S. The wooden base is arranged in an east and west direction so that the magnetic effect of the specimen may be measured.

The solenoid is connected in series with a small coil, C, and several accumulators, a variable resistance, an ammeter reading to 3 amperes, and a reversing switch. The coil, C, is so arranged that the magnetic effect produced by it on the magnetometer is in the opposite direction to that produced by the solenoid, S.

The position of C is adjusted so that whatever is the value of the current in the circuit there is no movement of the needle. In this way the whole movement of the magnetometer in the main experiment is due to the magnetism induced in the iron wire.

The specimen of iron to be examined should be shorter than the solenoid S, so that there is a space at each end of the solenoid not effective in the magnetization, in other words, the region where the field in the solenoid is reduced due to end-effect is not utilized. A wooden spacing cylinder may be arranged at the base of the solenoid to arrange these conditions, and at the same time cause the effective induced pole to be on the same level as the magnetometer needle.

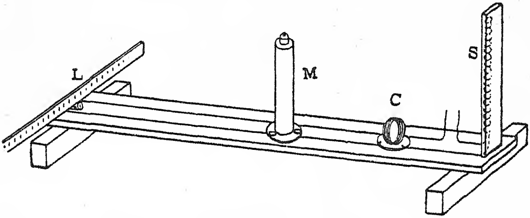


FIG. 311

The specimen of iron should be initially without residual magnetism. This condition may be brought about by heating it to red heat and beating with a hammer while the wire is at right angles to the earth's field. A further precaution against magnetization in the specimen may be taken, either by using another large solenoid or S itself.

An alternating current of about 1 ampere is sent through the solenoid in which the specimen is placed, and the value of the current is gradually reduced to zero by means of a controlling resistance. If alternating current is not available the process may be carried out using the current from an accumulator which passes through a commutator before reaching the solenoid. Rapid reversal of current is arranged to take place as the current is reduced by increasing a series resistance.

If after these precautions are taken it is found that a deflection is produced in the magnetometer when the specimen is introduced into the solenoid, this deflection is probably due to magnetization induced by the earth's vertical component. To neutralize this, an independent solenoid is wound about S and a current from a steady accumulator, controlled by a series resistance, is passed in a direction to produce a vertical magnetic field opposite to that of the earth's field. By suitable

adjustment the deflection may be reduced to zero and the current in this circuit is kept constant throughout the experiment.

A preliminary experiment on a second sample of the same iron wire enables a value of the current at saturation to be determined. The distance, MS, is adjusted so that a full-scale deflection is obtained when the sample is saturated. The original specimen is then introduced so that its lower end is level with the axis of the magnetometer magnet.

Starting with the soft iron unmagnetized specimen, the zero position of the magnetometer on the scale is observed. The value of the current is then raised to, say, 0·2 ampere and the deflection noted. Then

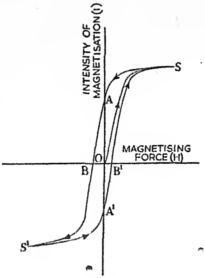


FIG. 312

without breaking the circuit the current strength is increased by increments of 0·2 to a final value of about 3 amperes, the exact value being determined by the saturation condition of the specimen.

Having increased the current to a suitable maximum value, it is then decreased by similar steps, again taking care that the circuit is not broken in the process, until no current circulates. The current is then reversed and increased step by step to the same maximum in the other direction. The current is again brought back in steps to zero, and having once more reversed the current it is raised to the maximum in the original direction.

Tabulating these results a connexion is found between the value of the current in amperes and the magnitude of the deflections or the value of the scale reading, which, when plotted, will show the general form of the curve above.

The results are then converted to the corresponding C.G.S. values for B and H.

In fig. 313, let *ns* represent the magnetized specimen which will produce at P, the position of the magnetometer, a field strength, F, at right angles to the earth's horizontal component, and which causes the deflection, θ, say.

We have $$F = H' \tan \theta,$$

where H' is the magnitude of the earth's horizontal component.

Let m be the pole strength induced in the specimen,
l be the length in cm.,
r the distance from the lower end of specimen to the magnetometer in cm.

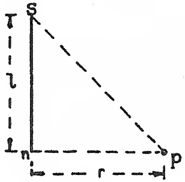


FIG. 313

Then

$$F = \frac{m}{r^2} - \frac{m}{SP^2}\cos SPn$$
$$= \frac{m}{r^2} - \frac{m}{r^2+l^2}\cdot\frac{r}{(r^2+l^2)^{\frac{1}{2}}} = m\left(\frac{1}{r^2} - \frac{r}{(r^2+l^2)^{\frac{3}{2}}}\right).$$

Whence

$$m = \frac{H' \tan \theta}{\left(\frac{1}{r^2} - \frac{r}{(r^2+l^2)^{\frac{3}{2}}}\right)};$$

and $$I = \frac{m}{s}, \quad B = H + 4\pi I.$$

Further, using a solenoid of n turns per cm. having a current of i amperes, the uniform field $H = \frac{4\pi ni}{10}$ is effective on the specimen.

Hence, for each reading in the table of results, I, B, and H, μ and k may be calculated. If the curve is drawn for I and H, the area enclosed is equal to the work done on the specimen in the cycle (i.e. the heat developed).

The B-H Curve for a Sample of Iron using a Ballistic Galvanometer

This method is specially applicable to the determination of the B-H curve for a specimen in the form of an anchor ring, or a very short

hollow cylinder. For such a specimen, magnetized by a magnetic force of, say, H, no free poles will be developed, and therefore no demagnetizing field will be set up. The value of the induction, B, will therefore correspond to H, and not some smaller field, H′, as in the previous case.

The experimental arrangements of this method, as shown in fig. 314, are such that a variable field, H, may be set up by passing a current through a primary winding, P, which is closely wound on the anchor

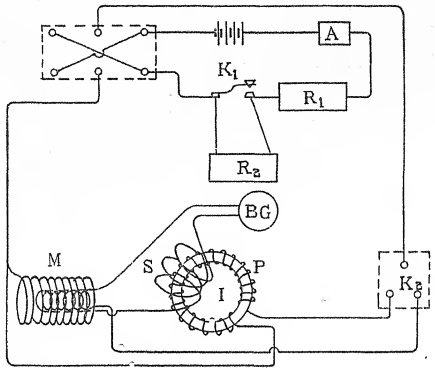


FIG. 314

ring. The method employed to measure B is to wrap a few turns of wire, S, round the anchor ring and primary winding and measure the quantity of electricity which passes through a ballistic galvanometer, BG, due to the change in the induction in the specimen for a known change in H. Since the ballistic galvanometer must be standardized, the secondary circuit is completed through a second small coil of a mutual inductance, M, used in the standardizing experiment. Thus the ballistic galvanometer is in a fixed resistance circuit in all measurements.

The current from an accumulator may be regulated by resistances, R_1 and R_2, and measured by an ammeter, A. By means of the Pohl commutator the current may be sent directly or reversed through the primary windings, P, when K_2 is to the left, or through the standardizing mutual inductance, M, when the key K_2 is closed on the right.

By closing the key K_1, the resistance R_2 may be cut out.

The value of H may be calculated, as explained later, from the value of the current strength as obtained from the ammeter, and B may be calculated from the observed throw of the ballistic galvanometer, BG. Since the galvanometer circuit is composed of a comparatively low resistance, a moving coil instrument may be too heavily damped to be serviceable (see p. 520). A moving needle instrument is used if this is the case.

As a preliminary experiment, the key K_2 is closed to the left, and R_1 and R_2 decreased, until on closing the Pohl commutator, the galvanometer gives a full-scale deflection from the zero, which should be the central graduation of the scale. The current required to do this is noted and is used as a maximum value in the main experiment. Usually from 2 to 3 amperes will suffice, the sensitivity of the galvanometer being adjusted to measure the throw produced.

Any residual magnetism in the specimen must first be reduced to zero. To do this the galvanometer circuit is broken and the resistances R_1 and R_2 reduced to a minimum. The current passing in P is then reversed many times and R_1 and R_2 gradually increased, until finally the current which is reversed is very small. It should be noted that R_1 and R_2 are such that changes in resistance may be brought about without breaking the circuit.

Another method of demagnetizing the specimen is to pass an alternating current through P and a liquid resistance in series. The alternating current is gradually reduced to zero by withdrawing the electrodes of the resistance.

The galvanometer is once more put in circuit with S, etc., by closing the key in the circuit (not shown in the diagram); K_1 is closed, and R_1 given the value corresponding to the maximum current as determined by the preliminary experiment. The commutator is then closed to the right, and the throw of the ballistic galvanometer, δ, noted. The reading of the ammeter is also noted. The throw will correspond to an induction, B_1, and the ammeter reading to a magnetizing force, H_1, represented by some point such as S in fig. 315. We now use this as a point of reference. The galvanometer circuit is again broken, and the commutator is reversed rapidly some 20 to 25 times, and finally left on the right, i.e. the iron is taken several times round the cycle represented in fig. 315 and is said to be in the 'cyclic state'. BG is again put in circuit. When all is steady, R_2 is given a small value and K_1 is opened. The magnetizing field is thereby decreased and the throw of the galvanometer, δ_2, noted. This throw corresponds to a decrease in induction, $B_1 - B_2$. The value of the current corresponding to H_2 is noted on the ammeter.

K_1 is now closed, the galvanometer key opened, the Pohl commutator reversed 20 to 25 times, and finally left to the right. The galvanometer is put in circuit; R_2 is given a larger value; K_1 is then opened and the

throw due to the change in the induction, say, $B_1 - B_3$, is noted. The ammeter is again read.

This process is repeated with the commutator to the right, until R_2 is infinite and consequently the current and H zero, i.e. the relation of B to H represented by the path, SA, of fig. 315, is investigated.

After each measurement, the iron, by the reversal of the maximum current, is returned to the state represented by S, which therefore becomes the reference point.

The key, K_1, being closed, the commutator is reversed some 20 times once more and finally left to the right, R_2 is given a large value and the galvanometer is again put in circuit.

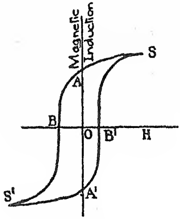


FIG. 315

The commutator is then thrown over to the left, and *at the same time* K_1 is opened, i.e. the current is reversed and at the same time made of small value. This gives a point on the part AB of the curve. The same starting-point, S, is taken, the drop in induction measured, and the change in H from a maximum position to a small negative value obtained.

This process is repeated in many steps until finally $R_2 = 0$, i.e. until the change in field, H, and induction, B, corresponds to a reversal of the maximum positive to maximum negative values.

At this stage we may assume that the curve is symmetrical, and obtain the full hysteresis loop, having obtained SABS′ experimentally, or we may proceed to find S′A′B′S by experiment, using S′ now as our reference point. The process is as before except that the commutator is now left on the left-hand side. K_1 is opened while R_2 has various values from 0 to infinity. At this stage (point A′) R_2 is given values from infinity to 0, and K_1 opened as the commutator is thrown over to the right-hand side.

It should be found that the value of the throw for the reversal, when the full current is passing through the primary is identical for the change

Current i in E.M. units (1)	Field H in E.M. units (2)	Throw of galvanometer (δ) (3)	Change of induction from S or S′ (4)	Induction (B) assuming symmetry (5)
		FOR CURRENT CHANGE	Drop from S	
0·24		0 to 0·24=2·0 cm.	0	10000
0·22		0·24 to 0·22=1·5 cm.	500	9500
0·20		0·24 to 0·20=1·8 cm.	700	9300
0·18		0·24 to 0·18=2·4 cm.	1000	9000
...		etc.		
...				
...				
+0·02				
0·0				
−0·02				
...				
...				
...				
−0·20				
−0·22				
−0·24		0·24 to −0·24=	20000, say	−10000

from S to S′ and S′ to S, and that either value is approximately twice the first recorded throw, δ_1.

The results may be tabulated as shown. The original direction of the current in the coil, P, is taken as positive and the current value may be recorded in column (1) and the corresponding throw in column (3).

To calculate H, imagine a unit magnetic pole to be taken round the axial circle of the solenoid. If H is the value of the magnetic field strength due to a current, i, in the solenoid, the work done on the pole for such a complete circular path is $2\pi r \cdot \mathrm{H}$, where r is the mean radius of the anchor ring.

If there are N turns in the winding, the unit pole is linked with each winding in the complete path, therefore doing work $4\pi i$ for each or a total of $4\pi \mathrm{N}i$, whence

$$4\pi \mathrm{N}i = 2\pi r\mathrm{H},$$

$$\mathrm{H} = \frac{2\mathrm{N}i}{r}, \qquad \ldots(1)$$

where i is the current strength in theoretical E.M. units (*not* amperes).

The value of the current in E.M. units shown in column (1) may be

converted to oersteds by multiplying by the factor $\frac{2N}{r}$ as seen in equation (1).

To calculate the induction, B, corresponding to the observed value of θ, an auxiliary experiment is necessary. The two-way switch, K_2, is closed on the right, so that a current may be sent through the long straight solenoid, M (about 40 cm. long and 4 cm. in diameter), which has m_1 turns per cm. Inside M is the second small solenoid which has been in series with S and the galvanometer throughout the preceding experiment.

For a current of i E.M. units flowing through M, since the solenoid is long, the field strength at the centre is $4\pi m_1 i$ oersteds.

Let the radius of the inner coil be r_2 cm., and N_2 the total number of turns in this coil, then the total flux in the inner coil is

$$4\pi m_1 i\,(\pi r_2{}^2 N_2)\ \text{maxwells.}$$

If now the current be reversed in M the change in induction in the central coil is

$$8\pi^2 m_1 i r_2{}^2 N_2\ \text{lines.}$$

Let R be the total resistance in the galvanometer circuit and i the value of the current in this circuit. Since the electromotive force is numerically $\frac{dN}{dt}$,

$$iR = \frac{dN}{dt} \quad \text{or} \quad \int i dt = \int \frac{dN}{R},$$

the change
$$Q = \frac{8\pi^2 m_1 N_2 r_2{}^2}{R} \cdot i. \qquad \ldots(2)$$

For an instrument of the moving-needle type:

$$Q = \frac{T}{\pi}\frac{H}{G}\sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right),$$

or
$$Q = K_1 \sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right),$$

where $K_1 = \frac{T}{\pi} \cdot \frac{H}{G}$ and is constant.

For small deflections, d, the scale deflection is proportional to θ and as $\sin\frac{\theta}{2} \simeq \frac{\theta}{2}$ we may write

$$Q = K_2 d,$$

where K_2 is a constant.

In the moving coil type of galvanometer the same relation holds, but the constant, K_2, has another value.

Substituting in (2) above

$$\frac{8\pi^2 m_1 N_2 r_2{}^2 i}{R} = K_2 d,$$

or

$$Kd = 8\pi^2 m_1 N_2 r_2{}^2 i,$$

where $K = K_2 R =$ constant.

The last equation, which connects the deflection, d, with the total change in flux, enables the calculation of the value of B corresponding to each field H to be made.

The change in the number of lines, threading the entire circuit, corresponding to 1 scale division deflection on the galvanometer scale is

$$K = \frac{8\pi^2 m_1 N_2 r_2{}^2 i}{d}.$$

The deflections δ_1, δ_2, etc., in column (3) therefore correspond to a change in flux equal to $K\delta_1$, $K\delta_2$, etc. This is a change in the total flux in the space of the secondary winding, S. If r_1 is the radius of cross-section of the iron anchor ring, approximately that of the secondary winding, S, m the number of turns in the secondary, and $(B_1 - B_2)$ the change in induction to be entered in column (4), we have, since B is the number of lines per sq. cm. of cross-section:

$$(B_1 - B_2)\, m \,.\, \pi r_1{}^2 = K\delta,$$

$$B_1 - B_2 = \frac{K}{m\pi r_1{}^2} \,.\, \delta$$

$$= \left(\frac{8\pi m_1}{dm} \frac{r_2{}^2}{r_1{}^2} N_2 i\right) \delta.$$

The term in brackets is evaluated and used as a reduction factor, converting δ to induction for column (4).

To obtain column (5), a symmetrical form for the curve may be assumed: that is, the induction for the maximum positive and negative current values are equally placed from the zero induction line, and equal to half the induction for a full current reversal, which is the mean of the extreme values in column (4). The method of calculation may be understood from the numbers shown in the suggested table.

The B-H curve could also be obtained for an anchor ring, using a fluxmeter in place of the ballistic galvanometer.

The calculation of H is as before. When the induction is large the number of turns in the secondary coil can be diminished, so that the fluxmeter reading is not excessive. The calculation of B from the observed deflection, θ, for any current change in the primary, is more direct than in the preceding case. Unit scale deflection on the fluxmeter corresponds to a flux of 10,000 maxwells. So that if the secondary coil used is of m turns, and the radius of each turn of the secondary is r_1,

the induction change $(B_1 - B_2)$ corresponding to the deflection θ, is given by:

$$m \times (B_1 - B_2) \times \pi r_1^2 = 10000\theta,$$

$$B_1 - B_2 = \frac{10000\theta}{m\pi r_1^2}.$$

The experiment is carried out in a manner similar to that described above.

The Susceptibilities of Paramagnetic and Diamagnetic Substances

In substances which fall into these classes the intensity of magnetization produced by an external field is much smaller than that which results in the case of ferromagnetic substances. In both cases the susceptibility is independent of the field strength, and since I is small in comparison with the applied field, no correction is required for the effect of demagnetization. In the equations given at the beginning of this chapter it is possible to write $H = H_0$ without appreciable error.

The small response in these cases calls for a different method of measurement. This will be illustrated in the following experiments, the first useful in the case of a solid, the second in the case of a liquid.

In order to work out the result, the force on the substance situated in a non-uniform field is required. Let a small volume, v, of the magnetic material be situated in a place where the field strength is H. If the field strength varies from place to place the substance will tend to move to places of lower intensity in the case of a diamagnetic substance, and to places of higher intensity in other cases. Thus, the substance will be subjected to a force. Let this force be denoted by F and suppose that the substance moves through a distance, dx. The work done by the force is Fdx and is equal to the change in potential energy which results from the displacement. A well-known example of this is that of a body falling under gravity. The force in this case is the weight of the body, and as it falls potential energy is lost. In the magnetic case, suppose that the material is at first outside the field, and that the medium occupying the field is of permeability, μ_0. The energy density of the medium is $\frac{\mu_0 H^2}{8\pi}$. Thus the volume, v, into which the specimen is to be placed contains energy of amount $\frac{v\mu_0 H^2}{8\pi}$. When the specimen is in position the energy is $\frac{\mu v H^2}{8\pi}$, where μ is the permeability of the material. Thus the energy arising from the replacement of a volume, v, of the medium by the same volume of the material is $\frac{(\mu - \mu_0)\, vH^2}{8\pi}$. If the specimen is moved through the distance, dx, the

energy change is $\frac{(\mu - \mu_0)}{8\pi} v \frac{d\text{H}^2}{dx} dx$, or in terms of susceptibilities,

$$\tfrac{1}{2}(k - k_0)\, v \frac{d\text{H}^2}{dx} dx.$$

This change of energy in the field is equal to the work done on the body. Thus

$$\text{F} = \tfrac{1}{2}(k - k_0)\, v \frac{d\text{H}^2}{dx}.$$

The Susceptibility of a Solid

Suitable substances for this experiment are wood, sulphur, or ebonite in the form of a rectangular slab.

The variable magnetic field is provided by an electromagnet with wedge-shaped pole pieces.

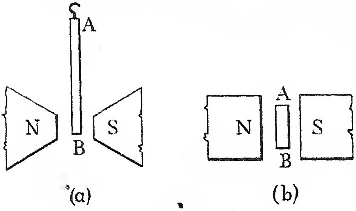


FIG. 316

The figure illustrates the specimen AB in the field between the pole pieces of the magnet. The field will vary rapidly along the vertical direction, due to the wedging of the pole pieces. There will be some variation in the horizontal direction, but if the pole pieces are sufficiently wide apart the central part of the field will be uniform horizontally. The size of the specimen should be such that it can be suspended in the uniform region. The vertical dimension of the pole pieces is smaller, and there will be a rapid falling-off in the value of H. Thus, the force on the specimen will be vertical. The specimen is suspended from one arm of a sensitive balance and is first counterpoised when it is situated in a place where there is no field. In this experiment the influence of the earth's field can be neglected. The material is then brought up to hang between the poles symmetrically, as shown, and the force resulting from the field is counterbalanced by altering the weights to counterpoise the specimen. If the difference is denoted by mg, this is equal to the force arising from the magnetic field.

Let the area of the horizontal cross-section of the specimen be A,

and let x be a distance measured vertically from the upper end. The force on an element of volume, $A dx$, is

$$dF = \tfrac{1}{2}(k - k_0) A dx \frac{dH^2}{dx}.$$

Thus, integrating for the whole extent of the specimen,

$$F = \frac{A}{2}(k - k_0)\int \frac{dH^2}{dx} dx = \frac{A}{2}(k - k_0)(H_1^2 - H_0^2),$$

where H_1 denotes the intensity at the lower end and H_0 that at the upper end of the slab. H_0 can be neglected, for the field falls off very rapidly from the poles. In the case of a paramagnetic substance the direction of this force is downwards into the region of greatest magnetic intensity. In this case the weights in the pan have to be increased, and in order to determine k we have

$$mg = \frac{A}{2}(k - k_0) H_1^2$$

and k_0, the susceptibility of air, may be assumed to be zero in this experiment.

The value of H_1 should be determined by means of the fluxmeter (p. 521).

The Susceptibility of a Solution

The method to be described is suitable for the measurement of the susceptibility of a solution of a magnetic salt, and of other magnetic liquids.

For the purpose of the experiment a solution of a ferrous or ferric salt, e.g. the chloride, may be taken.

The method consists in observing the rise of the liquid in a tube placed between the poles of a magnet. The conditions and theoretical basis of the experiment are similar to those described. The difference consists in the way the force is measured.

The solution is placed in a U-tube consisting of a wide and a narrow limb. The former limb is placed outside the field and the latter within the region between the poles of the magnet.

Suppose that the position of the liquid surfaces in the U-tube are A′B′ and C′D′ before the field is generated. The level of C′D′ must be observed by means of a micrometer microscope. When the field is present let the new levels be AB and EF, and let the position of EF be observed by the microscope. Thus the height, FD′, is obtained. Let this be denoted by h. The level in the wider limb has sunk to AB, so that EF stands at a height above the general level which is greater than h. This height has to be measured from the new level of AB. Suppose that A′B′ sinks through a distance, c, then a volume of liquid of amount cA has passed into the other limb and now lies above C′D′.

Thus $cA = ha$, where A and a are the areas of section of the wide and narrow limbs respectively.

Thus $c = \frac{ah}{A}$ and the corrected height is $\left(1 + \frac{a}{A}\right) h$.

This is the height of the column of liquid supported by the forces arising from the magnetic field.

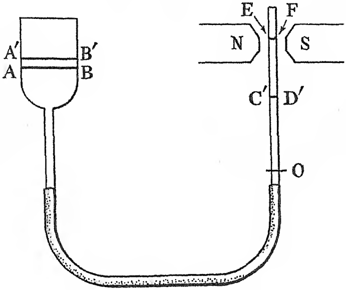


FIG. 317

Take some section at O where the field is negligible and calculate the force on the liquid above this point. It will be found to have the value $\frac{1}{2}(k - k_0)\, a\mathrm{H}_1{}^2$, where H_1 is the field intensity at the upper level.

Thus $$\tfrac{1}{2}(k - k_0)\, \mathrm{H}_1{}^2 = \left(1 + \frac{a}{A}\right) \rho g h.$$

It is thus necessary to determine the sections of the limbs of the U-tube and the density of the liquid.

The value of H_1 is measured as in the last experiment, and it is a good plan to arrange that the level of the liquid rises to the centre of the space between the magnetic poles and to measure the field strength at this level. Another method of making the measurement is to allow the liquid to rise in a capillary tube and to observe the further rise when the field is generated. The column should be long enough so that it projects below the poles of the magnet into a region of negligible magnetic intensity.

If the rise be h, the above formula without the term $\frac{a}{A}$ can be applied.

k_0 for air may be neglected.

It is interesting to study the variation of k with concentration of an iron salt. If m denote the mass of iron in solution per c.c., a graph

of k in terms of m should give a linear law. It will be found that the line cuts the axis $m = 0$ at a finite value of k. This is due to the contribution to the susceptibility of the solution which arises from the water. Recent results of experiment indicate a numerical value of this intercept of $0{\cdot}72 \times 10^{-8}$ at $20°C$.

CHAPTER XIX

AMMETERS, VOLTMETERS, AND GALVANOMETERS

THE most convenient form of current measurer is the ampere meter or ammeter, which indicates directly, on a graduated scale, the strength of the current passing through the circuit in which it is placed.

As a result of the method of construction, the range of usefulness of the instrument is limited, in the usual form, to measurement of current not less than a milliampere. By improving the suspension, etc., the two-pivot instrument has been manufactured capable of measuring one microampere; for example, a Weston ammeter is made with an open scale division in microamperes.

Similar conditions hold for the voltmeter.

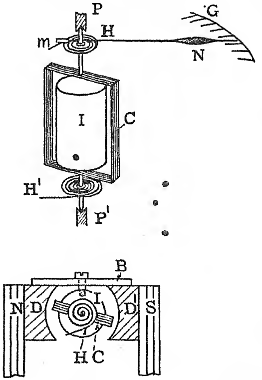


FIG. 318

The Voltmeter

This instrument consists essentially of a coil of thin copper wire, C, which is supported on an axis pivoted on jewelled pivots, P, P′, and is free to move in the cylindrical gap between the soft iron pole piece, DD′, of an aged steel permanent magnet, NS.

The field is strengthened and made approximately radial by the insertion of a soft iron cylinder, I, in the space inside the coil. This

cylinder, as seen in the lower fig. 318, is fixed to a brass bar, B, which forms a bridge between the pole pieces. The air gap is thus reduced and the coil moves freely in this space.

To establish a restoring couple, the two hair-springs, H, H′, are attached to the axis, and to points on the framework which are insulated from each other. These springs also serve as leads for the current, to and from the coil.

Attached near the upper spring is a light counterpoised pointer which moves over a scale, G. The centre of gravity of the whole system is arranged to coincide with the axis of suspension.

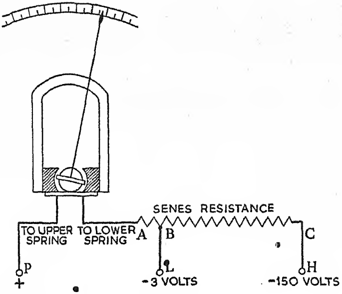


Fig. 319

When in use, the instrument is placed in parallel with the points whose potential difference is to be measured. The internal resistance of the instrument must therefore be large in order to avoid any appreciable rearrangement of current and potential drop in the circuit. The current passing through the voltmeter is therefore very small for such a high internal resistance, and therefore the heating of the coil is not very great.

The internal resistance is not made up entirely of that of the copper coil. In most forms the greater part of the internal resistance consists of a resistance in series with it (fig. 319). The chief reason for this is to avoid any error due to heating in the moving coil.

Such heating may be due to (1) atmospheric rise in temperature, (2) the Joule effect. An increase in resistance of the moving part would occur in either case unless the temperature coefficient of the wire were small. Manganin has a small temperature coefficient, but a higher

specific resistance, i.e. for the same coil resistance a less radiating surface is available for manganin than copper.

The effect of (2) in raising the temperature of the coil may be best eliminated by making the moving coil resistance fairly low, and hence the percentage change of the whole is reduced.

Hence the usual compromise is a copper moving coil of comparatively small resistance and a series manganin coil of high resistance. This series resistance is constructed of thicker wire than would be possible for the moving coil.

Having in this manner secured the best approximation to constancy for the internal resistance, it will be seen that the small current, i, passing through the instrument is proportional to the potential difference between the terminals. Now the deflection produced is proportional to the current for a large range of deflection when the field is radial, and hence the deflection produced is proportional to the potential difference between the terminals.

The instrument is made 'dead beat' by winding the moving coil on a copper frame.

The same instrument may be used to measure different ranges of potential. This will be seen from fig. 319. Thus, if a potential of 3 volts, when applied to PL, produces a full-scale deflection in the instrument, whose internal resistance (AB + coil) is 345·5 ohms, it will be seen that the current in the coil is 0·00865 ampere.

Now, if a higher potential, say, 150 volts, is to be the new value corresponding to a full-scale deflection, this may be connected to H and P so that a bigger series resistance, BC, is included, such that the total internal resistance, R, is given by

$$0{\cdot}00865 = \frac{150}{R},$$

$$R = 17270 \text{ ohms.}$$

The same current will flow through the coil, and hence the deflection will be again a full-scale deflection.

The graduations on the scale will therefore subdivide the 0 to 150 into equal increments, and each division corresponds to fifty times the value which corresponds to the lower voltage applied to PL.

For the lower range voltmeter, measuring potential of the order of a millivolt, the value of the internal resistance is smaller, for the deflection is proportional to Bi, where B is the magnetic flux in the gap and i the current. B is constant and so to produce the deflection, when a low potential is applied, R must be decreased.

In such a moving-coil instrument the direction of deflection depends upon the direction of the current. The higher potential should be connected to P which is marked +, otherwise the needle may be strained.

Ammeters

To measure the current in a circuit, the measuring instrument used should be of low resistance, unless some account be taken (as in galvanometers) of the resistance introduced by it. For example, if a small resistance of known magnitude, r, be included in the circuit, the current strength, i, may be found if the value of the potential drop, V, along r, be determined by means of a millivoltmeter, for $i = \frac{V}{r}$. This arrangement of a voltmeter shunted with a low resistance is utilized in the ammeter. The fixed-range ammeter usually contains the shunt inside the case. The value of the shunt resistance is small, and therefore the resistance of the whole instrument is of the same order. The moving coil of the ammeter is often, also, provided with a series resistance as in the voltmeter to minimize temperature variations as described above.

Many of the better forms of ammeter are not provided with fixed shunts, but require an external one. The value of the shunt resistance determines the range of the instrument.

The greater the range, the smaller is the resistance of the shunt. Suppose the maximum scale reading is obtained for a current, i, through the coil; this is proportional to the potential difference between the ends of the shunt. If the external current be doubled the drop of potential along the shunt will be doubled. In order to keep the same current, i, in the coil, the shunt resistance must be halved. The same principle applies for other multiples of the external current and the range of the instrument can be varied by means of a series of appropriate shunt resistances.

The shunts are made of manganin, which has a low-temperature coefficient. The dimensions required in using a given manganin strip can be calculated. If it is found that to produce a negligible heating the shunt width has to be excessive, it is usual to construct the shunt of several strips in parallel.

The instrument should never be used without the appropriate shunt for the current to be measured.

The ammeters and voltmeters described above have the advantage of being direct reading on a calibrated scale; they are robust and do not require any adjustment. But as indicated at the outset, the general type of instrument is not sufficiently sensitive to measure currents of less than a milliampere or potentials less than one millivolt. By more delicate construction and general improvement they can be made to measure to one microampere and one microvolt. In such a case the extra sensitivity entails very precise work, and makes the cost of the instrument somewhat higher than for the ordinary range (i.e. one millivolt or milliampere).

Some forms of instrument are available which combine the voltmeter and ammeter. The necessary shunts and series resistance are contained inside the case, and by connecting to the proper terminals, the instrument may be used either as an ammeter, or as a voltmeter of several ranges (see, for example, fig. 320, at p. 1).

Unipivot Instruments

To increase the sensitivity of the above types of instruments, a modification of the support of the moving part was introduced by R. W. Paul. The pivot friction was reduced very considerably by the use of the one-pivot method of suspension, and at the same time all the advantages of the form of double-pivot suspension were retained, so that a sensitivity corresponding to one subdivision per microampere is obtainable for the Unipivot instrument.

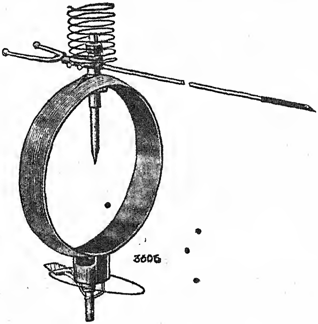


FIG. 322

The construction is shown in figs. 321 (at p. 1) and 322. A circular coil is suspended about a spherical core of soft iron between the poles of a permanent magnet.

Fig. 322 shows the detail of the coil support. A vertical spindle carries a light counterpoised pointer and rests on a polished jewel at the bottom of a cylindrical hole drilled in the soft iron sphere.

The cylindrical spring at the upper end has a very slight lifting effect on the coil, and produces a restoring couple when the coil is deflected; it also serves as a lead for the current to the coil. The current leaves the coil by the flexible wire shown at the lower extremity.

The centre of gravity of the moving part is at the point of support.

A simple device is included to raise the point off the jewel when the instrument is not in use. This is shown under the coil in fig. 321, which gives the general appearance of the instrument when one pole piece and the magnet are removed.

For many purposes it is necessary to be sure that the ammeter or voltmeter used is reliable to a known degree of accuracy. The instrument may be calibrated in the laboratory (e.g. by potentiometer and standard cell), but it is here suggested that each laboratory be provided with one form of ammeter or voltmeter which has been tested at the National Physical Laboratory, and is provided with a correction certificate. Such instruments should be retained as laboratory standards and the working instruments checked against them.

Galvanometers

When smaller currents are to be measured, use must be made of some form of galvanometer. This instrument is not as robust as the above, it must be levelled before use, and the current must be calculated from the observed deflection produced by it.

Thus it is not as convenient and simple to use as the ammeter, but has a sensitivity which is impossible to attain in the latter.

The increase in sensitivity is largely produced by a more sensitive method of suspension. The friction of the pivot is entirely removed by the use of a fine suspension of silk or phosphor-bronze. The suspension carries a small concave mirror by means of which small movements of the moving part may be magnified. The two common methods of producing such magnification are by use of:

(1) a lamp and scale,
(2) a scale and telescope.

(1) Lamp and Scale Method

In this method, a beam of light from an incandescent lamp or Nernst filament is directed by means of a lens on to the concave mirror, which reflects it to a scale some distance away. The scale should be placed a distance away from the mirror equal to its radius of curvature. The condensing lens, over which is stretched a vertical wire, acts as an illuminated object whose image, a circular patch of light with vertical black line, is used to measure deflection. The greater the distance, D, between mirror and scale, the greater the magnification produced. When the mirror rotates through an angle, θ, the reflected beam moves through twice that angle, causing a movement of the spot of light, say, d cm. on the scale.

Hence $$\tan 2\theta = \frac{d}{D}.$$

If the mirror is plane, a wire or part of the filament of the lamp is focused by a lens on the scale after reflection on the mirror.

To measure the deflection it is essential that the mirror should produce a clear image. For this reason, with the size and character of the mirror available on such a suspension, the usual maximum value of D is one metre.

(2) The Telescope and Scale Method

A scale is set up horizontally at about one metre from the galvanometer mirror, and a telescope, usually under the central graduation of the scale, is turned towards the mirror. When the mirror, which should be a plane one for this method, is parallel to the scale, the latter will be seen in the telescope. The reading of the scale in coincidence with the cross-hair in the eyepiece of the telescope is noted. When the mirror moves, a second scale reading will coincide with the cross-hairs. The difference for an angular rotation, θ, of the moving system may be accurately measured.

If this is equal to d cm., then $\tan 2\theta = \dfrac{d}{\mathrm{D}}$.

It is advisable, for simplicity of reading, to use either a telescope with an erecting lens or prism or to use a scale provided with inverted graduations, so that the scale appears the right way up.

The Constants of a Galvanometer

(1) The Determination of the Resistance

The first method to be described is useful when the relation between the current through the instrument and the scale deflection is linear.

It is more readily applicable when the galvanometer can be shunted so that the resistance of the galvanometer and shunt in parallel is practically that of the shunt.

In the diagram, G and S denote the galvanometer and shunt resistance and R is a resistance which can be varied in series with G; both R and G being shunted by S.

The rheostat, R′, is used to keep the current constant in the main circuit, this constancy being checked by the ammeter, A. If G and R together are much greater than S, it will not be found necessary to alter R′ very much as R is varied. The current in G is given by

$$i_g = \frac{\mathrm{S}i}{\mathrm{G}+\mathrm{R}+\mathrm{S}} = k\theta,$$

where i denotes the current in the main circuit.

Thus
$$\frac{\mathrm{G}+\mathrm{R}}{\mathrm{S}} + 1 = \frac{i}{k\theta}.$$

If S is very small compared with $(\mathrm{G}+\mathrm{R})$, unity on the left-hand side may be neglected.

On drawing a graph of R against $\frac{i}{\theta}$ we obtain (G + S) from the intercept on the R axis.

The graph will indicate whether the linear relation holds and thus whether the method is applicable.

A special case of this method may be used to give the value of G directly provided that a change in (G + R) makes no appreciable change in i. The value of S is adjusted to give a full-scale deflection when R is zero. R is then increased until the deflection is halved, and in this case R = G, unity in the above equation being neglected.

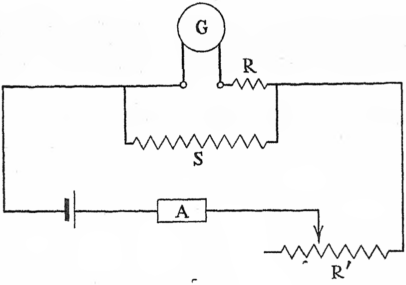


FIG. 323

The second method can be used when the linear relation does not apply. In this case the deflection of the galvanometer is kept constant. Arrange the circuit as in the first method and let i be changed to various values, fi, where f is some number greater than unity. Adjust R to preserve the same deflection in G.

Beginning with R at zero and with the main current, i. With the current, fi, let R be changed so that the galvanometer deflection is the same as before.

The condition satisfied is

$$\frac{Si}{G + S} = \frac{Sfi}{G + S + R}$$

or

$$f(G + S) = G + R + S.$$

S can usually be neglected, so that

$$G(f - 1) = R.$$

Thus a graph of R against f will be linear and the intercept on the axis of R will give G.

In the case in which the current is doubled $R = G$. If it should be inconvenient to begin with $R = 0$, the first reading can be taken with $R = R_0$, when

$$f(G + R_0) = G + R.$$

The graph again gives the value of G as the intercept on the R axis. Other methods are given on p. 528.

(2). *The Logarithmic Decrement. Resistance by Method of Damping*

The logarithmic decrement of the galvanometer is one of its important constants and is required for the correction of deflections in some cases.

The method of finding the logarithmic decrement has already been described (p. 152). This can be applied in the present case. The determination of this quantity will be considered here in connexion with another method for finding the resistance of the galvanometer. Suppose that the moving coil of the instrument oscillates in a magnetic field of strength, H, when its terminals are joined by a resistance which, with the galvanometer resistance, gives a total of R ohms.

The resistance to the motion of the coil consists of the twist of the suspension, the air resistance, and the resistance produced by the application of Lenz's law to a circuit through which the magnetic flux is changing. The first of these is proportional to the deflection, θ, the second to the angular velocity, $\dot{\theta}$, and the third is proportional to the current in the coil. Thus, in the case of a rectangular coil carrying a current, i, in a field, H, the force per unit length on the vertical sides is iH. The current is proportional to the angular velocity and inversely proportional to the resistance. The third term is therefore of the form $\frac{a\dot{\theta}}{R}$, and with the usual notation the equation of motion can be written

in the form $$I\ddot{\theta} + \left(\frac{a}{R} + b\right)\dot{\theta} + \mu\theta = 0.$$

In the notation of p. 150, this becomes

$$\ddot{\theta} + k\dot{\theta} + n^2\theta = 0.$$

Thus $$\lambda = \frac{kT}{4} = \frac{T}{4I}\left(\frac{a}{R} + b\right).$$

On open circuit, R being infinitely great, λ is known as the log. dec. of the galvanometer. Its value is $\lambda_0 = \frac{Tb}{4I}$, and the damping is due to air resistance.

The experiment should be carried out by finding λ for various values of R. A resistance, r, should be placed in series with the galvanometer of which the resistance is G. λ_0 is first determined and then λ for various values of r.

Since
$$\frac{1}{\lambda - \lambda_0} = \frac{4I}{Ta}(G + r),$$

a linear graph will result from plotting $\frac{1}{(\lambda - \lambda_0)}$ against r and G will be given by the intercept on the axis of r.

(3) The Period of Vibration

This is another important constant which should be carefully measured and recorded.

(4) The Current Sensitivity

For many reasons it will be apparent that the instrument cannot be calibrated once for all. The scale distance is variable, the suspension may of necessity be replaced, and so on; there is, therefore, no permanent direct reading scale as in the case of ammeters and voltmeters.

The usual method of converting scale readings to the corresponding currents is to obtain the sensitivity of the instrument. The sensitivity also gives an indication of the possibilities of the instrument.

Current sensitivity may be defined in either of the two following ways:

(*a*) The number of millimetres deflection produced on a scale 1 m. away by 1 microampere, i.e. by 10^{-6} ampere, or

(*b*) the number of microamperes required to produce 1 mm. deflection on a scale 1 m. from the galvanometer mirror.

The second definition is from some points of view the better, but (*a*) gives a value which varies directly with the property measured. It must be understood that a sensitivity of 1000 mm. per microampere means that a current produces a deflection in the proportion of 1000 mm. per microampere assuming the deflection to be proportional to the angular movement of the coil. The deflections measured never actually exceed from 5° to 8°.

Volt sensitivity may be similarly defined, substituting microvolt for microampere in the above.

It will be seen that for a fixed current strength the current sensitivity is greater for a higher coil resistance, that is, for a greater number of turns.

Volt sensitivity, which deals with a fixed potential applied to the galvanometer, is greater the smaller the value of the resistance of the coils.

The measurement of current sensitivity is now simply carried out, if the instrument is not too sensitive, by placing the galvanometer, a megohm (10^6 ohms) and a steady cell of known E.M.F., E, in series. The deflection produced, δ mm., on a scale 1 m. away is observed directly and the current in microamperes is calculated from

$$i_g = \frac{E}{10^6 + G} \cdot 10^6,$$

where G is the resistance of the galvanometer, which may be found by one of the methods given on pp. 499 et seq. If G is small in comparison with 10^6 ohms the sensitivity is $\frac{\delta}{E}$ mm. per microampere.

A more precise value of the sensitivity may be found using the circuit shown in fig. 324. This method may be used for instruments of all sensitivities.

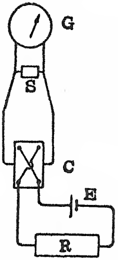


FIG. 324

A steady accumulator whose electromotive force, E volts, is known accurately—either by a potentiometer comparison with a standard cell or by measurement with a calibrated voltmeter—is connected in series with a high-resistance, R ohms, through a commutator, C, to a low resistance, S, which is in parallel with the galvanometer of resistance, G.

The mean deflection, δ, say, for both positions of the commutator is obtained.

The current causing the deflection is readily calculated for the effective resistance of the shunt and galvanometer is $\frac{SG}{S + G}$; the main circuit current is therefore equal to*

$$\frac{E}{R + B + \frac{SG}{S + G}}, \qquad \ldots(1)$$

where B is the battery resistance which, if an accumulator is used, is

* N.B.—If S is much less than R, the factor $\frac{SG}{S + G}$ may be omitted as this is a little less than S.

negligible compared with R. Hence, neglecting B in (1), the current through the galvanometer is

$$i_g = \frac{E}{R + \dfrac{SG}{(S+G)}} \times \frac{S}{(S+G)} \times 10^6 \text{ microamperes}$$

$$= \frac{E \,.\, S \times 10^6}{R\,(S+G) + SG} \text{ microamperes} \qquad \ldots(2)$$

Hence the sensitivity is $\frac{\delta}{i_g}$ mm. per microampere.

Thus the sensitivity having been obtained, the deflection produced in any case may be converted to the corresponding current values so long as the suspension remains the same and the scale is at the same distance, one metre, from the galvanometer mirror.

STEADY CURRENT MEASUREMENT

Galvanometers for the measurement of steady direct current may be subdivided into two general classes according to the nature of the moving part: (*a*) moving needle, (*b*) moving coil.

(a) Moving-needle Galvanometers

The student will be familiar with the simple tangent galvanometer. In this form a small magnet is suspended at the centre of a coil of wire of n turns. When the current of C, expressed in electromagnetic units, passes through the coil, placed in the magnetic meridian, the magnetic field set up causes a deflection, θ, such that the restoring couple due to the horizontal component of the earth's field, H, balances the couple due to the magnetic field of the current, and we have

$$i = \frac{Hr}{2\pi n} \tan \theta, \qquad \ldots(3)$$

where r is the mean radius of the coil.

In general the assumptions upon which the formula is developed are not justified, and corrections should be applied to allow for the width of the coil, etc.; such corrections are but seldom applied, and the galvanometer is not used as a small current measurer.

A modification, known as the Helmholtz galvanometer, consists of two similar coils placed on a common axis at a distance apart equal to the radius of either.

A short magnet is supported at the mid-point between the centres of the coils. As with the tangent galvanometer, the plane of the coils is placed in the magnetic meridian, then using the same notation as before,

$$i = \frac{5\sqrt{5}r^2H}{32\pi n} \tan \theta; \qquad \ldots(4)$$

n is the number of turns in each coil.

These two forms are such that the absolute value of the current may be simply calculated, but they have no claim to great sensitivity.

Sensitive Current Detectors

The result expressed in (3) for a simple tangent galvanometer shows that, for this form of instrument, the deflection for a given current may be increased by:

(α) decreasing H, the control field,

(β) decreasing r,

(γ) increasing n.

There is a limit to methods (β) and (γ) which is prescribed by the practical problem presented.

Further, as r is decreased and n increased, the conditions upon which (3) was developed no longer hold.

(α) may be carried out by the use of an external control magnet which neutralizes the value of the horizontal component of the earth's field. Further, by using an astatic pair of magnets or group of magnets the control effect may be reduced without interfering with the magnitude of the deflecting couple.

The above conditions are embodied in the Thomson (or Kelvin) and Broca galvanometers.

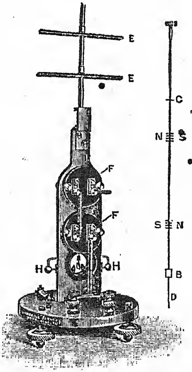


FIG. 325

The Thomson Galvanometer

The Thomson galvanometer is illustrated in fig. 325. The control magnets, EE, enable the galvanometer to be used in any position

independently of the earth's field; the suspended astatic system is shown at the right of the diagram. Each set of magnets is placed at the centre of a pair of coils, FF. The magnets are mounted on a light rod, the whole being supported from a torsion head by an unspun silk or a quartz fibre.

The magnitude and direction of the control field may be varied by alteration of the positions of EE along the vertical rod shown.

Sets of coils of different resistances are usually supplied. Using high-resistance coils a sensitivity of about 600 mm. per microampere may be attained, i.e. the instrument will detect currents of the order of 10^{-9} ampere.

The movement of the magnet is detected by the small mirror, B.

The Broca Galvanometer

This instrument is shown in fig. 326. The moving astatic pair is made up of two vertical magnets having consequent poles as shown. The control field is due to the magnet, which moves in a ball socket to any desired position.

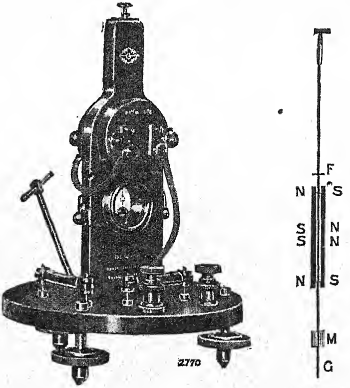


Fig. 326

The coils may be of any suitable resistance, say, 10, 100, or 1000 ohms, as required for sensitivity in the circuit. They are connected in series, one at each side of the space occupied by the centre magnet, and arranged to produce a field in the same direction.

As in the Thomson galvanometer, an aluminium vane, G, moves between two parallel plates which may be adjusted by the rods terminating in the metal knobs, HH. The damping of the system may be altered by an adjustment of the distance between these plates. Currents of the order of 10^{-10} ampere may be detected.

The Thomson and Broca galvanometers are most advantageously employed as very sensitive current detectors whose sensitivity may be rapidly adjusted over a wide range by the control magnets.

Both forms require levelling before using.

(b) Moving-coil Galvanometer

The moving-coil galvanometer is constructed in a manner very similar to the ammeter already described. A phosphor-bronze strip,

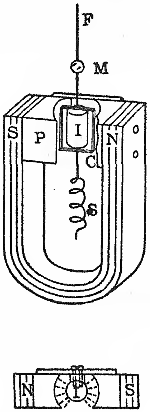


Fig. 327

F (fig. 327) acts as suspension to the coil, C, which is free to move in the gap between the pole pieces of a permanent magnet and a soft iron cylinder, I, which is screwed to a brass plate as seen in the lower figure.

In this case the magnetic field in the air gap is approximately radial, as shown by the broken lines in the lower figure (fig. 327).

The current enters the coil via the phosphor-bronze strip, F, and

leaves at the lower end by means of a helix of phosphor-bronze, S. The control is mainly due to the twist of F in such a case.

If B is the magnetic flux in the air gap, A the area of one turn, n the number of turns, and τ the restoring couple per unit angular twist of the suspension, F, we have, for the case of a radial field when a current, i, passes, causing a deflection, θ,

$$\mathrm{B}n\mathrm{A}i = \tau\theta, \qquad \ldots(5)$$
$$\mathrm{G}i = \tau\theta,$$
$$\mathrm{G} = \mathrm{B}n\mathrm{A}$$

and is the couple on the coil for unit current and is called the *galvanometer constant*,

or $$i = k\theta,$$

where k is a constant and equal to $\dfrac{\tau}{\mathrm{B}n\mathrm{A}}$.

For small deflections θ is proportional to d, the deflection on the scale.

From (5) above it will be seen that to increase the sensitivity, τ must be made as small as possible, and B, n, and A as large as possible.

To decrease τ we may decrease the cross-section of the strip or increase its length.

For a circular wire it was shown on p. 86 that

$$\tau \propto \frac{\mathrm{R}^4}{l}.$$

The most profitable method of decreasing τ is therefore to reduce the cross-section, i.e. use a fine suspension. This is limited by the fact that very fine suspensions are also very fragile and somewhat difficult to use. From this point of view phosphor-bronze is the most satisfactory material and is almost universally employed. The usual type of moving coil is supported by a fibre of the greatest convenient length.

The increase of B, n, and A must be considered together as the terms are interdependent.

Assuming the magnet to be fully saturated, the value of B in the air gap depends on the size of the gap. The smaller the air space, the larger is B. The effect of increasing nA is to increase the size of the coil, and consequently the air gap, and hence decrease B.

In practice it is usual to use a standard size of air gap which allows a frame of such size to be wound with wire and move freely in the available space, that the best compromise between these two effects is obtained for a maximum value to the product BnA.

An example of galvanometers of this type is shown in fig. 328 (at p. 1).

Such instruments may have a sensitivity as great as 1500 mm. per microampere, i.e. the instrument will measure currents of the order 10^{-10} ampere.

Adjustment

The moving-coil instrument must be adjusted before use. The coil is first released, then the instrument is levelled by means of the levelling screws, so that the coil does not touch either the pole pieces or the iron core, and is thus able to swing freely.

The instrument may be used in any position, and the coil is turned by means of a torsion head, which carries the suspension, until the plane of the coil is approximately parallel to the sides of the magnet. It is inadvisable to make this adjustment, unless the reflected beam does not fall on the scale, for there is a danger of breaking the suspension.

Onwood Moving-coil Instruments

The Onwood galvanometer differs somewhat from the above general types in that it requires no levelling, is not so fragile, and is more convenient to move.

The difference is in the method of suspension, as seen in fig. 329. NS is the permanent magnet with a soft iron core, *d*, which is drilled

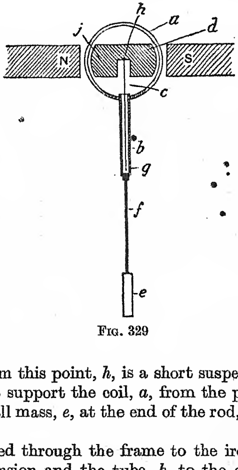


FIG. 329

to the centre. From this point, *h*, is a short suspension, *c*, which is of sufficient length to support the coil, *a*, from the point, *g*, clear of the fixed parts. A small mass, *e*, at the end of the rod, *f*, keeps the system vertical.

The current is led through the frame to the iron core, and thence through the suspension and the tube, *b*, to the coil, which it leaves by means of a flexible ligament at the base of the coil, but not shown in the diagram.

Since the coil is supported at the centre it will be equidistant from pole pieces and iron core alike for all positions of the instrument not too far removed from the horizontal, i.e. the instrument does not require levelling.

A small spring, not shown, at the upper end of the suspension protects the latter against sudden shocks.

The galvanometer is a very small one and the sensitivity claimed ranges from 20 to 500 mm. per microampere according to construction details.

THE CHOICE OF A GALVANOMETER FOR THE MEASUREMENT OF A STEADY DIRECT CURRENT

To select the galvanometer most suitable for a particular experiment many factors must be considered. To decide firstly between the two general types described above, their relative advantages and disadvantages are discussed below.

Moving-needle Type

(1) The value of the control field is affected by the proximity of external magnetic fields. The instrument may register a deflection when no current circulates through the coils, when magnets are moved in the laboratory or dynamos set in motion or even by a passing electric tramway car. Thus there is some uncertainty unless such stray fields are eliminated.

(2) The moving magnet may have its moment altered by the field set up when a current passes through the coil.

(3) Unless the instrument is made 'dead beat' the needle takes a long time to come to rest and is therefore somewhat troublesome to use. This may be overcome to some extent by using an external damping circuit, consisting of a solenoid through which a current may pass when the circuit is closed. This solenoid is placed near the case of the instrument and the circuit closed momentarily as the needle swings, in such a way as to produce a field which opposes its motion.

(4) The damping of the instrument is independent of the resistance of the circuit in which it is placed.

(5) The needle may be supported by means of a quartz suspension which has the property of returning after deflection to its original position. The slight torsion in the fibre is usually negligible.

Moving-coil Type

(1) The moving-coil galvanometer has a large permanent magnet, and is therefore practically unaffected by stray external fields.

(2) No demagnetizing effect is possible on the moving part.

(3) The movement of the coil may be very rapidly arrested, even when the instrument is not 'dead beat', by connecting the ends of the coil together through a very low resistance as the coil passes its rest position.

(4) The electromagnetic damping varies with the value of the resistance of the external circuit. If used in series with a low resistance this damping is very high, and the coil may take several minutes to attain the full deflection, a fact which is very often overlooked when the instrument is used.

(5) The zero-keeping quality of the suspension depends upon the degree of sensitivity attained. The torsion in the fibre is taken into account in the expressions for the current.

Having decided, from the conditions of the experiment, the type of instrument most suitable, the next problem is to decide on the most satisfactory galvanometer resistance.

The Order of Resistance of a Moving-coil Galvanometer which is most sensitive in a given Circuit

In general terms we may state that if the current is to be of a fixed value, independently of the galvanometer, considering all factors, the sensitivity will be proportional, approximately, to $\sqrt{G}$, i.e. will increase with increasing resistance.

If a fixed potential difference is to be measured, the sensitivity will obviously be greater the smaller the resistance; approximately, the sensitivity is proportional to $\frac{1}{\sqrt{G}}$, i.e. high resistance for detecting current, and low for detecting potential differences.

The usual problem is to find the best value of the galvanometer resistance, G, when dealing with a fixed external resistance, R, and electromotive force, E.

We saw (p. 408, equation 5) that the couple due to the current i is $nABi$, and if E is the electromotive force in the circuit

$$i = \frac{E}{R + G} \qquad \ldots(6)$$

Further, in a galvanometer there is but a limited space between the pole pieces. Fig. 330 shows a cross-section of the frame which will just move freely in this space. Let the cross-section of the whole of the windings be a sq. cm. and assume that the windings entirely fill the space with copper. If p be the mean perimeter of the coil windings, we have length of wire used $= np$, cross-section of the wire $= \frac{a}{n}$.

Therefore
$$G = \frac{\sigma np}{\frac{a}{n}} = \frac{n^2 p\sigma}{a}.$$

where σ is the resistivity or specific resistance of the wire, usually copper,

i.e.
$$n = \sqrt{\frac{Ga}{p\sigma}}. \qquad \ldots(7)$$

Hence the couple due to the current $= \mathrm{B}n\mathrm{A}i$

$$= \mathrm{BA}\sqrt{\frac{Ga}{p\sigma}} \cdot \frac{E}{R+G}.$$

from (6) and (7).

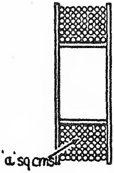


FIG. 330

The condition for the couple to be a maximum is that

$$\frac{\mathrm{BAE}\sqrt{a}}{\sqrt{p\sigma}} \cdot \frac{\sqrt{G}}{R+G}$$

be a maximum.

We have seen that a and p are constant, so that the condition is

$$\frac{\sqrt{G}}{R+G} \text{ is to be a maximum}$$

or that
$$\frac{R}{\sqrt{G}} + \sqrt{G} \text{ be a minimum,}$$

i.e.
$$R = G.$$

That is, under the circumstances stated, the maximum sensitivity is obtained *when the galvanometer resistance is equal to the total external resistance.*

It should be noted that the galvanometer resistance in this discussion refers to the copper coil resistance only. The resistance of the suspension should be included in the value of R. This being so, it is apparent that there is a lower limit beyond which the resistance of a coil may not be reduced with any advantage.

If the galvanometer is to be chosen as a detector or measurer of small differences of potential the most suitable instrument will be one of low resistance (e.g. for thermocouple work).

Other qualities of galvanometers to be considered when making a selection of galvanometers are:

(1) Damping

The damping of the moving part in a galvanometer apart from external artificial agency may be considered to be due to two causes.

(*a*) *The damping due to the viscosity of the air.* This is present in moving coil and needle alike, and is approximately proportional to the angular velocity of the system. It is always present, but is usually small.

(*b*) *Electromagnetic damping.* In the case of a moving magnet the amount of damping due to this cause is very slight when the magnet is in a non-metallic case, and when the coils are wound on wood or ebonite.

This is the reason for the long and troublesome wait which occurs before the needle returns to its rest position. This may be reduced as described below. However, in either case the amount of damping is obviously independent of the external circuit.

In the case of the moving-coil instrument, the suspended coil, moves in a strong magnetic field. When the circuit is closed the movement of the coil is opposed by the current induced. The damping produced in this way is known as electromagnetic damping.

The value of the damping current depends on the magnitude of the external resistance, and may become very great for a low series resistance.

For many purposes it is necessary or convenient to have a galvanometer such that the moving part returns to the zero position without oscillation after being deflected. A galvanometer having this property is said to be 'dead beat'.

In both types of instrument this may be brought about by increasing the electromagnetic damping. In the moving-needle type this is accomplished by encasing the needle in a copper, or similar metal, case. The movement of the needle sets up eddy currents in the copper and the magnet is rapidly brought to rest.

The moving coil may be rendered dead beat by winding it on a metal 'former' or frame. This constitutes a closed metallic circuit, and the desired result is obtained. Alternatively, if the galvanometer is wound, for ballistic purposes, on a non-metallic frame, e.g. bamboo, the same result is obtained by facing the edge of the coil with a thin sheet of copper foil.

(2) Period and Constancy of the Zero

A galvanometer which is very sensitive has a long time of swing. Further, due to the very fine suspension, trouble may be caused by the 'creep' of the zero. However, the kind of instrument used in experiments in this book will not be of the extremely sensitive type with which this trouble arises. The main cure for the trouble lies in

the selection of the suspension, and that really involves the selection of an instrument-maker who will take the trouble to *minimize* this fault. Beyond this, the correction of residual effect must be solved by the ingenuity of the experimenter applied to the particular experiment concerned.

Quartz for moving magnets and phosphor-bronze for moving coils cause least trouble in this respect.

For accurate measurements a period of five to ten seconds is suitable, and for less precise measurements the period need be no more than two seconds.

MEASUREMENT OF QUANTITY OF ELECTRICITY

The Ballistic Galvanometer

A galvanometer suitable for measuring a *quantity* of electricity is called a *ballistic galvanometer*, and has the following essential features:

(1) The periodic time of swing, T, of the moving part is fairly large.

(2) Damping of the moving part is very small.

The first condition is fulfilled by making the moment of inertia of the needle or coil which forms the moving part as large as practicable and by reducing the controlling forces, for

$$T = 2\pi\sqrt{\frac{I}{\tau}},$$

where I is the moment of inertia of moving part, and τ is the restoring couple per unit angular displacement.

Thus, by increasing I and decreasing the restoring forces, T, the time for one complete swing, is increased.

The second factor, damping, is reduced in a way which depends on the instrument (needle or coil)

The electromagnetic damping only may be reduced (p. 513). The air damping is usually small.

(3) A third condition is, that when used to measure a quantity of electricity, the whole of the transient current shall pass before the needle or coil moves from the zero position. Should there arise a case in which the quantity of electricity to be measured takes a longer time to traverse the instrument, due, for example, to inductance in the circuit, the time of swing of the needle must be increased, by loading it, so that this third condition is fulfilled.

As indicated above, the galvanometer may be of the moving-needle or moving-coil type. We shall develop a relation between the throw or angular deflection in each type, and the quantity of electricity which passes.

Moving-needle Type

This type of ballistic galvanometer consists of a needle suspended

by a fine quartz or unspun silk fibre, at the centre of a coil, through which the quantity of electricity, Q, passes.

The control in this case is either the earth's field or a control magnet. The needle in its zero position is arranged at right angles to the axis of the coils, so that when a current passes a field is set up at right angles to the control field.

Let G be the galvanometer constant, i.e. the field due to the coils for unit current circulating through them,
H the value of the control field strength,
I the moment of inertia of the magnet about the axis of suspension,
M the magnetic moment of the magnet.

Suppose a current of strength, i, to pass through the coils for a very small interval of time. Since the third condition above holds, the needle will be at right angles to a field of strength Gi, and will experience a turning moment, GiM.

This couple will produce an angular acceleration, $\ddot{\theta}$, in the needle. Hence (p. 37)

$$I\ddot{\theta} = GiM,$$

$$I\dot{\theta} = GM\int idt = GMQ. \qquad \ldots(8)$$

At the end of the swing, the needle having turned through an angle θ_0, the kinetic energy of the moving needle, $\frac{1}{2}I\dot{\theta}^2$, has been reduced to zero in doing work against the magnetic force, Hm (m being the pole strength of the needle) at each pole.

The work done is
$$2Hm\left(\frac{l}{2} - \frac{l}{2}\cos\theta_0\right)$$
$$= Hml\,(1 - \cos\theta_0)$$
$$= 2MH\sin^2\frac{\theta_0}{2},$$

where l is the distance between the poles.

Thus
$$2MH\sin^2\frac{\theta_0}{2} = \tfrac{1}{2}I\dot{\theta}^2,$$

$$I\dot{\theta}^2 = 4MH\sin^2\frac{\theta_0}{2}. \qquad \ldots(9)$$

Hence, squaring (8) and dividing by (9),

$$I = \frac{Q^2M^2G^2}{4HM\sin^2\dfrac{\theta_0}{2}}. \qquad \ldots(10)$$

We have also for the period, T, of the suspended needle in the control field, H

$$T = 2\pi\sqrt{\frac{I}{MH}},$$

$$I = \frac{T^2MH}{4\pi^2} \qquad \ldots(11)$$

Equating (10) and(11):

$$\frac{Q^2M^2G^2}{4HM\sin^2\frac{\theta_0}{2}} = \frac{T^2MH}{4\pi^2},$$

Hence

$$Q = \frac{T}{\pi}\cdot\frac{H}{G}\cdot\sin\frac{\theta_0}{2}. \qquad \ldots(12)$$

In developing this result we have assumed that the original kinetic energy of the needle is wholly used in moving the magnet against the field, H, through an angle, θ_0. If, however, there are any frictional forces, i.e. the needle is damped, some energy will be required to overcome this force, and the result will be that the observed angle of swing, say, α, is not truly of the magnitude given in the undamped case. The true value for Q is therefore dependent not on the observed value, α, of the angle, but on some slightly bigger angle, θ_0.

It will be shown that $\theta_0 = \alpha\left(1 + \frac{\lambda}{2}\right)$, where λ is the logarithmic decrement of the suspended system,

$$Q = \frac{T}{\pi}\frac{H}{G}\sin\left[\frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right)\right]. \qquad \ldots(12a)$$

Moving-coil Ballistic Galvanometer

In the moving-coil type of ballistic galvanometer, the electromagnetic damping is reduced by using a coil wound on a bamboo frame; T is increased by using a fine phosphor-bronze strip suspension.

To obtain the connexion between Q, the quantity of electricity discharged through the galvanometer, and θ_0, the first throw, we may first of all assume that there is no damping.

Let G be the galvanometer constant, i.e. the couple acting on the coil when unit current passes through it.

τ the restoring couple due to the suspension, for unit angular displacement.

I the moment of inertia of the suspended system about the axis of suspension.

As for the moving magnet galvanometer (p. 515), let i be the value of the current at a time, t, then

$$Gi = I\ddot{\theta}. \qquad \ldots(13)$$

Integrating, $$G \int i dt = GQ = I\dot{\theta}. \qquad \ldots(14)$$

If $\dot{\theta}$ is the original angular velocity given to the coil by the impulse due to the discharge of a quantity of electricity, Q, the initial kinetic energy is $\frac{1}{2}I\dot{\theta}^2$.

This energy is used in twisting the suspension. At any angular displacement, θ, the restoring couple is $\tau\theta$. To twist through a further angle, $d\theta$, the work done is $\tau\theta \,.\, d\theta$; i.e. the total work done in deflecting the coil is

$$\int_0^{\theta_0} \tau\theta \,.\, d\theta = \frac{\tau\theta_0^{\,2}}{2}. \qquad \ldots(15)$$

Thus we have $$\frac{\tau\theta_0^{\,2}}{2} = \frac{I\dot{\theta}^2}{2}.$$

$$I\dot{\theta}^2 = \tau\theta_0^{\,2} \qquad \ldots(16)$$

Squaring equation (14) and dividing by (16)

$$I = \frac{G^2Q^2}{\tau\theta_0^{\,2}}. \qquad \ldots(17)$$

Again the time of swing of the coil is given by

$$T = 2\pi\sqrt{\frac{I}{\tau}},$$

or $$I = \frac{T^2\tau}{4\pi^2}.$$

Substituting this value in (17)

$$\frac{T^2\tau}{4\pi^2} = \frac{G^2Q^2}{\tau\theta_0^{\,2}}$$

or $$Q = \frac{T}{\pi}\cdot\frac{\tau}{G}\cdot\frac{\theta_0}{2}, \qquad \ldots(18)$$

or if α is the observed first swing, and λ is the logarithmic decrement,

$$Q = \frac{T}{\pi}\cdot\frac{\tau}{G}\cdot\frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right). \qquad \ldots(18a)$$

Suppose now we send a steady current of known magnitude, i, through the galvanometer and observe the steady deflection, φ,

$$Gi = \tau\varphi$$

or $$\frac{\tau}{G} = \frac{i}{\varphi}.$$

Hence $$Q = \frac{T}{\pi} \cdot \frac{i}{\varphi} \cdot \frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right). \qquad \ldots(19)$$

It should be noted that the preceding paragraph gives a very ready method of finding the ballistic reduction factor for the galvanometer. This reduction factor, k, is defined by the relation $Q = k\theta_0$.

Thus $$k = \frac{T \times \text{steady current}}{2\pi \times \text{steady deflection}}.$$

Subsequent deflections may be converted to quantity by multiplying these deflections by k.

A similar factor may be deduced for the moving-magnet ballistic galvanometer, using equation (23a).

The introduction of the factor $\left(1 + \frac{\lambda}{2}\right)$ in the ballistic galvanometer formulae follows from the discussion on damped oscillations (p. 150).

It was shown that the expression for the deflection at any time, t, is

$$\theta = Ce^{-\frac{kt}{2}} \sin \sqrt{n^2 - \frac{k^2}{4}}\, t.$$

In the case when there is no damping, $k = 0$, and the amplitude is C or θ_0 in the notation used in the case of the galvanometer.

When damping occurs the first swing, α, is observed after a quarter of a period, $t = \frac{T}{4}$.

Thus $$\alpha = Ce^{-\frac{kT}{8}} = \theta_0 e^{-\frac{\lambda}{2}},$$

or $$\theta_0 = \alpha e^{\frac{\lambda}{2}} = \alpha\left(1 + \frac{\lambda}{2}\right),$$

when the damping is small. This approximation is justified in the case of galvanometers.

λ may be obtained by one of the methods of pp. 152 to 154, and the correcting factor, $\left(1 + \frac{\lambda}{2}\right)$, calculated. The galvanometer must be in the circuit of the experiment when λ is obtained, so that the damping conditions are the same.

A shorter method of arriving at the correction, although not so accurate, is to observe the first swing and the next swing on the same side of the zero. Let these be denoted by α_1 and α_3, which are given by

$$\alpha_1 = \theta_0 e^{-\frac{kT}{8}}, \qquad \ldots(20)$$

$$\alpha_3 = \theta_0 e^{-\frac{5kT}{8}}.$$

Thus $$\frac{\alpha_1}{\alpha_3} = e^{\frac{kT}{2}}$$

and therefore, substituting in (20), we have

$$\theta_0 = \alpha_1 \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}}. \qquad \ldots(21)$$

The correction for damping obtained in this way will be sufficiently correct for many experiments.

Of course α_1, α_2, and α_3 are not measured as angles. The corresponding scale deflections, δ_1, δ_2, and δ_3, are measured.

Now $$\frac{\delta_1}{\delta_3} = \frac{\tan 2\alpha_1}{\tan 2\alpha_3} \text{ for small deflections,}$$

and since under these circumstances the value of $\tan 2\alpha$ is approximately the same as 2α,

$$\frac{\delta_1}{\delta_3} = \frac{\alpha_1}{\alpha_3}.$$

So that (19) for the moving-coil galvanometer becomes

$$Q = \frac{T}{\pi} \cdot \frac{i}{\varphi} \cdot \frac{\alpha_1}{2} \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}}, \qquad \ldots(22)$$

and (12a)

$$Q = \frac{T}{\pi} \frac{H}{G} \cdot \sin \left\{ \frac{\alpha_1}{2} \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}} \right\}. \qquad \ldots(23)$$

Equation (23) may be modified to conform with (22). For if a steady current, i, produces a deflection, φ,

$$iG = H \tan \varphi,$$

i.e. $$Q = \frac{T}{\pi} \frac{i}{\tan \varphi} \cdot \sin \left\{ \frac{\alpha_1}{2} \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}} \right\}. \qquad \ldots(23a)$$

The Quantity Sensitivity of a Ballistic Galvanometer

It will be supposed that the galvanometer is of the moving-coil type to which the formula (18a), p. 517, applies.

The quantity sensitivity means the number of millimetres scale deflection produced by a charge of one microcoulomb when the distance between mirror and scale is one metre.

It is the object of this experiment to determine the ratio $\frac{\alpha}{Q}$ expressed in units in accordance with the definition. The value of Q can be obtained by a flux method, the theory of which has been given on p. 485.

A coil of N_2 turns and of cross-section $A = \pi r_2{}^2$ is connected to the terminals of the galvanometer. This is placed well within a long solenoid containing m_1 turns per cm. This solenoid is connected through a reversing key to a supply of electric current, the magnitude of which is adjusted to give a full-scale deflection in the galv on reversal. If the current has the value i E.M. units the

in the secondary coil due to reversal is $8\pi^2 m_1 N_2 r_2^2 i$. If the resistance of the circuit is R E.M. units, the quantity expressed in these units is

$$\frac{8\pi^2 m_1 N_2 r_2^2 i}{R}.$$

In practice, i will be expressed in amperes and R in ohms, and if it be remembered that i amperes $= \frac{i}{10}$ E.M.U. and R ohms $= 10^9$R E.M.U., it follows that with i in amperes and R in ohms the quantity is

$$Q = \frac{8\pi^2 m_1 N_2 r_2^2 i}{10^9 R} \text{ coulombs.}$$

To Reduce the Excessive Damping in a Moving-coil Ballistic Galvanometer used in a Low-resistance Circuit

It was shown in the discussion of damping that the moving-coil galvanometer has a large amount of electromagnetic damping when in a closed circuit of low resistance. This would make the use of a moving-coil ballistic galvanometer inadmissible in many experiments unless some means were taken to reduce the damping.

The quantity of electricity passing through the instrument is due to some current change. If, therefore, the galvanometer is in circuit

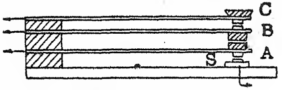


FIG. 331

whilst such change takes place, and is then allowed to swing freely when it has received the impulse due to that cause, the damping is almost eliminated.

By use of a compound key, such as that shown in fig. 331, this may be brought about. The three brass strips mounted on an ebonite block (shaded) are connected to separate terminals, as is the stop, S. The battery circuit is connected to A and S, and the galvanometer to C and B. When the key is depressed, C and B, A and S are connected, but B and A remain insulated by the ebonite stops shown (shaded). When the key is released A and S are broken, the impulse is given to the galvanometer, and then C and B are disconnected by the upward movement of the key. If the time interval between the break of the battery and the galvanometer circuits is small compared with the period, T, of the coil of the galvanometer, the latter will swing an amount which is almost independent of the electromagnetic damping of the low-resistance circuit.

If such a key is used for a moving-coil instrument the damping correction used should be obtained from observations of the coil when swinging freely in open circuit.

The Grassot Fluxmeter

The Grassot fluxmeter is an instrument which performs the same function as a ballistic galvanometer, as, for example, in experiments on pp. 369, 459, 597, 606, etc. However, the instrument is specially designed for measuring magnetic field strengths directly.

It is a suspended coil instrument which depends entirely on the electromagnetic damping for control. The coil, D, is supported by a single cocoon silk fibre, which has a negligible torsional control, from a flat spiral, E (fig. 332), to eliminate the effect of shocks. The current

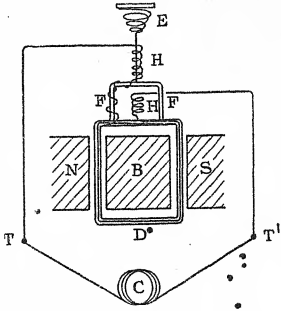


FIG. 332

enters and leaves the coil by the silver strip coils, H, as shown in the figure. The usual iron core, B, and permanent magnet pole pieces, N, S, supply a constant magnetic field in which the coil swings.

The light frame, FF, carries a pointer and a concave mirror, not shown in the figure. The points, T, T′, correspond to terminals on the case of the instrument.

In general use a search coil, C, of known mean area and number of turns is connected to T, T′.

The search coil is placed into the magnetic field to be measured, and the induced electromotive force causes a current to flow in the closed circuit and so produces a deflection which may be measured either by a lamp and scale arrangement, using the concave mirror, or, if sufficiently large, by direct reading of the pointer over a scale.

✓The instrument produces a deflection which is proportional to the quantity of electricity which passes through it.

Fig. 333 (at p. 1) shows the general appearance of the instrument in use to measure the magnetic distribution along a magnet.

The method of calibration of the graduated scale and general possibilities of the instrument will be apparent from a consideration of the theory of the instrument.

Let R = the resistance of the circuit, i.e. of the search coil, C, and leads, and the suspended coil, D,

L = the self-induction of the whole circuit,

I = the moment of inertia of the coil, D, about the axis of suspension,

$$\left.\begin{array}{l} E = \text{the E.M.F. induced in C} \\ i = \text{the current in the circuit} \\ \omega = \text{the angular velocity of the coil} \end{array}\right\} \text{at any instant,}$$

K = induced E.M.F. set up in D for unit angular velocity,

G = the galvanometer constant, i.e. the couple exerted on the coil due to unit current passing through it,

A = the couple due to air resistance for unit angular velocity.

Expressing Ohm's law for the circuit, we have

$$i\mathrm{R} = \mathrm{E} - \mathrm{L}\frac{di}{dt} - \mathrm{K}\omega, \qquad \ldots(24)$$

i.e.

$$i = \frac{1}{\mathrm{R}}\left(\mathrm{E} - \mathrm{L}\frac{di}{dt} - \mathrm{K}\omega\right).$$

The equation of motion of the coil is

$$\mathrm{I}\frac{d^2\theta}{dt^2} = \mathrm{G}i - \mathrm{A}\omega. \qquad \ldots(25)$$

Substituting the above value for i and remembering that $\omega = \frac{d\theta}{dt}$,

$$\mathrm{I}\frac{d\omega}{dt} = \frac{\mathrm{GE}}{\mathrm{R}} - \frac{\mathrm{LG}}{\mathrm{R}}\frac{di}{dt} - \left(\frac{\mathrm{GK}}{\mathrm{R}} + \mathrm{A}\right)\omega.$$

Now L, G, R, K, and A are constants; further, if the coil starts from its rest position when no current passes, and becomes deflected through an angle, α, by the quantity discharged through it, the coil is at rest at the end of the swing when no current again passes through it.

Therefore, integrating the last equation over the whole swing with respect to t, we have

$$\mathrm{I}\Big[\omega\Big]_0^0 = \frac{\mathrm{G}}{\mathrm{R}}\int \mathrm{E}dt - \frac{\mathrm{LG}}{\mathrm{R}}\Big[i\Big]_0^0 - \left(\frac{\mathrm{GK}}{\mathrm{R}} + \mathrm{A}\right)\Big[\theta\Big]_0^\alpha,$$

i.e.

$$\int \mathrm{E}dt = \frac{\mathrm{R}}{\mathrm{G}}\left(\frac{\mathrm{GK}}{\mathrm{R}} + \mathrm{A}\right)\alpha,$$

or

$$\int \mathrm{E}dt = \left(\frac{\mathrm{AR}}{\mathrm{G}} + \mathrm{K}\right)\alpha. \qquad \ldots(26)$$

Thus it appears that the value of the deflection, α, is determined by $\int E dt$, since the other terms are constant. An important case occurs when the coil is placed in a magnetic field, thus setting up an induced E.M.F. It follows, as is shown below, that the total deflection is independent of the speed of motion of the coil.

Now in general the value of A is small and the term $\frac{AR}{G}$ may be neglected in comparison with K. Hence we have

$$\int E dt = K\alpha,$$

approximately.

With this approximation the deflection α is independent of the value of R, i.e. the search coil may be replaced by another, provided that R does not become very large, in which case the approximation is not justifiable.

It is worthy of note that the deflection is, as for the ballistic galvanometer, proportional to the quantity of electricity, $\int \frac{E}{R} \cdot dt$, which passes through the coils.

The E.M.F., E, arises from a change in the magnetic flux passing through the search coil. Thus, since we are not concerned with the direction of the E.M.F. we can write

$$E = \frac{dN}{dt},$$

from which $\int E dt = N_1 - N_0$, where N_1 denotes the number of lines of induction passing through the coil at the end of the movement and N_0 the number at the beginning.

If N_0 is zero, $\quad N = K\alpha.$

In a fluxmeter the scale, which is clearly a linear one, is calibrated to give the flux in maxwells corresponding to the deflection of the pointer.

For many of the experiments in which the instrument is used the graduated scale is too coarse. Use is then made of the mirror, using a lamp and scale arrangement as in the ordinary galvanometer. A full scale deflection is produced in the instrument by a definite flux change in a search coil; the position of the reflected spot of light is noted, as is the movement of the pointer on the graduated scale. By repetition, the corresponding values are obtained. From such observations the value of the flux change, corresponding to 1 mm. scale deflection (at, say, 1 m.), is deduced from the graduated scale deflection.

MEASUREMENT OF VARYING CURRENT

The preceding account has included the method of measuring (1) steady direct current, (2) quantity of electricity.

We now consider briefly methods available for the measurement of varying currents. Such currents may be subdivided into (*a*) currents of short duration at regular or irregular intervals, (*b*) alternating currents.

The Einthoven Galvanometer

The principle of this instrument may be understood from fig. 334. A 'string', CC, of fine platinum or tungsten wire is supported vertically between the poles of an electromagnet. If a current be sent down the string, the latter will be deflected in a direction parallel to the face of the pole pieces. In the diagram, the direction of movement is shown by the arrow, *a*, for a field in the direction, NS.

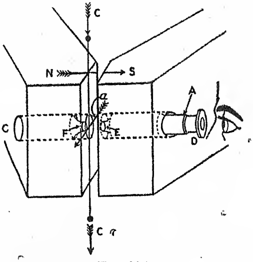


FIG. 334

To observe the movement, the pole pieces of the magnet are drilled, as shown by the broken lines, in a direction parallel to that of the magnetic field. Light from a bright point source is concentrated by a condenser, CF, placed in one hole, and the movement of the illuminated string is magnified by a telescope, DE, placed in the other.

The string is usually attached to the ends of two small springs which keep it stretched. The tension may be altered to any desired amount by a micrometer adjustment at one end. The string and tension-varying device are supported on a frame which may be removed bodily from the gap between the pole pieces.

The deflection produced for a given current depends upon the tension on the string and the strength of the magnetic field. The former may be varied as indicated above, whilst the latter may be adjusted by regulation of the current flowing through the coils of the electromagnet. The sensitivity of the instrument may be therefore

varied over a wide range quite simply. At a fixed sensitivity the deflection is proportional to the current.

The usual method of detecting deflections is to use a photographic arrangement. The beam of light emerging from ED, if allowed to fall on a screen, forms a shadow image of the string. This is reflected on to a cylindrical lens which forms an image of the central portion of the line, i.e. if the string is still and a photographic plate is moved vertically at the focus of the lens, a straight white line is produced on the plate when developed. If a current passes, the string moves, and a shift of the shadow image results. The point focus of the cylindrical lens is thereby deflected. This results in a lateral displacement of the line image on the photographic plate. The magnitude of the displacement gives a measure of the current strength. The natural period of the string is small and it rapidly returns to the rest position when the current ceases to flow (not more than a few hundredths of a second are required). Thus, if a succession of small currents pass in the circuit, the instrument detects them, even when but a few hundredths of a second elapse between successive currents, whereas an ordinary galvanometer would not distinguish the interval between them.

To measure the interval of time between successive impulses, or the duration of one of them, a time scale is imprinted on the record of the current by means of a time marker, as shown in fig. 335 (at p. 1). This consists of a device for intercepting the light at regular intervals and thereby making transverse white lines across the photographic plate on which the current is recorded.

A small motor is made, as shown in the figure with a soft iron armature of, say, ten teeth. Intermittent current is supplied to the electromagnets by connecting them in series with the circuit of an electrically maintained tuning-fork, the impulses given to the motor

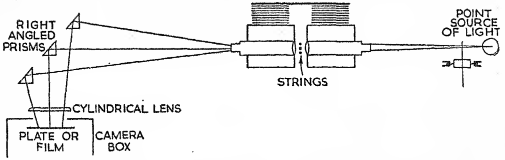


FIG. 336

are therefore regular. Suppose the fork vibrates fifty times per second, the synchronous motor will rotate five times per second. This drives a circular disk provided with projecting arms, as shown in the figure. These arms are usually allowed to move across the beam of light illuminating the apparatus. The figure shows five such spokes, the fifth being broader than the others. A spoke intercepts the light

twenty-five times per second and makes a time scale on the photographic plate of $\frac{1}{25}$ second.

By having twenty spokes to the wheel $\frac{1}{100}$ second graduations may be obtained. The width of a division may be varied by varying the speed of movement of the plate. For a continuous record a cinematograph film may be used.

A number of strings may be used, and the image focused simultaneously on the same film by placing a 45° right-angled prism in the path of each image. The whole is therefore concentrated on to the slit in the camera box. The arrangement of the parts is seen in fig. 336, which shows a plan of the apparatus.

The Duddell Oscillograph

Fig. 337 shows the essential features of this form of instrument. A thin phosphor-bronze strip, *ss*, is supported over a small ivory bobbin, P, and fastened at the lower ends. The tension on the strip may be adjusted by regulation of the tension on the spring suspension of P.

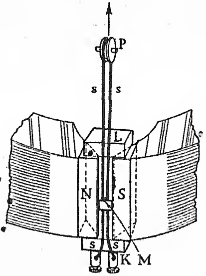


FIG. 337

If a current is passed through the loop, *ss*, the two strips will suffer a deflection in opposite directions, and consequently rotate a mirror, M, which is attached to both. If the current is reversed, the direction of rotation is also reversed.

Thus, using alternating current, the mirror will rotate backwards and forwards, provided that the natural period of the loop is small compared with that of the alternating supply.

The image of a source of light reflected by the mirror on to a scale or screen by M will therefore be drawn out into a line.

If the image is focused on to a photographic plate in a camera, and the plate is moved at constant speed in a direction normal to the beam

of light and the direction of vibration of the image, the trace on the plate will be approximately a sine curve. The form of the current-time curve may then be investigated from the record.

To obtain the wave-form visually, the photographic arrangement can be dispensed with, and a mirror made to rotate and reflect the beam on to a screen.

The amplitude of the curve gives an indication of the maximum current strength which is proportional to it.

This instrument can also be employed in many cases as an alternative to the Einthoven galvanometer.

CHAPTER XX

RESISTANCE MEASUREMENTS

The Wheatstone Bridge

The student will be familiar with the Wheatstone net as shown in fig. 338. When the bridge is balanced the relation

$$\frac{P}{Q} = \frac{R}{S} \qquad \ldots(1)$$

holds. For maximum sensitivity, using a fixed galvanometer and battery and measuring a resistance, R, it has been shown* that P, Q, and S should be chosen so that

$$Q^2 = BG, \quad P^2 = RG\frac{R+B}{R+G}, \quad S^2 = RB\frac{R+G}{R+B},$$

where B is the resistance of the battery. If a choice of galvanometer

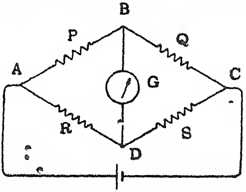


Fig. 338

is possible, it should have a resistance comparable with the other arms. When $P = Q = R = S$ maximum sensitivity is obtained when $G = P$.

Measurement of the Resistance of a Galvanometer

This may be done in several ways, of which representative methods are given below.

(1) *Kelvin's Method*

In this method the galvanometer acts as its own detector of balance in a Wheatstone net. The galvanometer is placed in the arm, DC (fig. 338), and the galvanometer of that figure is replaced by a single-way key, so that B and D may be connected together when the key is depressed.

* See Gray, *Absolute Measurements*, Vol. I, p. 331 (1888).

When the battery circuit is completed, a steady current flows through the galvanometer. P, Q, and R are adjusted until on joining B to D through the key, no change is produced in that deflection, then $\frac{P}{Q} = \frac{R}{G}$, where G is the resistance of the galvanometer.

The usual difficulty with a sensitive galvanometer is that the steady current is too large. The galvanometer may not be shunted in this experiment, but the potential applied may be reduced, e.g. the cell may be connected through a high resistance, and leads from the end of a small fraction of the resistance may be taken to A and C instead of the battery directly applied. This, however, decreases the sensitivity of the method.

For a moving-magnet instrument it is better to apply the cell directly, and to reduce the steady deflection to zero by adjustment of the control magnet, or, if that is not sufficiently strong, by the adjustment of an external bar magnet. The sensitivity is thereby retained. This method has historic interest, but it is not in general to be recommended.

(2) For a moving-coil instrument where no adjustment is available, the simplest course is to clamp the coil of the instrument and find its resistance using another galvanometer as detector in the usual way.

(3) The method given on p. 499 may be used for a simple and approximate evaluation. In fig. 323 R is made equal to zero and R′ adjusted until a deflection of 30 or 40 cm. is obtained, the shunt, S, being adjusted to a suitable value. R is then increased until the deflection is reduced to half its initial value. G is then approximately equal to R.

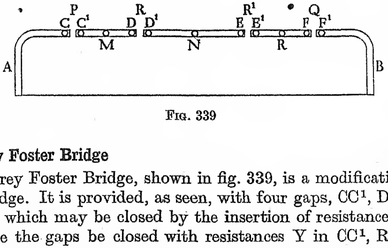


FIG. 339

The Carey Foster Bridge

The Carey Foster Bridge, shown in fig. 339, is a modification of the metre bridge. It is provided, as seen, with four gaps, CC^1, DD^1, EE^1, and FF^1, which may be closed by the insertion of resistances.

Suppose the gaps be closed with resistances Y in CC^1, R in DD^1, R^1 in EE^1, Z in FF^1, as shown in fig. 340. The battery, E, and galvanometer, G, are connected to the points AC and BD respectively, which correspond to the points in the theoretical net diagram, fig. 338. If

the point D is chosen such that no current passes through the galvanometer, we have

$$\frac{R}{R^1} = \frac{Y + r_1 + x_1\rho}{Z + r_2 + (100 - x_1)\rho}, \qquad \ldots(2)$$

where

ρ is the resistance per cm. of bridge wire,
x_1 the length, SD,
DT $= (100 - x_1)$ if ST is one metre,
r_1 is the value of the resistance at the soldered junction, S,
r_2 the resistance at T.

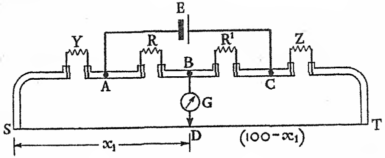


FIG. 340

If the simple metre bridge were used to compare R and R^1, i.e. $Y = Z = 0$, we should have a balance at, say, l_1 cm., such that, neglecting r_1 and r_2 for the moment,

$$\frac{R}{R^1} = \frac{l_1\rho}{(100 - l_1)\rho}.$$

Now suppose $Y = y\rho$ and $Z = z\rho$, equation (2) becomes

$$\frac{R}{R^1} = \frac{(y + x_1)\rho}{\{z + (100 - x_1)\}\rho},$$

if r_1 and r_2 be neglected.

It is obvious from a comparison of the two results that the Carey Foster bridge functions as though the length of the wire were increased, i.e. the same error in obtaining a balance point gives rise to a less percentage error in the Carey Foster bridge determination than in the simple metre bridge.

Comparison of the British Association Ohm and the Legal Ohm

The comparison of two resistances, very nearly equal, serves to show a common use for this bridge. In the following method it will be seen that the end resistances are eliminated.

The resistances R and R′ are made approximately equal to the values of Y and Z. Connecting as in fig. 340 with, say, Z, a B.A. ohm, and Y, a legal ohm, whilst R and R′ are each 1 legal ohm, we should obtain a balance at a point, D, x_1 cm. from S, so that

$$\frac{R}{R'} = \frac{Y + r_1 + x_1\rho}{Z + r_2 + (100 - x_1)\,\rho}, \qquad \ldots(3)$$

If now Y and Z are interchanged, a balance can be obtained at a distance x_2 from S, from which we have

$$\frac{R}{R'} = \frac{Z + r_1 + x_2\rho}{Y + r_2 + (100 - x_2)\,\rho}. \qquad \ldots(4)$$

Equation (3) may be rewritten

$$\frac{R}{R + R'} = \frac{Y + r_1 + x_1\rho}{Y + Z + r_1 + r_2 + 100\rho},$$

and (4) similarly becomes

$$\frac{R}{R + R'} = \frac{Z + r_1 + x_2\rho}{Y + Z + r_1 + r_2 + 100\rho}.$$

Equating numerators of these equations, we have

$$Y + r_1 + x_1\rho = Z + r_1 + x_2\rho,$$

$$Y - Z = (x_2 - x_1)\,\rho, \qquad \ldots(5)$$

i.e. the difference between Y and Z is equal to the resistance of the bridge wire between the two points of balance. It is independent of the value of r_1 and r_2, and of the total length of the bridge wire.

To obtain the value of the B.A. ohm in terms of the legal ohm, we have, Y being the legal ohm,

$$\text{B.A. ohm} = \{1 - (x_2 - x_1)\rho\}\ \text{legal ohm}.$$

The value of the resistance of the bridge wire between the points of balance may be obtained by calibrating the bridge wire (see p. 533), or, if the wire is uniform, the following simple method will serve.

Resistance of Unit Length of Bridge Wire (ρ)

(1) The bridge connexions of the main experiment above remain as before. R and R′ are approximately equal, and may very well be the same as above. Y and Z are replaced respectively by a fraction of an ohm, say, Y′, and a stout strip of copper of negligible resistance, say $Z' = 0$.

Following the same procedure as before, a balance is obtained at a distance x_1' from S, when Y′ and Z′ are in the positions of Y and Z in fig. 340. When Y′ and Z′ are interchanged the balance will move to another point, x_2', from S′.

Then by equation (5)

$$Y' - Z' = (x_2' - x_1')\rho,$$

or since $Z' = 0$, and Y is known,

$$\rho = \frac{Y'}{x_2' - x_1'}.$$

Hence, using this value of ρ, the difference between Y and Z in the first case may be evaluated.

This method may obviously be applied to any similar case, and a comparison between two nearly equal resistances obtained.

Experimental Details

The resistances and Y, Z, R, and R′ are connected by means of stout copper strips. These will have practically zero resistance, and the small difference to be measured will be inappreciably affected by them. It is also essential, of course, that all connexions be very tightly screwed.

In performing the second part of the experiment, i.e. to find ρ, two or three sets of observations should be made.

For example, Y′ should be made 0·1, 0·2, 0·3, 0·4 ohm successively, and ρ calculated in each case, from which a mean value is obtained. It may be found that, 0·4 ohm acting as Y′, no balance is obtainable. In that case the total resistance of the bridge wire is less than 0·4 ohm.

The following set of observations shows the order of the result obtained in such an experiment.

$$\text{Using } Z = 1 \text{ B.A. ohm}, Y = 1 \text{ legal ohm},$$
$$x_2 = 51{\cdot}2 \text{ cm.}, x_1 = 48{\cdot}2 \text{ cm.}$$

EXPERIMENTS TO DETERMINE ρ

Y′ in ohms	0·1	0·2	0·3	0·4
x' in cm.	36·35	23·76	11·2	no balance
x_2' in cm.	63·75	76·34	88·9	
ρ ohms per cm. $= \dfrac{Y'}{x_2' - x_1'}$	0·00365	0·00380	0·00386	

Mean value of $\rho = 0{\cdot}00377$.
Hence since

$$Y - Z = (x_2 - x_1)\,\rho,$$

Legal ohm — B.A. ohm = (51·2 — 48·2) (0·00377),

$$\text{B.A. ohm} = 1 - 3 \times 0{\cdot}00377 = 0{\cdot}989 \text{ legal ohm.}$$

(2) Another equally simple method for finding the resistance per cm. of bridge wire may be used. Suppose Y, Z, R, and R′ (fig. 340) are all 1-ohm coils of the same kind, e.g. B.A. ohms. R and R′ are set up as before. To introduce a small difference between Y and Z, one of them is shunted with, say, 10 ohms, i.e. the net result is $\frac{10}{11}$ ohm. If now the process described above is carried out, interchanging the

1 ohm and effective $\frac{10}{11}$ ohm, we have two balance positions, l_1 and l_2 cm., say, and

$$Y - Z = (l_1 - l_2)\,\rho,$$

i.e.

$$\tfrac{1}{11} = (l_1 - l_2)\,\rho.$$

Hence ρ is determined.

To Construct a Resistance Coil of Known Magnitude

To construct, for example, a 1-ohm coil, a wire with small temperature coefficient between 10° and 20° C. is selected, e.g. manganin, and the resistance per cm. of the specimen available is obtained by finding the resistance of about 100 cm.

The length of wire required to have a resistance not less than 1·1 ohm is calculated and cut off. The insulation covering is removed from the ends and, using a non-corrosive flux (e.g. resin), the two ends are soldered to two stout copper wires which are soldered to flat copper forks, A and B (fig. 341). A and B and the rods are then fastened to

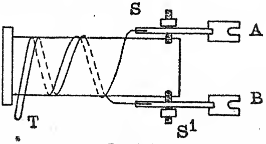


FIG. 341

opposite sides of a wooden bobbin by means of terminals as shown, and the wire wrapped, as in the diagram, in a non-inductive manner round the bobbin. The middle of the wire, T, is freed from the silk cover. The resistance between A and B is obtained by the Carey Foster method. The middle, T, is twisted with pliers, cutting out the end loop until, compared with a standard 1-ohm coil by the method of p. 530, the balance is near the centre of the bridge. T is then soldered in position, and the value of this copy of the standard ohm when completed is compared with the true standard as already described.

CALIBRATION OF A BRIDGE WIRE

The simple metre bridge, briefly referred to on p. 530, the Carey Foster bridge, the potentiometer and similar instruments, depend upon measurement to a point of balance on a stretched wire. For simplicity it is often assumed that the wire is of uniform cross-section, and that its resistance per cm. is consequently constant throughout the length. Further, the soldered end and the thick copper connecting strip are assumed to be of zero resistance.

In a practical measurement it is better to make no such assumptions, but to determine the variation due to these causes by a preliminary calibration of the bridge.

Determination of the End Correction of the Bridge Wire

For this determination the outer gaps of the bridge (CC^1, FF^1 in fig. 339) are closed by short, clean, thick copper strips of negligible resistance, and in the inner gaps are two unequal resistances, say a 10- and a 1-ohm coil (R and R^1).

Let a balance be obtained at x_1 cm. from A. The 10-ohm and 1-ohm coils are interchanged, and a second balance is obtained at x_2 cm. Now, if the end resistance at A is equivalent to l_1 cm. of the bridge wire, and if the end resistance at B is equivalent to l_2 cm. of bridge wire, we have

$$\frac{R}{R^1} = \frac{x_1 + l_1}{100 - x_1 + l_2},$$

and

$$\frac{R^1}{R} = \frac{x_2 + l_1}{100 - x_2 + l_2},$$

from which l_1 and l_2 may be calculated.

Calibration of the Wire

This may be done in many ways, of which we will consider two.

The object of these experiments is to find, at different points along the wire, lengths having the same resistance, usually equal to that of a gauge employed. Knowing the total resistance, S, of the wire, the mean value of the resistance of such a length is readily calculated, and hence the correction to be applied at each segment taken, to reduce to the mean value, may be determined. Alternatively, having obtained the lengths which have the fixed resistance, the method of analysis used in the calibration of a tube may be applied (p. 27). Suppose we wish to test every 5 cm. of the wire, and this will be quite sufficient for most cases, the following methods may be used.

(1) Carey Foster's Method

The object of the calibration of the wire of a metre bridge is to enable the readings of the bridge to be reduced to what they would have been had the wire been uniform throughout its length.

In Carey Foster's method a standard resistance is taken and lengths of the bridge wire are measured which have the resistance of this standard. It is convenient to take a strip of wire identical with the bridge wire as the standard. For this purpose approximately 7 cm. should be cut and 1 cm. at each end should be soldered to thick pieces of copper, leaving as nearly as possible 5 cm. between them (fig. 342).

Except for the variations which are to be determined this will make it possible to examine the wire in 5-cm. strips from one end to the other.

The standard is first placed across the gap at Y and Z is closed by a thick strip of copper of negligible resistance.

In order to cause the balance points to move along AB, a second wire CD, is introduced, as in fig. 343, and the galvanometer is connected

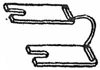


FIG. 342

to a point, H, of it. The ends of CD are connected by thick copper strips to the bridge. This wire replaces the part, UWV, of the bridge, the point, H, now taking the place of W, and the strips, CH and HD, replace the resistances which usually fill the gaps at U and V. By sliding the contact H along CD, the effect is to vary the resistances of the gap and thus to vary the point of balance at F. If the wire, CD,

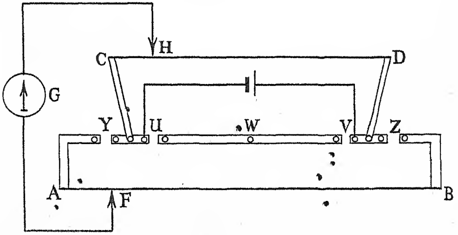


FIG. 343

is similar to that of AB, a displacement of H will cause an equal displacement of F. In practice there is not exact similarity, but the two displacements will not be very different, and if CD is of the length of AB it will be possible to cover the whole wire, AB, as a result of successive displacements of H.

Begin by placing the standard at Y and zero resistance at Z. Adjust F to lie as closely as possible to A and obtain a balance by adjusting H. With H fixed, interchange the standard and zero resistances in Y and Z and restore the balance by displacing F. If the length of the wire between the two positions of F be l_1, the resistance of this length of the bridge is $(Y - Z)$ by equation (5), p. 531. Thus, at this region of the bridge the length l_1 has the resistance of the standard.

In order to proceed to the next measurement let F remain at the last position of balance and bring the standard back to Y, placing the zero at Z. Restore the balance by displacing H. If the interchange at Y and Z be made a new position of F will be found about 5 cm. farther on. Suppose the new lengths of wire between the positions of F be l_2. In this region length l_2 has the resistance of the standard. The procedure should be repeated step by step until a last adjustment brings us within 5 cm. of B.

In order to smooth out the lack of uniformity of the bridge wire a table is constructed as illustrated below.

Region of wire examined	Length of wire of resistance of standard	Mean length equal to standard	Difference from mean	Corrections to reduce to equivalent uniform wire
cm.				
0– 5	l_1		$l-l_1=s_1$	s_1
5– 10	l_2		$l-l_2=s_2$	s_1+s_2
10– 15	l_3		$l-l_3=s_3$	$s_1+s_2+s_3$
—	—	$l=\frac{\Sigma l}{n}$	—	—
—	—		—	—
—	—		—	—
90– 95	l_{n-1}		$l-l_{n-1}=s_{n-1}$	$s_1+s_2+\ldots+s_{n-1}$
95–100	l_n		$l-l_n=s_{n-2}$	$s_1+s_2+\ldots+s_n$

The table is constructed in order to make it possible to reduce the results to those of an equivalent uniform wire. In order to see how this is brought about it is to be noted that if we construct a wire made of a succession of strips of length l, sufficient to extend from the beginning to the end of the scale, a uniform bridge wire has been constructed of the same length but with the irregularities smoothed out. With this wire the first balance would have been obtained at l instead of at l_1, the second would have been obtained at length $2l$ from the end instead of at $(l_1 + l_2)$, and so on. The inaccuracy of the bridge lies in these facts. The reading at l_1 is incorrect on account of the lack of uniformity, and to get the correction we add s_1 to l_1 in order to get l. In the case of the strip of length $(l_1 + l_2)$ we add $(s_1 + s_2)$ in order to get $2l$, and so on. This addition is to be taken in the algebraic sense, since some of the corrections may be negative.

A curve should be drawn with the numbers in the fifth column of the table as ordinates and the lengths, l_1, $(l_1 + l_2)$, etc., as abscissae. The graph obtained in this way can be used to give the corrections

to the various lengths, and it has the advantage that it provides an interpolation for points between those actually investigated.

In this experiment it is more than usually important to pay attention to the avoidance of contact resistances. This does not mean that all screws should be turned as in a vice; it means that no wires should be loosely held in screw terminals and no threads of insulating material should separate metal surfaces. If there is any suspicion of rust or corrosion at metal faces which are to come into contact, careful cleaning is necessary.

It should be borne in mind that the corrections to the bridge obtained in this way do not account for end corrections which have to be applied in addition.

The graph representing the correction curve as a result of the errors at the ends will be a straight line joining the ends of the ordinates representing the amounts to be added at the zero end and at the other end. Corrections at intermediate points are read from the graph, and in obtaining the total correction at any point the ordinates of the graph for lack of uniformity and of the graph for end corrections must be added algebraically.

(2) Direct Calibration by a Potentiometer Method

The wire, ST, of the bridge is connected, as shown in fig. 344, in series with an adjustable resistance, R_2, and a 2-volt accumulator

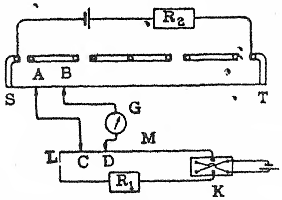


FIG. 344

which is in good condition and well charged. A similar accumulator is connected in series with a second adjustable resistance, R_1, and a length of, say, 20 cm. of stretched wire of the same material and approximate cross-section as the bridge wire. G is a high-resistance galvanometer.

C is soldered to the wire, M,* and with A and B in contact with ST,

* Alternatively, the potentiometer may be made up of two resistance boxes, R_1 and R_1', in place of the wire M and box R_1.

say, between the 0 and 6 cm. readings, and about 5 cm. apart, D is adjusted until no deflection is given in the galvanometer. D is then soldered to M.

It should be noted that convenient resistances for R_1 and R_2 are of the order 20 to 50 ohms. The adjustment of these resistances is made until the currents are about equal.

Leaving B in contact with ST, reverse the current in M by means of the commutator, K, and move A to the other side of B; adjust till again no deflection is produced. Repeat this process over the length of ST.

If the latter is uniform, then of course the length AB will be the same. Variations of the cross-section will be made apparent by the different values of AB.

Tabulate the length of AB as before, and deduce the correction curve *for the wire.*

This method may also be used for any stretched wire, e.g. potentiometer, etc.

DETERMINATIONS OF THE VALUE OF A HIGH OR LOW RESISTANCE

The ordinary Post Office box method of finding resistance is a very ready method, but for resistances of a million (10^6) ohms or more it is unsuitable. The ordinary Post Office box enables a magnification of 1000 to 10 only to be obtained, and in the adjustable arm 10,000 ohms is the maximum resistance available, so, for a resistance greater than 10^6 ohms a special method is required.

Corresponding limitations for very small resistances make it necessary to employ other methods in this case.

Some of these special methods are described below.

Low Resistance

(1) The Direct Deflection Method

To find the resistance, r, of a low resistance wire, AB, the following simple method gives a fair approximation. AB is joined in series with a known low resistance, R, an accumulator (2-volt), and an adjustable, fairly large resistance, S (fig. 345).

Having by means of S adjusted a small current, i, through the circuit, there will be a drop of potential, ir, across AB, and iR between the ends of R. If a high-resistance galvanometer is used, the current in the main circuit is almost unaltered, and the resulting current in the galvanometer is proportional to the potential ir or iR applied to it. If this causes a small deflection, d_1 cm., using the usual lamp and scale method, $ir \propto d_1$ approximately.

The same galvanometer will have deflection d_2 cm. when connected to the ends of R. Again, $iR \propto d_2$ approximately, i.e.

$$\frac{d_1}{d_2} = \frac{ir}{iR} = \frac{r}{R}.$$

Hence $$r = \frac{d_1}{d_2} \,.\, R. \qquad \ldots(6)$$

This method of observing deflections may be used to find the specific resistance of copper.

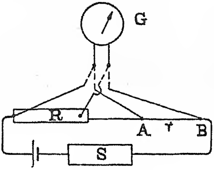


FIG. 345

A length of copper wire is soldered to two terminals about 1 metre apart on a wooden base, and is connected in series with 0·1 or 0·01 ohm, a constant source of potential such as a steady 2-volt accumulator, and a resistance box of 0 to 100 ohms. (It is not advisable to have less than 20 to 30 ohms in the circuit.)

The ends of the wire AB (r) and of the $\frac{1}{10}$ ohm (R) are connected to a double pole two-way switch (indicated by the broken lines in fig. 345), which is connected to a high-resistance galvanometer, G, the deflections of which may be observed by the usual lamp and scale method.

The deflection given when the potential differences between the ends of r and R are applied is measured by taking the reading of the deflected spot of light in each case. The battery is then reversed and the reading is obtained on the other side; half the difference in readings giving d_1 and d_2.

S is adjusted and the experiment repeated for 2 or 3 values of the current, and the mean value of r is obtained.

Hence, putting $R = \frac{1}{10}$ or $\frac{1}{100}$ in (6), r is evaluated.

(2) Potentiometer Method

A steady lead accumulator, E_1, is set up in the potentiometer circuit, AB.

The two small resistances to be compared, R and r, are joined in series with a third variable resistance, S.

A galvanometer, G, is connected as shown, and a double pole two-way switch is used. This should be a mercury-cup key, using well amalgamated copper connecting strips.

A balance is obtained at H when cups 1 and 4, 2 and 5 are joined together. A second balance is obtained at H′ when 2 and 4, 5 and 3, are joined together.

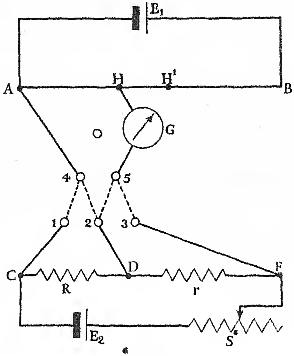


Fig. 346

The P.D. between C and D is iR, where i is the steady current in the circuit CFS; in the same way the P.D. between D and F is ir. So that if AH $= l_1$ and AH′ $= l_2$,

$$\frac{iR}{ir} = \frac{l_1}{l_2} \quad \text{or} \quad \frac{R}{r} = \frac{l_1}{l_2}.$$

The length, l_1, is obtained, then l_2, and finally l_1 is checked; if any difference is found the mean of l_1 and l_1' is compared with l_2, in the usual way. The experiment is repeated with different currents in the circuit, CFS, obtained by varying the resistance, S.

(3) The Kelvin Bridge

A more accurate comparison of two small resistances may be made with the Kelvin bridge, shown diagrammatically in fig. 347. In the more commonly used form of bridge the resistance, r_1, is made equal to r_2 and r_3 to r_4 or $\frac{r_1}{r_2} = \frac{r_3}{r_4} = 1$, where the resistances are of the order

of 1 or 2 ohms. To increase the range of measurement, however, the bridge is also constructed so that $\frac{r_1}{r_2} = \frac{r_3}{r_4} = n$, when n is usually 10, 100, 0·1, or 0·01.

The two low resistances to be compared, DE (r) and FG (r'), are connected in series with an accumulator and key. If r and r' are rods or thick wires they are joined at B in a mercury cup and a fairly large current is passed through them. The parts DE and FG of the rods are compared, and it can be shown that the resistance between E and F

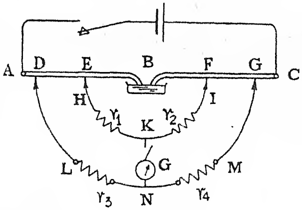


FIG. 347

has no effect on the final result. One of the contacts on the rods, say G, is a movable knife-edge and the other contacts, D, E, and F, are fixed. To compare r and r', G is moved along BC until on closing the battery key and then the galvanometer key no deflection in the galvanometer results, in which case

$$\frac{r}{r'} = \frac{r_1}{r_2} = \frac{r_3}{r_4} = n. \qquad \ldots(7)$$

The relation shown in (7) may be simply deduced by assuming the condition of balance, in which case the current in the galvanometer (i_g) is zero. Let i be the current in r (DE), i_1 the current in r_1, and i_2 the current in r_3. Then, since $i_g = 0$ the current in r_2 is i_1, in r_4 is i_2, and in r' (FG) is i.

At balance the potential difference between K and N is zero, or:

$$ir + i_1 r_1 = i_2 r_3$$

and

$$i_1 r_2 + ir' = i_2 r_4$$

or

$$\frac{ir + i_1 r_1}{i_1 r_2 + ir'} = \frac{r_3}{r_4},$$

i.e.

$$i\,(rr_4 - r_3 r') = i_1\,(r_2 r_3 - r_1 r_4),$$

and since by (7) $r_2 r_3 = r_1 r_4$ and i is not equal to zero,

$$rr_4 = r_3 r'$$

or $$\frac{r}{r'} = \frac{r_3}{r_4} = \frac{r_1}{r_2} = n.$$

The resistances r_1, r_2, r_3, r_4 should be made of wire of the same material or wires of similar temperature coefficients, and should be placed near each other to ensure a common temperature throughout the experiment. When the battery circuit is closed it should be for a minimum time to test for balance, in order to reduce the heating in the network, and especially in AB and BC, where the wires presumably have different temperature coefficients.

The galvanometer used should be a sensitive one of low resistance and E and F should be arranged to be near B.

To find the specific resistance of brass, as an example of the method described, a brass rod of radius A is placed at BC and a copper rod

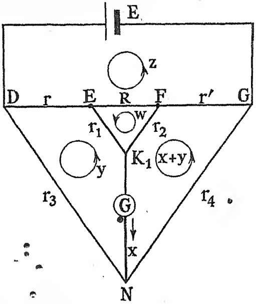


Fig. 347 (*a*)

of radius a is placed at AB. The connexions from D, E, F, and G are made with thick copper wire. The connexion D should be a soldered joint and the leads from r_1 and r_2 should be dropped into the mercury cup (i.e. E, B, and F coincide). The variable contact at G should be a knife-edge clamped in the final position to the brass rod. When the balance is found, suppose the length of brass rod FG = L cm. and the length of copper rod DE = l cm., then

$$\frac{r}{r'} = \frac{r_1}{r_2} \quad \text{and} \quad \frac{r}{r'} = \frac{slA^2}{SLa^2},$$

where s and S are the specific resistances of copper and brass respectively. The value of s from drawn copper at 18°C. is $1{\cdot}78 \times 10^{-6}$ ohm-cm., and hence S may be calculated.

The value of S may be checked by repeating the experiment, using a standard 0·01 ohm resistance for r and the brass rod for r'. If in this case the length of brass for balance is L,

$$S = 0{\cdot}01\,\frac{r_2}{r_1}\cdot\frac{A^2}{L}.$$

It is obvious that if we are to find the value of the resistance of a fixed length of wire or of a fixed small resistance, the value of r must be variable. This can be brought about by having a variable length of copper wire at DE (i.e. have D movable) or DE may be made up of a number of coils of, say, 0·01 ohm resistance in series with the wire. In the standard Kelvin bridge this latter method is employed, as illustrated in figs. 348 (at p. 1) and 349.

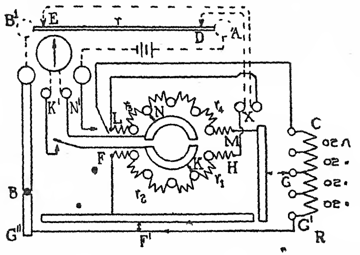


FIG. 349

For measuring the specific resistance of a wire of low resistance, the special clamping contact arrangement shown at the back of fig. 348 is used. For other forms of low resistance, the terminals X of that figure and fig. 349 are used directly.

Such an apparatus has a wider range and is more adjustable than the simpler form already described, but is, in principle, identical with the bridge illustrated in fig. 347. To emphasize this, fig. 349 has been drawn showing the internal arrangements of fig. 348, lettered to agree with fig. 347.

The proof that the resistance, R, between E and F does not affect the final result is as follows.

Let the diagram of fig. 347 (a) denote the network and let the current through the galvanometer be denoted by x.

If the circles denote the direction of the currents in the various elements of the network and the letters attached to them the amount of current in the arm of a unit which is not

units, the application of Kirchhoff's laws leads to the following equations, b denoting the battery resistance and E its E.M.F.:

$$zb + (z - y)\,r + (z - w)\,\mathrm{R} + (z - x - y)\,r' = \mathrm{E};$$
$$(y - z)\,r + yr_3 - x\mathrm{G} + (y - w)\,r_1 = 0;$$
$$x\mathrm{G} + (x + y)\,r_4 + (x + y - z)\,r' + (x + y - w)\,r_2 = 0;$$
$$(w - z)\,\mathrm{R} + (w - y)\,r_1 + (w - x - y)\,r_2 = 0.$$

If these equations are arranged in the form $(\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D}) = 0$, they can be solved and the value of x will be found to be:

$$x = \begin{vmatrix} \mathrm{E} & -(r + r') & (b + r + \mathrm{R} + r') & -\mathrm{R} \\ 0 & (r + r_1 + r_3) & -r & -r_1 \\ 0 & (r' + r_2 + r_4) & -r' & -r_2 \\ 0 & -(r_1 + r_2) & -\mathrm{R} & (\mathrm{R} + r_1 + r_2) \end{vmatrix} \div \Delta,$$

where Δ is the determinant consisting of the terms like A, B, C, D, in the rows.

Balance occurs when $x = 0$. On working out the term in the numerator of x it is found that it will vanish for any value of R, provided that

$$\frac{r}{r'} = \frac{r_1}{r_2} = \frac{r_3}{r_4}.$$

The coils which make up r_1, r_2, r_3, r_4, are accurately adjusted to ensure the condition of (7) above, and may give values of $n = 0{\cdot}01$, 0·1, 1, 10, or 100.

As shown in the figure, r is the unknown resistance, and r' may be one or more of the four coils each of 0·02 ohm, together with a part, F′G′, of the graduated and calibrated wire, G′G″. The length of the wire is about 450 mm. and each millimetre has a resistance of 0·00005 ohm, i.e. 40 cm. of the wire add 0·02 ohm to the value taken from CG′ (0·04 in the diagram = GG′). The wire is graduated in fractions of an ohm.

A comparison of figs. 349 and 347 will show the general arrangements.

For example, suppose an unknown resistance r were balanced when $\frac{r_1}{r_2}$ was made = 0·01, by 0·04 ohm from the coils, and 290 units of the slide wire, as shown approximately in fig. 349,

$$\begin{aligned} r = r'\left(\frac{r_1}{r_2}\right) &= (0{\cdot}04 + 290 \times 0{\cdot}00005)\,0{\cdot}01 \\ &= (0{\cdot}04 + 0{\cdot}0145)\,0{\cdot}01 \\ &= 0{\cdot}000545 \text{ ohm}. \end{aligned}$$

This instrument will measure resistances of 10 to 0·00001 ohm.

Fig. 351 shows a second form of Kelvin bridge made from the potentiometer illustrated in fig. 372, and a double ratio box (fig. 350,

at p. 1). This enables a range of measurement from 1·5 ohms to 0·00001 ohm, about.

The double ratio box contains resistances (fig. 351), LN, NN′, N′N″, and N″M, which are respectively $\frac{1}{2}$, $\frac{9}{22}$, $\frac{90}{1111}$, and $\frac{1}{101}$ of the total resistance of LM. A similar arrangement holds for the parts of HI, so that the resistances LN : NM = HK : KI (1 : 1) or LN′ : N′M = HK′ : K′I (= 10 : 1), etc. The scheme of connexions is lettered to conform with the letters of fig. 347. A comparison with that figure shows the reason for this scheme: r', the unknown resistance, is equal to the

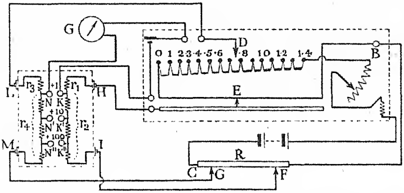


FIG. 351

resistance between E and D when the galvanometer is connected to NK as in the case taken, or is $\frac{1}{100}$ED when the galvanometer is connected at N′K′, etc.

The coils to which D is tapped are each 0·1 ohm and the resistance of the slide wire is 0·001 ohm per small division. Thus, using N″K″ as galvanometer terminals, a resistance of 0·00001 may be measured.

High-resistance Measurement

As already stated, a special method is required to determine resistances of greater values than 1 megohm (i.e. 10^6 ohms). One of the simplest ways of evaluating resistances of this order is the method of substitution. Fig. 324, p. 503, shows the scheme of connexions.

A steady accumulator, E, is connected in series with the unknown resistance, R, and connected to a commutator, C, whence the current may be sent in either direction through a galvanometer, G, which is provided with a variable shunt, S.

Let the steady deflection due to the current be d cm. on a scale one metre away (corresponding to a movement of the suspended system of $\theta°$). A known adjustable resistance is substituted in place of R, and if on adjusting the known resistance a steady deflection, d cm., is obtained again, the known resistance is of the same value as the unknown. If no variable known resistances of the order of the unknown

resistance is available, the galvanometer is shunted so that one-thousandth part of the current is allowed to pass through it. When a large known resistance (e.g. 10,000 to 20,000 ohms from two Post Office boxes in series) is placed in series with it and the steady accumulator, a deflection of the same order as that given when the unknown resistance is placed in the circuit may be obtained.

Under these circumstances, suppose that d_1 is the deflection caused by the battery when in series with the unknown resistance, R ohms, and d_2 when in series with r, the known resistance.

Then if E is the E.M.F. of the accumulator, B the resistance of the battery, and G the resistance of the galvanometer, we have, where k is the galvanometer constant,

$$kd_1 = \frac{E}{B + G + R},$$

$$kd_2 = \frac{E}{\left(B + \dfrac{SG}{S + G} + r\right)} \frac{S}{S + G};$$

$$\therefore \quad \frac{d_1}{d_2} = \frac{(r + B)(S + G) + SG}{S(B + G + R)} \simeq \frac{r\,\dfrac{S + G}{S} + G}{G + R}, \qquad \ldots(9)$$

since B may be neglected in comparison with r or R.

The experiment may be carried out, using an adjustable known resistance and a shunt of known value across the galvanometer. The known resistance, r, is adjusted until $d_2 = d_1$, i.e. equal deflections are obtained. Then, if G is small compared with R,

$$R = 1000r, \qquad \ldots(10)$$

if the shunt is $\frac{1}{999}$ G.

Alternatively, if the known resistance is not sufficiently adjustable to cause equal deflections, the values of d_1 and d_2, both of the same order, are noted, then we have:

$$R = \frac{d_2}{d_1}\frac{S + G}{S} \cdot r + G\left(\frac{d_2}{d_1} - 1\right).$$

Determine by the above methods the value of the resistance of a leaky condenser. Another suitable high resistance to be measured by these methods is made by taking a sheet of ebonite, say 20 cm. by 5 cm. Two holes are drilled about 15 cm. apart, and the surface of the ebonite blackened round the holes with a black lead pencil. Two terminals are screwed down in the holes, and a fine black lead pencil line is ruled between them. The sheet is covered with a thin sheet of ebonite to prevent any accidental change in the dimensions of the line.

The apparatus should be set up as in fig. 324, and the values of R obtained.

Using a thin black lead pencil line, the following results were obtained:

$d_1 = 1{\cdot}95$ cm., $\quad R = ?$ $\quad$ no shunt;
$d_2 = 9{\cdot}30$ cm., $\quad r = 10{,}000$ ohms, $\quad$ shunt $\frac{1}{999}$,

Hence

$$R = \frac{9{\cdot}3}{1{\cdot}95} \times 10000 \times 1000 = 47{\cdot}7 \times 10^6 \text{ ohms.}$$

Using a thicker line on ebonite, the method of equal deflections gave the following results:

deflection 10 cm. when R was in circuit, no shunt;
deflection 10 cm. when 3640 ohms replaced R, shunt $\frac{1}{999}$.

Thus, $\quad R = 3640 \times 1000 = 3{\cdot}64 \times 10^6$ ohms.

Determination of High Resistance by Leakage, using a Ballistic Galvanometer

On p. 652 a method is given for finding the value of a high resistance by leakage. The method here described is identical in principle but employs a ballistic galvanometer in place of the quadrant electrometer of fig. 427.

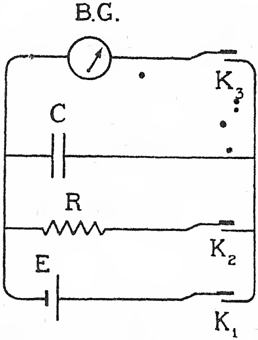


Fig. 352

Fig. 352 shows the arrangement of apparatus. C is a standard capacity (usually $\frac{1}{3}$ to 1 microfarad), BG the ballistic galvanometer, R the high resistance, E a steady lead accumulator, and K_1, K_2, K_3 are condenser keys which do not leak appreciably.

K_1 is depressed and the condenser is charged. K_1 is then raised and by depressing K_3, the quantity of electricity Q_1 (see p. 654) passes through the galvanometer causing a deflection θ_1. K_1 is again depressed and released, K_2 is then closed for a measured number of seconds, t_1; it is then opened and K_3 again closed, and Q_2, the quantity of electricity remaining in the condenser is measured by means of a deflection, θ_2. The time is estimated from preliminary experiments, so that θ_2 is about half of θ_1, and, if d_1 and d_2 are the corresponding scale readings, we have as on p. 654, equation (3),

$$R = \frac{t}{2{\cdot}303C \log_{10}\left(\frac{d_1}{d_2}\right)}.$$

The value of t may be varied and a plot of t and $\log \frac{d_1}{d_2}$ made; from the slope of the curve, knowing C, R may be found.

The graph will be found to deviate from a straight line if certain high resistances are used (e.g. carbon lines and 'grid leaks'). This should be investigated.

Measurement of Small Intervals of Time

To measure the interval of contact of, say, a metal pendulum bob and a metal plate against which it strikes, the bob and plate are cleaned, and the bob is suspended by a copper wire. The plate is arranged so that it is just in contact with the ball as the latter is in its equilibrium position. The whole arrangement is then put in place of K_2. R is given values sufficiently small that d_2 is of the order of $\frac{d_1}{2}$, where d_1 is obtained as before, and d_2 is the scale deflection after the plate has been struck *once* by the ball. In this experiment all the factors except t are known and therefore t may be calculated.

To make sure that one contact only is made between the ball and plate, it is better to place the two *between* R and K_2; depress K_2 until the noise of contact is heard, then release K_2, and depress K_3 to find the amount of electricity left in C.

In this way find how the time of contact varies with the velocity of the ball at the time of contact.

THE VARIATION OF RESISTANCE WITH TEMPERATURE

The value of a resistance of a specimen of wire, in most cases, increases with temperature.

The relation between R_0, the resistance at 0° C., and R_t, the resistance at another temperature, t° C., measured on the air scale is

$$R_t = R_0 (1 + \alpha t + \beta t^2), \qquad \ldots(11)$$

where α and β are constants.

The constant, β, is small, and therefore over small ranges of temperature the resistance is practically a linear function of the temperature, i.e.

$$R_t = R_0 (1 + \alpha t) \qquad \ldots(12)$$

expresses the relation, for small temperature ranges, between resistance and temperature (t should not exceed 100° C. for (12) to be valid).

From (12) we have

$$\alpha = \frac{R_t - R_0}{R_0 t},$$

and α may be called the '*coefficient of increase of resistance with temperature*' for the limited range taken.

Determination of the Resistance of a Wire at Temperatures from 0° C. to 100° C.

A length of thin platinum wire of about 1 ohm resistance is mounted non-inductively on a mica frame, as shown in fig. 353. The ends of

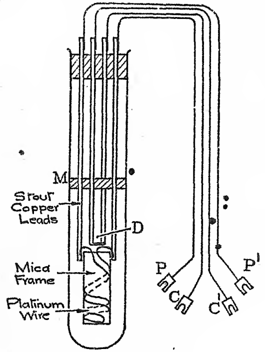


Fig. 353

this wire are soldered to two thick copper wires whose resistance is negligible compared with that of the platinum. This arrangement is mounted rigidly in a glass tube, with mica or rubber supports, as shown at M. The free ends of the copper wires are soldered to two long leads, and two identical leads, cut from the same wire, are joined (D) together. The four free ends of the leads are soldered to copper connecting strips, shown in fig. 357, P, P′, C, and C′. These ensure good contacts with terminals under which they are fixed.

The pair, P, P′, are the ends of the leads from the platinum wire; C, C′ are the compensating leads.

The inner gaps in the bridge are occupied by two equal resistances of the same order as that of the platinum wire; in the case taken these resistances, R_1 and R_2, are both made 1 ohm (fig. 354).

P and P′ are connected in one of the outer gaps and the compensating leads in series with a resistance box, S, fill the fourth.

It must be remembered that R_1, R_2, and S should be connected to the bridge by means of copper strips, and every source of uncertain resistance, such as a bad or dirty connexion, must be eliminated.

All the resistances should be non-inductive, for in this method it will be seen that the galvanometer is permanently connected in the circuit, and the battery is connected to the sliding contact.

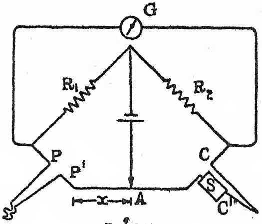


FIG. 354

This is essential, for we must balance the resistance of the platinum at the temperature which is fixed by the surroundings, so that the current should not pass for any appreciable time and cause heating in the spiral.

S is a resistance box in which are all values from 0·1 to 10 or more ohms.

The function of the compensating leads will be apparent from the theory of the bridge given below. Since the material and length of the leads are identical with those of the leads to the platinum, their resistances are equal, and since the two pairs of leads are side by side, variation in the resistance of this part of the circuit due to changes of temperature affect each in the same way.

The value of S is so adjusted that A is near the middle of the wire. In obtaining a balance the current must not be allowed to flow for more than an instant, i.e. balance is obtained by ensuring no immediate deflection of the galvanometer for contacts of short duration. If the current continues to flow, a change in the resistance of the platinum

wire, due to the heating by the current, will cause a deflection in the galvanometer.

Suppose the glass tube containing the platinum wire specimen is immersed in a constant-temperature bath and a balance is obtained at x cm. from the end of the bridge, we have, if R_t is the resistance of the platinum at this temperature $t°$,

$$\frac{R_1}{R_2} = \frac{R_t + r + x\rho}{r + S + (100 - x)\rho},$$

where r is the resistance of either pair of leads, ρ is the resistance of 1 cm. of bridge wire.

Since

$$R_1 = R_2 = 1 \text{ ohm},$$

$$R_t + r + x\rho = S + r + (100 - x)\rho,$$

$$R_t = S + (100 - 2x)\rho. \qquad \ldots(13)$$

Hence, if ρ be known, R_t may be calculated.

The value of R_0 is obtained by immersing the resistance in melting ice. After twenty minutes or half an hour the whole of the tube will have attained the temperature of 0° C.; the balance is obtained near the centre of the bridge by adjusting S at a value S_0. A is x_0 cm. from the end. If after five minutes the balance is still at x_0, we may safely assume that the resistance is at 0° C.

The value of ρ may be obtained at this stage by varying S_0 to S_0' and finding the new balance at x_0'. S_0' is again adjusted to a third value, S_0'', and the balance is now at x_0''. The platinum wire is meanwhile at 0° C. and therefore its resistance remains R_0,

i.e.

$$R_0 = S_0 + (100 - 2x_0)\rho,$$

$$R_0 = S_0' + (100 - 2x_0')\rho,$$

$$R_0 = S_0'' + (100 - 2x_0'')\rho.$$

From these equations, taken in pairs, three values of ρ may be calculated. The mean value is substituted in the first equation to give a value of R_0.

The glass tube and contents are then placed in a hypsometer, and a balance is obtained once more, when the temperature of the wire has acquired that of the steam which is made to pass round it in the hypsometer.

The value of the resistance at, say, 40° C., 60° C., 80° C., is obtained in the same way by immersing the container in a water bath maintained at steady temperatures near these points. In each case time is allowed for the platinum to acquire the temperature of the bath.

The values of R are plotted against temperature. The result will be approximately a straight line. From the slope of the straight line drawn through the observed points, calculate α as defined above.

In the theoretical account of the balance given above it was assumed that there were no end corrections to the bridge wire.

If l_1 and l_2 are the equivalent lengths obtained in the bridge calibration (see p. 534), equation (13) would become

$$R_t + r + x\rho + l_1\rho = S + r + (100 - x)\rho + l_2\rho,$$
$$R_t = S + (100 - 2x)\rho + (l_2 - l_1)\rho.$$

Since l_1 and l_2 are known, the effect produced may be allowed for in this way throughout the determinations.

The values of x and $(100 - x)$ should be corrected from a calibration curve as obtained in the manner given on pp. 534–7.

The Platinum Resistance Thermometer. Callendar-Griffiths Bridge

The platinum resistance thermometer is a temperature-measuring instrument which depends for its action on the variation of the resistance of a wire with temperature, investigated in the last experiment.

The temperature of any enclosure is calculated from the observed value of the resistance of a calibrated specimen of platinum wire.

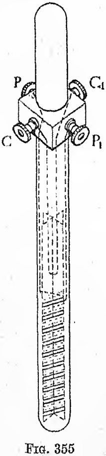


FIG. 355

The instrument consists of a length of fine platinum wire, wound non-inductively on a mica frame. The ends of this thin wire are soldered to two thick leads which are connected at the other ends to two terminals on the cap of the porcelain tube which contains them.

Arranged parallel to these thick leads is a second pair, of identical material and size, soldered to two more terminals on the cap. The other ends of this second pair are joined together, as shown in fig. 355.

The terminals are usually marked in a distinctive manner, P, P_1, C, C_1; C, C_1 being the ends of the compensating leads.

We have already seen (p. 548) that the resistance, R_t, of a given specimen of wire, at a temperature t° C., may be expressed in terms of the resistance at 0° C., R_0, and two constants α and β in the following manner:

$$R_t = R_0(1 + \alpha t + \beta t^2). \qquad \ldots(11)$$

Thus, if the resistance of the platinum wire were measured at 0° and two other known temperatures, we should have two equations from which α and β could be calculated. The wire would then be standardized, so that if R_t were measured at an unknown temperature the latter could be calculated from equation (11) above or from the values of R_0, α, and β, a calibration curve could be drawn showing the relation between R and t, and hence the temperature corresponding to any resistance could be obtained.

Another way of using the measurements obtained is to make use of the platinum resistance temperature scale.

Let R_0 be the resistance of the platinum wire at the temperature of melting ice and R_{100} that at the temperature of steam at standard pressure; $R_{100}-R_0$, the increase in resistance for 100° rise in temperature is called the 'fundamental interval'.

The platinum scale makes the size of the degree such that each degree rise in temperature on this scale corresponds to an equal increase of resistance of the specimen of wire of amount

$$\frac{R_{100} - R_0}{100}.$$

The platinum scale so defined coincides therefore with the gas scale at 0° and 100°, but will differ at other points since equation (11) above expresses the true relation between R and t on the gas scale.

Let R be the resistance of the wire at any temperature, t° C., or t_p on the platinum scale. By definition

$$t_p = 100 \frac{R - R_0}{R_{100} - R_0}. \qquad \ldots(14)$$

The difference between t_p and t has been found to be given by

$$t - t_p = k \left(\frac{t}{100} - 1\right) \frac{t}{100}, \qquad \ldots(14a)$$

where $k = 1{\cdot}5$ for pure platinum, but its value for any particular specimen may be obtained by measuring the resistance R_0, R_{100}, and R_t at 0° C., 100° C., and a third standard temperature, say, the boiling-point of sulphur for which t° is 444·7° C. The value of t is calculated by (14) and k is deduced from (14a).

The values of the resistances may most conveniently be determined by using the Callendar-Griffiths bridge.

The Callendar-Griffiths Bridge

Fig. 356 (at p.1) shows the general appearance of an example of this type of bridge.

Reference to fig. 357 will make the principle of construction clear.

R_1 and R_2 are equal resistances. EF an adjustable resistance capable of giving 5, 10, 20, 40, 80, 160, 320, 640, 1280 arbitrary units of resistance. ML is a straight, stretched wire chosen by reason of its uniformity, and T a second parallel wire of the same material, which may be connected to ML by means of a short length of the same wire. Thus the possibility of thermo-electrical effect when the galvanometer is connected to the wire is eliminated.

Suppose the thermometer be balanced at a temperature, t_p, against S units from EF, when the contact is made at O, the centre of ML, then using the same notation as before,

$$R_t + r + \text{resistance LO} = S + r + \text{resistance OM}.$$

or $$r_t = S.$$

If now at a second slightly different temperature, t'_p, the thermometer has resistance R_1, and is balanced by moving the contact 1 unit of length to the right,

$$R_1 + r + \text{resistance of } (OL - 1) = S + r + \text{resistance of } (OM + 1),$$

i.e. $$R_1 = S + 2 \text{ resistance of 1 cm. of LM.}$$

If the resistance per cm. of ML is half an arbitrary unit in which S is measured,

$$R_1 = (S + 1) \text{ arbitrary units.}$$

The scale on which measurements of MO, etc., are made is usually inscribed in arbitrary units.

If the balance is to the right of the mid-point of LM, the number of units must be added to S; if to the left the number is taken from S.

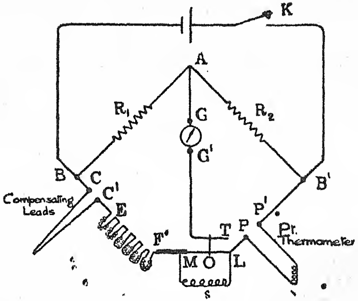


FIG. 357

The length of the wire, LM, is usually sufficient to allow about 15 units each side of the centre.

To arrange this simple relation between the length of the wire and the arbitrary units, a coil, *s*, of suitable resistance is shunted across the bridge wire. The coil is adjusted to give to the bridge wire the desired value of resistance per cm.

The arbitrary unit of resistance often chosen is deduced from the value of the fundamental interval of the thermometer (i.e. the change in resistance for a change in temperature from 0° to 100° C.).

The platinum thermometer is constructed with such a resistance that its fundamental interval is 1 ohm and one-hundredth part of this is taken as the unit. The coils in S have, therefore, resistances of 0·05, 0·1, etc.

The resistance per cm. of bridge wire in the example taken would

therefore be $\frac{1}{2}$ (0·01) or 0·005, the effect of 1 cm. change in balance being $2 \times 0{\cdot}005 = 0{\cdot}01$, i.e. a change of balance of 1 cm. corresponds to a change in resistance of the platinum wire of 1 unit.

The form of Callendar-Griffiths bridge illustrated in fig. 356 is provided with a scale for the slide wire bridge which is graduated from 0 to 15 units, i.e. the readings are continuous from one end of the wire to the other. The balance points give directly the number of units to be added to S to give the equivalent resistance of the thermometer. The half, MO, of the wire is in this case within the instrument in the form of a coil.

A more recent feature of the bridge is the use of mercury cup contacts instead of the usual plug contacts in the adjustable arm, S. Fig. 358 shows an enlarged view of one of the contacts.

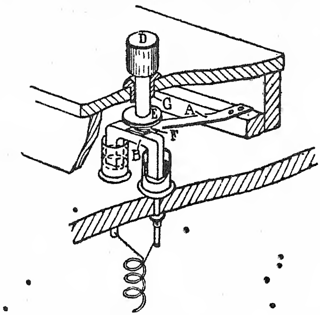


FIG. 358

The coil is connected to two mercury cups inside a tightly fitting cover.

A plug, D, when inserted in the hole, G, corresponding to this resistance, strikes a thin circular sheet, E, which is covered with baize and is ordinarily held tightly against the hole by means of the spring, A, and a spiral spring, F, thus keeping dirt and dust from the mercury. When D is allowed to depress this arrangement, the copper connecting strip, B, which is amalgamated at the ends, dips into the cups and the resistance is thereby cut out. The total value of the resistance, S, is therefore the sum of the numbers opposite the holes without plugs.

The makers suggest that when not in use all plugs should be in to maintain the amalgamation of the connecting strip, B.

The balance point on the bridge wire may be maintained by a rough

movement of the slider, followed by a fine adjustment by means of the small lever attachment. The position of the balance is read on a vernier which enables one-tenth of the small scale division to be measured, i.e. if graduated in lengths corresponding to a unit and subdivided into tenths, one may read to $\frac{1}{100}$ of the unit.

Reference to fig. 359 will make clear the internal wiring of the bridge. The lettering in this diagram corresponds to that of fig. 357. A 2-volt accumulator or Daniel cell is connected to BB′ and a galvanometer to GG′. PP′ and CC′ are gaps for the thermometer and compensating leads. It will be noted that R_1 and R_2 are contained in the bridge.

The point A in fig. 359 is capable of slight adjustment in many forms of the bridge, so that, if in error, R_1 may be made equal to R_2.

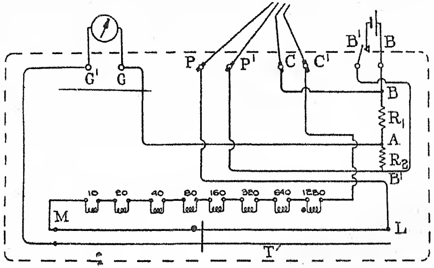


FIG. 359

Before using the bridge with the thermometer it is advisable to check some of the points of construction which have already been described.

(*a*) See that the zero of the scale is truly the mid-point of the bridge. This is readily done by inserting equal resistances, e.g. two thick copper strips, in the gap provided for the thermometer and compensating leads. With $S = 0$, find the balance-point on the wire. This is the mid-point and should coincide with the zero graduation; if this is not so, then probably the values of R and R′ are not truly the same; this may be further checked by using two equal resistances of 5 or 10 ohms in the gaps PP′ and CC′. A balance is obtained. The coils are then interchanged and a second balance obtained. The mean of the two readings should be the zero of the scale. This method cannot be very well used in the second form of bridge wire described above.

However, for the purpose of comparing resistances the slight error in equality will not affect the comparison.

(*b*) Verify the relation between arbitrary units of resistance in S and the resistance per unit length of bridge wire, and see that the resistances in S are consistent among themselves.

The gap CC′ is closed by means of a copper connecting strip, and an external resistance is connected by copper strips to the gap PP′.

The coil 1280 is unplugged and balanced against an external resistance of, say, 12·9 ohms. Suppose x_1 be the reading on the bridge at balance.

Replace the 1280 coil by the rest of the coils, i.e.

$$640 + 320 + \ldots 10 + 5,$$

and again balance these coils (nominal value 1275 units) at a point x_1', using the same external resistance. Let l_1 be the resistance corresponding to a change in balance $x_1' - x_1$.

Then $$\text{coil } 1280 - \text{sum of the rest} = l_1. \qquad \ldots(15)$$

Carry out this test using the 640 coil: balance against an external resistance making the point of balance at x_2, say. Cut out the 640 coil and, again leaving the external resistance the same, balance the rest of the coils $(320 + 160 + 80 + 40 + 20 + 10 + 5)$ at a second point x_2' cm. Let the resistance corresponding to the change $x_2' - x_2 = l_2$, then

$$\text{resistance (coil } 640 - \text{the rest)} = l_2. \qquad \ldots(16)$$

Carry out this test with each coil in turn.

Finally,

$$\text{resistance (coil } 5 - 0) = l_9.$$

Nine equations are obtained in this way for the bridge described.

The difference between the first and second equations, where each term represents the corresponding resistance, is

$$\text{coil } 1280 - \text{coil } (640 + 320 \ldots + 5)$$
$$- \{\text{coil } 640 - \text{coil } (320 + \ldots + 5)\} = l_1 - l_2,$$

i.e. $$\text{coil } 1280 - 2 \text{ coil } 640 = l_1 - l_2$$

or $$\text{coil } 640 = \frac{\text{coil } 1280}{2} + \frac{l_2 - l_1}{2}. \qquad \ldots(17)$$

In the same way, taking the third equation from the second

$$\text{coil } 320 = \frac{\text{coil } 640}{2} + \frac{l_3 - l_2}{2}, \qquad \ldots(18)$$

and so on. Substituting in (18) the value of coil 640 in terms of 1280 from equation (17):

$$\text{coil } 320 = \frac{\text{coil } 1280}{4} + \frac{l_2 - l_1}{4} + \frac{l_3 - l_2}{2}. \qquad \ldots(19)$$

This process is carried on throughout the range. We may express each coil in terms of the largest one. Equation (17) and (19) give

the corrections to be applied to each coil to make them consistent with the largest.

For a perfect set of coils $l_1 = l_2 = \ldots = l_8 = l_9 =$ the length of wire having 5 units of resistance.

It is important to note that the unit often taken is 0·01 ohm, so that for safety the 1280 coil should be out as a permanent addition when testing the 80-, 40-, 20-, 10-, and 5- unit coils.

The results may be tabulated as shown above, the last column giving the correction to each coil.

External resistance	Coil	Bridge wire reading x units resistance	Difference l	Correction
	1280 640+320+ . . . +5			
	640 320+160+ . . . +5			
	320 160+80+ . . . +5			
	160 80+40+ . . . +5			
	80 (+1280) 40+20+10+5 (+1280)			
	40 (+1280) 20+10+5 (+1280)			
	20 (+1280) 10+5 (+1280)			
	10 (+1280) 5 (+1280)			
	5 (+1280) 0 (+1280)			

The above calibration gives the mean value per unit length of bridge wire.

For some purposes, and especially when the bridge has had considerable use, it is necessary to calibrate the bridge wire. This may be done by a slight modification of the method given on p. 534. But a new bridge wire should be uniform to within 0·3 per cent, and therefore will give readings correct to the second decimal place in degrees on the platinum scale.

Calibration and Use of the Platinum Thermometer

Having tested the bridge and calibrated the coil and wire as indicated above, the thermometer is placed in melting ice and allowed to remain there until the resistance remains constant. The value of this resistance is noted.

The thermometer is next placed in a hypsometer, in which water is boiled, and again the steady resistance, at the temperature of the steam, is obtained after being in the steam for 20 to 30 minutes.

The temperature of the steam corresponding to the atmospheric pressure is determined from tables ($= b°$C. say) then, if R_b is the resistance obtained in the steam, and R_0 in ice at 0° C.,

$$\frac{R_b - R_0}{b}$$

is the change in resistance per degree centigrade.

The thermometer is next immersed in the vapour from boiling sulphur (see below for method), and from the balance of the bridge, when steady, the value of R, the resistance at the temperature of the sulphur vapour, is obtained.

The boiling-point of sulphur in degrees on the platinum scale is from (14):

$$t_p = \frac{b\,(R - R_0)}{R_b - R_0}.$$

Now the value of the boiling-point of the sulphur at 76 cm. of mercury pressure is 444·7° C., and the value at any other pressure, p mm., is

$$t = 444{\cdot}7 + 0{\cdot}0904\,(p - 760) - 0{\cdot}000052\,(p - 760)^2.$$

The value of t, the boiling-point at the pressure which obtains during the experiment is calculated and hence, substituting t_p and t in (14*a*), the value of k may be calculated.

It will be found to be approximately 1·5. The thermometer is now fully calibrated and may be used, for example, to find the B.P. of aniline in degrees on the platinum scale. The corresponding value in degrees centigrade may be found by (14*a*).

It will be noted that in equation (14*a*) the correction to be made to t_p in order to give t is expressed in terms of t. The value of the air scale temperature is unknown, and the procedure is to obtain it by successive approximations. t_p and t differ by a small amount and the first approximation is found by substituting $t = t_p$ on the right-hand side. This will give a value for t which can again be substituted on the right-hand side. If this procedure is continued, it will be found that the stage is soon reached at which the approximation is as close as the conditions of the experiment allow.

To Obtain the Boiling-point of Sulphur

The apparatus of fig. 360 is convenient. This is readily made by taking a length of iron tube about two inches diameter, and about four or five inches longer than the thermometer. The lower end of the tube is closed by brazing on a circular disk of iron.

The lower end of the tube is covered completely with asbestos paper and then wound with nichrome wire of suitable gauge and length to produce sufficient heating. For example, the one shown in the figure was made by winding with nichrome wire of 0·092 cm. diameter and

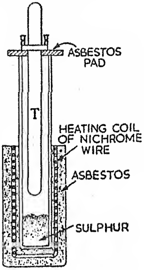


Fig. 360

6 m. in length. This was satisfactory for use with a series resistance on 100-volt mains. The length of the same gauge wire may be varied to suit the potential of the mains available. It may be found advisable, in addition to the thick layer of asbestos around the nichrome winding, to surround the boiler with dry sand. The sulphur vapour condenses on the upper walls of the tube and runs back.

The boiling-point under the pressure which obtains during the experiment may be obtained from the formula given on p. 559.

The Variation of the Resistance of a Bismuth Spiral in a Magnetic Field

In this experiment a bismuth spiral such as that shown in fig. 361 is placed in a variable magnetic field, and the resistance is determined for each value of the field strength. The measurements involved are therefore (1) the strength of the field, (2) the resistance of the spiral.

The magnetic field should be of as large a range as possible, and may be obtained by using a large magnet as shown in fig. 362. A direct current is passed through a series of frame resistances, R, to the magnet, and an incandescent lamp, L, in parallel. The current passing through the windings of the magnet may be measured on a calibrated ammeter,

A, and may be reversed by the commutator, C. In order that the ammeter is not affected by the magnet they are placed a considerable distance apart.

By varying the frame resistance, the current passing may be of values from, say, 0 to 5 amperes.

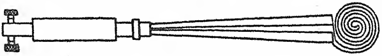


FIG. 361

The bismuth spiral is placed centrally between the poles of the magnet with the plane of the spiral normal to the magnetic field of the magnet. The resistance of the spiral is measured either by a Post Office box, or by the Carey Foster method. In the latter case the scheme of connexions shown in fig. 354 may be used, where the gap,

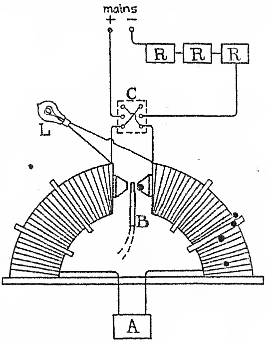


FIG. 362

PP′, is closed by the spiral and the resistance, S, may be of any value from 0 to 50 ohms. R_1 and R_2 are given convenient values. For a spiral of resistance 15 to 20 ohms R_1 and R_2 may be 20 ohms.

Its magnitude is determined for values of the field corresponding to currents in the electromagnet windings ranging from 0 to 5 amperes, by, say, 0·3 ampere steps.

The initial value of the resistance will depend upon the past history of the specimen, but will increase with increasing magnetic fields. The order of this increase will be 10 per cent of the original value, depending

upon the field change which a current variation of 0 to 5 amperes creates.

The values of the current and resistance are tabulated.

The next part of the experiment is an estimation of the magnetic field corresponding to the currents used.

This may be carried out in either of the following ways:

(1) *Fluxmeter Determination of the Field Strength* (*H*)

The fluxmeter is set up, away from the electromagnet, and a search coil of the same area as the bismuth spiral is made. This is connected to the fluxmeter and introduced between the poles of the magnet. The deflection is noted, either directly or on a scale one metre away (for the weaker field this method is essential). The scale readings are converted to maxwells, as described on p. 523.

If a is the area of the search coil per turn and there are n turns, the total flux recorded in maxwells is

$$m = na\mathrm{H}, \quad \text{or} \quad \mathrm{H} = \frac{m}{na} \text{ oersteds.}$$

As explained in dealing with the theory of the fluxmeter (p. 521), the total deflection is independent of the speed of insertion of the search coil.

The field is found in this way for each of the current values used in the resistance determination and the relation between the current and the field strength is represented graphically.

(2) *Ballistic Galvanometer Method of Finding H*

In this method a similar search coil is made and connected to a ballistic galvanometer. This forms a low-resistance circuit, so that a moving-needle type would be preferable to avoid heavy damping. This, however, should be removed to a very great distance from the magnet and ammeter.

It will generally be better to use a moving-coil instrument, also removed from the magnet, though not necessarily as far away as the needle type. In such a case a key such as described on p. 520 should be used so that the circuit may be broken immediately after the passage of the discharge.

Under such circumstances, when the current is passed through the windings of the electromagnet, the search coil being in the gap between the pole pieces of the magnet, replacing the bismuth coil, a transient E.M.F. is set up in the galvanometer circuit through which a quantity of electricity, Q, will be discharged. The quantity, Q, as shown earlier may be expressed by equation (19), p. 557.

$$\mathrm{Q} = \frac{\mathrm{T}}{\pi} \cdot \frac{i}{\varphi} \cdot \frac{\alpha_1}{2} \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}},$$

where

T is the periodic time,
α_1 is the first deflection,
α_3 is the second deflection on the same side,
i is a small steady current,
φ is the corresponding steady deflection.

The correction for damping $\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}}$ will only be applicable for the moving-coil instrument when the circuit is broken as soon as the impulse is given to the galvanometer.

The value of φ corresponding to a current, i, may be obtained by passing a current from a 2-volt accumulator through one megohm, and the galvanometer shunted by a small resistance. The deflection is observed and the current, i, calculated, as on p. 519. T, α_1, α_3 are observed in the usual way.

Now suppose that the field strength to be measured is H oersteds, and let i' be the current passing through the coil at any instant, t, after the circuit is made. During a small interval of time, dt, let $d\mathrm{N}$ be the number of lines threading the circuit, then

$$\text{E.M.F} = \frac{d\mathrm{N}}{dt} \text{ (numerically)},$$

or

$$ri' = \frac{d\mathrm{N}}{dt},$$

where r is the total resistance of the galvanometer circuit (galvanometer, leads and search coil),

i.e.

$$\int ri'dt = \int d\mathrm{N},$$

$$\mathrm{Q}r = \Big[\mathrm{N}\Big]_0^{\mathrm{H}.a.n},$$

when a is the area of one turn ($= \pi \mathrm{R}^2$), and n is the number of turns, in the search coil,

i.e.

$$\mathrm{Q} = \frac{\mathrm{H}an}{r},*$$

$$\mathrm{H} = \left(\frac{\mathrm{T}}{\pi}\frac{i}{\varphi}\frac{r}{2an}\right)\alpha_1\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}}.$$

The bracket term is constant during the experiment and may be evaluated and used as a reduction factor throughout.

The observations described above may be carried out successively. The search coil is placed in the gap between the pole pieces; α_1, α_3 are observed. The bismuth spiral is then introduced and its resistance

* If the search coil is placed in the gap between the magnets and the current reversed in the windings, $\mathrm{Q} = \frac{2an\mathrm{H}}{r}$.

measured. The search coil is replaced and the current cut off. The values, α_1', α_3', are again noted. A new value of the current strength is obtained by an adjustment of the frame resistance, and the current switched on. The throws are observed and then the bismuth spiral resistance is again measured. The check value for the throws is again obtained when the current is switched off, and so on.

With either method of working compile a table showing the relation between the magnetic field strength and the resistance, and plot a curve showing this relation.

The form of the curve will suggest that the relation between R and H is of the form

$$R_H = R_0 + bH + cH^2,$$

where R_H is the resistance in a field of strength, H,

R_0 the resistance in zero field,

b, c are constants.

By substituting values for two points, find b and c.

Calculate the expected value of R for other magnetic fields, in the range of the experiment, as far removed from the first two points as convenient, and test the results experimentally.

Determination of the Absolute Resistance of a Metal Rod. (Lorentz's Method)

This method of measuring resistance in absolute units is one in which a steady drop of potential at the ends of a rod, as determined by Ohm's law, is balanced against one set up by induction. The measurements are reduced to those of length and time.

Fig. 363 shows the details. The sketch shows a copper disk, mounted on a horizontal axis about which it may be rotated, either by hand or by a motor. A metal brush, C, makes contact at the rim of the disk, and a second contact to the disk is made at the axle.

At an adjustable distance, d, from the plane of the disk is a coil, D, of N turns of wire, of about the same radius as that of the disk (a cm.).

By regulating the position of the movable base, S, the distance, d, may be made any length within the limits of the length of the base. Fig. 363 also shows, in a diagrammatic form, the connexions used in the experiment. AB is the metal rod, say, of brass. A single accumulator is connected in series with the coil, D, and the resistance, AB.

If R is the resistance of the rod in absolute E.M. units, the drop in potential at the ends of AB is iR, when i is the current in these units.

The rod is also connected, by copper wires, in series with the disk and a galvanometer.

When the current from the accumulator, E, circulates through the coil, a definite number of lines of magnetic force cut the disk. If M is the coefficient of mutual inductance of the disk and the coil, there is

a flux $4Mi$ lines through the disk. If, therefore, the latter is made to rotate, an E.M.F. will be set up in the disk circuit, which will depend on the direction and speed of rotation of the disk.

The direction of rotation is chosen such that the E.M.F. set up by induction opposes the E.M.F., Ri, due to the steady current of the coil circuit. By adjusting the speed of rotation, or by altering the distance, d (usually by both these methods), we may balance the potential due to the two causes. Suppose there are n revolutions per

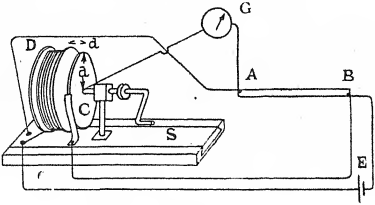


FIG. 363

second when this balance occurs, as shown by no deflection in the sensitive galvanometer, G.

In the disk circuit we have an E.M.F. due to the induction equal to

$$Min,$$

since Mi lines are cut by any radius for one revolution, there are n revolutions per second. Min is the number of lines cut per second, and opposing this is an E.M.F., Ri. For balance

$$Min = Ri,$$

or

$$R = nM.$$

The resistance, R, is therefore determined if M, the coefficient of mutual induction, is known, and n, the number of revolutions per second, is counted.

Maxwell's formula for the coefficient of mutual inductance in the case of two circuits of one turn each, of radius, a, and separated by a distance, d, is

$$m = 4\pi a \left\{ \log_e \frac{8a}{d}\left(1 + \frac{3d^2}{16a^2}\right) - \left(2 + \frac{d^2}{16a^2}\right) \right\}.$$

The coefficient, M, above is thus Nm,

i.e.

$$R = nmN,$$

or

$$R = 4\pi a N n \left\{ 2{\cdot}303 \,.\, \log_{10} \frac{8a}{d} \,.\, \left(1 + \frac{3d^2}{16a^2}\right) - \left(2 + \frac{d^2}{16a^2}\right) \right\}.$$

The resistance in ohms, where the ohm is defined as 10^9 E.M.U., is $R \times 10^{-9}$.

Experimental Details

To avoid the complication of the earth's field in the induction of the E.M.F. in the disk, its plane is turned so that it is in the magnetic meridian.

The Broca, or other sensitive moving-magnet galvanometer, works very well in this experiment.

If no motor is available, a fair result is obtained by rotating the disk by hand. The speed of rotation which can be best maintained steady is found by a trial experiment, and the coil and disk placed at a convenient separation, d. When all is ready, the disk is rotated until the spot of light from the galvanometer is brought back to the zero reading. At this stage the rotation is maintained constant. The number of revolutions made is counted and timed, with a stop-clock, over as long a period as the light spot can be kept at zero.

From this a knowledge of n may be obtained.

d and a are measured in cm., and the value of R calculated.

If the length and cross-section of the rod are measurable, the value of the specific resistance may be also calculated.

The following is an experimental result for a determination made with such an apparatus:

400 revolutions of the disk in 172·5 seconds;

$$\therefore \; n = 2{\cdot}325 \text{ revolutions per second.}$$
$$a = 11{\cdot}8 \text{ cm.,}$$
$$d = 5{\cdot}0 \text{ cm.,}$$
$$N = 100 \text{ cm.}$$

$$R = 100 \times 2{\cdot}325 \times 148{\cdot}1 \left\{2{\cdot}303 \times 1{\cdot}2761 \left(1 + \frac{75}{2227{\cdot}8}\right) - \left(2 + \frac{25}{2227{\cdot}8}\right)\right\}$$
$$= 35372 \text{ absolute E.M. units.}$$

Specific Resistance:

Length of rod $= 16{\cdot}7$ cm., area of cross-section $= \pi(0{\cdot}64)^2$,

$$\text{Specific Resistance} = \frac{35372 \times 1{\cdot}29}{16{\cdot}7}$$
$$= 2732 \text{ E.M. units}$$
$$(\text{or } 0{\cdot}000002732 \text{ ohm}).$$

Conductivity of Salt Solutions (Electrolytes)

When a direct current is passed through an electrolyte, the resulting polarization causes an increase in the resistance of the electrolyte.

Further, for continued passage of current through such a liquid, the resulting decomposition also causes a change in the resistance, due to alteration of the concentration of the solution.

The usual special methods adopted to measure such resistances are designed to overcome these difficulties, and are either a potentiometer method, or one making use of an alternating current.

The second is the one most generally employed. If the alternating current is small, and the area of the electrodes large, the polarization effect is reduced to a negligible amount. This is brought about more completely when rapidly alternating current is used.

Thus, whereas it is impossible to obtain reliable values for the resistance of salt solutions by the Wheatstone bridge method in the

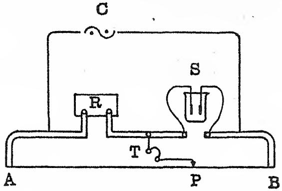


FIG. 364

ordinary way, by using alternating current in conjunction with a Wheatstone net, the value of the resistance of the electrolyte may be found. Of course in such an arrangement an ordinary galvanometer is useless as detector; a telephone replaces it in the usual modification.

It is not advisable to introduce any inductance or capacity into the net, so for this reason the wire bridge is preferable to the Post Office box type, as the one or ten metres of wire have less self-inductance than the coils of the Post Office box.

The scheme of connexions is shown in fig. 364.

AB is a stretched wire of one or ten metres in length,
R an adjustable resistance,
S a vessel containing the solution,
T a telephone,
C a small induction coil.

For a simple experiment the supply may consist of the output of an induction coil giving a low E.M.F.

The telephone, T, should be a head-piece receiver.

The vessel, S, shown in fig. 364, should be surrounded by a water bath, and be provided with platinum electrodes. The temperature of

The resistance in ohms, where the ohm is defined as 10^9 E.M.U., is $R \times 10^{-9}$.

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$= 35372$ absolute E.M. units.

Specific Resistance:

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$$\text{Specific Resistance} = \frac{35372 \times 1{\cdot}29}{16{\cdot}7}$$

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(or $0{\cdot}000002732$ ohm).

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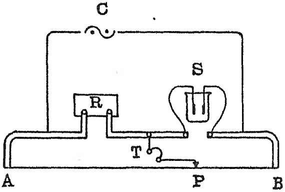

FIG. 364

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The vessel, S, shown in fig. 364, should be surrounded by a water bath, and be provided with platinum electrodes. The temperature of

the water jacket is maintained constant, as the resistance of the solution varies rapidly with temperature.

The platinum electrodes are coated with finely divided platinum to increase their effective area, and to decrease the back electromotive force due to polarization. If this has not already been done, the electrodes should be immersed in a solution made by taking 1 part platinum chloride, 30 parts water, and 0·008 part lead acetate, and a small current passed first in one direction, and then the reverse. To prevent an absorption of salt from the solution, the platinized electrodes are then raised to dull red heat and allowed to cool.

The solution of known concentration is placed in S, and R is adjusted so that a point P is obtained near the centre of the bridge, such that the sound in the telephone is entirely cut out or, as more often happens, until the sound is reduced to a minimum, when the usual Wheatstone result may be applied,

$$\frac{R}{S}=\frac{AP}{PB}.$$

It is usual to express the results in terms of the *specific conductivity* of the solution, i.e. the reciprocal of the resistance of one centimetre, one square centimetre in cross-section.

If r is the resistance in ohms between the electrodes,
s the specific conductivity,

then
$$s \propto \frac{1}{r},$$

or
$$s = \frac{A}{r},$$

where A is a constant depending on the dimensions of the vessel used.

A may be obtained by finding r for a liquid of known specific conductivity. We may take potassium chloride as a standardizing solution, using the data in the table shown on p. 569, where the specific equivalent conductivity is the quotient of the specific conductivity and the number of gramme molecules of the salt per litre.

It will be seen that for a dilute solution this quantity is nearly independent of the concentration.

Make up a solution of KCl containing a definite number of gramme molecules per litre, and find the resistance, r, of the solution when filling the vessel, S. Dilute this solution so that the solution contains half this number of gramme molecules. The value of the specific conductivity for these two strengths may be taken from the table, and the mean value of A calculated.

Then, using this calibrated vessel, find the resistance of several solutions of NaCl from a concentration of, say, 29·25 gm. per litre $\left(\frac{N}{2}\right)$ to 0·2925 gm. per litre, by diluting the concentrated solution first made.

Plot a curve showing the relation between specific conductivity and concentration, and specific equivalent conductivity and concentration.

The variation of the resistance of one of the solutions with temperature may also be investigated by heating the water baths surrounding the cell, S, and the value of the temperature coefficient may be calculated.

Using a container for the solution, of more measurable dimensions, the specific conductivity, and hence equivalent conductivity, may be calculated in the way usually employed in solids. For example, in fig. 365, a uniform tube connects two bottles in each of which is

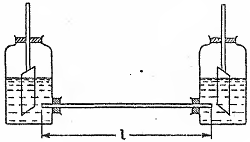

FIG. 365

immersed an electrode; the column of liquid conveying the current is practically one coinciding with the tube, i.e. of length l cm. and cross-section equal to that of the internal cross-section of the tube. The resistance is measured as above; the specific resistance is then directly calculated in the usual way, and the experiment proceeds as already described.

Parts by weight of KCl in 100 parts of solution	Gramme mol. per litre	Specific conductivity at 18°C.	Specific equivalent conductivity
0·00746	0·001	$0{\cdot}125\times10^{-3}$	125×10^{-3}
0·0746	0·01	1·206	120·6
0·743	0·1	11·300	113·0
1·48	0·2	22·00	110·0
3·64	0·5	50·60	101·6
7·43	1·0	99·10	99·1
13·7	2·0	188·8	94·4
19·7	3·0	274·2	91·4

Instead of using a single length, l, of tubing, it is preferable to measure the resistance, R, for various lengths and to plot R against l. Any end correction will then be apparent from the intercept on the axis of R and the resistance per unit length may be obtained from the slope of the graph.

The simple oscillator described above should, when possible, be substituted by a valve oscillator which generates an E.M.F. of a single frequency without harmonics. In addition, the standard resistances used should be pure ohmic resistances.

Under these conditions it will be possible to obtain silence in the telephones at the point of balance. In the case of the induction coil the wave-form is complex, and it is impossible to obtain the condition of complete silence. The balance has to be obtained by reducing the sound in the telephones to a minimum and accuracy is then poor.

CHAPTER XXI

MEASUREMENT OF POTENTIAL

Standard Cells

THE two standard cells in common use are the Weston or Cadmium cell and the Clark cell; of these, the former is better for most purposes.

The two cells have each a constant electromotive force at one temperature; the value of the E.M.F. varies with temperature.

The Weston cell is shown in diagram form in fig. 366.

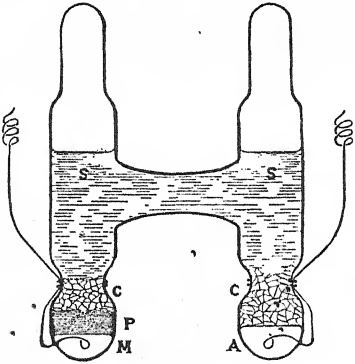

FIG. 366

Two tubes are arranged as shown, each being provided with an external lead which is in contact with the bottom layers. These layers consist of *pure* mercury, M, and an amalgam of pure mercury and cadmium, A, respectively. Above the pure mercury is a layer of a paste of mercurous sulphate, P, made as described later. Above this and the cadmium amalgam is a layer in each tube of pure cadmium sulphate crystals, CC. Finally, a layer of a saturated solution of pure cadmium sulphate occupies the upper parts of the tubes.

To make a Cadmium Cell

The following is a method of construction for the Weston cell; for a more permanent and exact cell the specification given in the Report

of the British Association Meeting, 1905, on p. 98, by F. E. Smith, should be consulted.

(*a*) Mercury is at first obtained in as pure a condition as possible commercially. It is then passed through a dilute solution of nitric acid, drop by drop. To bring this about take a tube of about three-quarters to one inch in diameter, and about 60 to 100 cm. long; draw out the end to a smaller diameter and bend this smaller tube so as to leave a short U-tube at the end; the shorter end is bent over as seen in fig. 367.

A second length of the wide tube is drawn out at one end to a fine capillary of such diameter, that when the tube is filled with mercury

Fig. 367

the latter will just emerge as very small drops. This tube is inserted at the upper end of the first, which is filled with dilute nitric acid. The mercury is collected in the U-tube and passes over into a collecting vessel.

The process is repeated and the partially purified mercury is next distilled under reduced pressure, air being bubbled through it during this process. To carry out this distillation the mercury is placed in a round-bottomed flask, provided with a side tube which leads through a condenser to a second round-bottomed flask, itself connected to a good water filter pump, as shown in fig. 368.

Into the first flask, through a tightly fitting cork, a narrow glass tube passes under the mercury. A clip regulates the inflow of air. When the pump has reduced the pressure inside the system, the mercury is heated on a sand bath, and the condensed mercury is collected in the second flask.

The air is allowed to enter through the mercury as a slow succession of bubbles. This oxidizes such impurities as zinc and minimizes the risk of their distillation with the mercury.

(*b*) The amalgam is made by dissolving pure cadmium in pure mercury, to make a 12 to 13 per cent cadmium amalgam (i.e. about in the proportion of 1 gm. of cadmium to 7 gm. of pure mercury).

(*c*) The paste is made in a mortar by grinding together pure mercurous sulphate, cadmium sulphate, and purified mercury in the proportion of 8 : 4 : 1. The mixture is made into a thin paste by the addition of a solution of cadmium sulphate.

Before filling the cell the platinum wires must be amalgamated. In the form of tube shown in fig. 366, this is done by the electrolysis of mercuric nitrate in the glass tubes, using the wires as electrodes and reversing the current.

Another form of glass container for the cell consists of a double tube with open upper ends. Corks are selected to fit the open ends tightly,

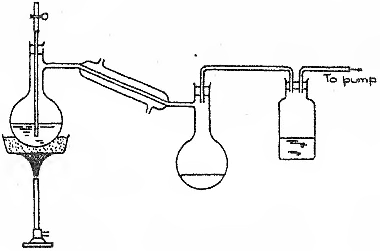

FIG. 368

and through holes in the corks a narrow glass tube may be inserted, one in each tube, to carry the wire through the cell content to the mercury or amalgam.

Such platinum wires can be amalgamated by heating to red heat and dipping in mercury.

Having amalgamated the platinum wires, the tubes are next carefully filled, as shown in fig. 366. If the open-tube type is used, the corks should be finally coated with marine glue or a mixture of beeswax and resin.

The E.M.F. of the cell so formed will be found constant at constant temperature. *On no account must a current of any appreciable magnitude be taken from the cell.*

The International Conference on Electrical Units and Standards, 1908, adopted the following formula as giving most accurately the E.M.F. of the cell:

$$E_t = 1{\cdot}0184 - 4{\cdot}06 \times 10^{-5}\,(t - 20) - 9{\cdot}5 \times 10^{-7}\,(t - 20)^2 + 10^{-8}\,(t - 20)^3 \text{ volt}, \quad \ldots(1)$$

where t is expressed in degrees centigrade.

The temperature coefficient is therefore small.

The Clark Cell

This is made in a manner identical with that described above, with the exception that cadmium is replaced in this case by zinc, cadmium sulphate by zinc sulphate, etc. Proceeding as above, again using pure salts and mercury, the standard cell so constructed has an electromotive force expressed by the formula:

$$E_t = 1{\cdot}4328 - 1{\cdot}19 \times 10^{-3}\,(t - 15) - 7 \times 10^{-6}\,(t - 15)^2. \quad \ldots(2)$$

Thus this cell has a larger temperature coefficient than the Weston, a fact which explains the more general use of the latter. When using either of these standard cells in potentiometer work, a large resistance, represented by r in fig. 337, should be in series until the point of balance is nearly reached. The key, K_3, can then be closed and the exact balance point obtained. This precaution is necessary, for if these *cells supply more than a small current they are subject to polarization* and the value of the E.M.F. becomes uncertain.

Comparison of Electromotive Force

The potentiometer method of comparing two electromotive forces is the most satisfactory one. It is assumed that the reader is familiar with the direct comparison of two E.M.F.s, using a stretched wire potentiometer. In this method a steady accumulator is arranged as at E in fig. 369, and E_1 and E_2 whose E.M.F.s are to be compared are in

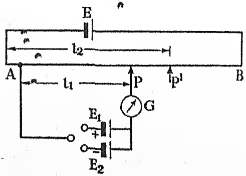

FIG. 369

turn placed in series with the galvanometer, and a point of balance obtained at l_1 and l_2 cm. respectively,* then, if the wire is of uniform resistance,

$$\frac{E_1}{E_2} = \frac{l_1}{l_2}.$$

Using one of the methods given on p. 533, sqq., the wire, AB, can

* l_1 is the mean of, say, three observations, and l_2 the mean of two observations taken alternately; this eliminates the effect of the variation of E.M.F. of E.

be calibrated and then, if l_1' and l_2' are the corrected lengths corresponding to l_1 and l_2,

$$\frac{E_1}{E_2} = \frac{l_1'}{l_2'}.$$

For the actual value of the E.M.F. of either cell a third balance can be obtained at, say, l_3 for a cadmium cell whose E.M.F. is known at the temperature of the experiment. Whence E_1 and E_2 may be obtained in volts.

For making such a comparison, the accuracy of the determination depends on the accuracy of obtaining the balance point. If instead of using a 1-m. potentiometer, a wire of 10 m. be used, then each cm. of wire has a potential drop equal to one-tenth the drop in the simpler potentiometer, i.e. a movement of 1 mm. in the single wire bridge would correspond to 1 cm. movement in the 10-m. instrument. Hence, by using the 10-m. potentiometer the true balance point may be more nearly estimated.

It is often convenient to make the potentiometer a direct reading instrument. To do this we arrange that between the ends of a definite length there is a fixed potential difference, say, 10^{-3} volts.

Using a 10-m. potentiometer, the most convenient length to employ to correspond to 10^{-3} volts is 5 mm. This is brought about as follows:

A standard cadmium cell (E.M.F. = 1·0184 volts at 20° C.) is connected to A, and the jockey makes contact with the wire at 1018·4 units of length from the end.

We have chosen the unit for this purpose as 5 mm., i.e. P is fixed at 509·2 cm. from A, as in fig. 370. R is now adjusted until the galvanometer gives no deflection, i.e. the current in the wire AB, due to E, is such as to cause 10^{-3} volt drop per 5 mm. of the wire (assumed uniform). Leaving R fixed, any other E.M.F. may be found by balancing on the potentiometer at, say, l cm. from A or $2l \times 5$ mm., i.e. $2l \times 10^{-3}$ volts is the value of the balanced E.M.F. The above assumes

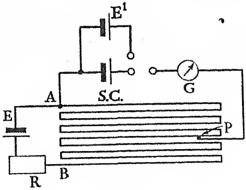

FIG. 370

that the accumulator E remains steady. This should be checked at intervals, if a series of comparisons are to be made, by reinserting the standard cell, SC, and adjusting R to effect a balance at 509·2 cm.

For many purposes it is convenient to replace the wire by variable standard resistances when comparing potentials. A suitable arrangement of apparatus using such a method is seen in fig. 371. R_1 and

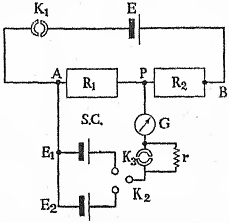

FIG. 371

R_2 are two resistance boxes, each having a resistance up to, say, 10,000 ohms, e.g. two Post Office boxes.

E is a steady accumulator which is connected in series with R_1 and R_2. The cells to be compared are connected to A and P through a galvanometer, G.

For direct comparison the resistance $(R_1 + R_2)$ is kept constant, say 10,000 ohms, and R_1 and R_2 are varied until no deflection is obtained when K_2 is closed. The value of R_1 is noted.

The process is repeated with the second cell, say, a cadmium cell, a balance being obtained for a resistance R_1' in the box between AP.

Then, as before
$$\frac{E_1}{E_2} = \frac{R_1}{R_1'},$$
for the drop in potential between AP due to E is
$$\frac{ER_1}{R_1 + R_2 + B},$$
where B is the resistance of the cell E. When a balance is obtained
$$E_1 = \frac{R_1}{R_1 + R_2 + B} \cdot E;$$
similarly,
$$E_2 = \frac{R_1'}{R_1' + R_2' + B} \cdot E,$$
and
$$R_1 + R_2 = R_1' + R_2'.$$
Hence
$$\frac{E_1}{E_2} = \frac{R_1}{R_1'}.$$

Another way of using the above form of potentiometer is similar to the direct-reading method of using the wire potentiometer.

One of the cells to be balanced against E is a cadmium cell. If the temperature of the experiment is 20° C., the electromotive force of the cell is 1·0184 volts.

R_1 is given the value 1018·4 ohms. Knowing the approximate electromotive force of the cell, E, the value of R_2 may be estimated such that the potential drop in AP $\simeq$ 1·0184 volts. The standard cell is placed in series with the galvanometer G; K_3 being open, r, of about 10^4 ohms is placed in series with SC, to avoid damaging it during the preliminary balancing. R_2 is now adjusted until no deflection is noted in G. K_3 is closed and the final balance verified. The total value of $R_1 + R_2$ under these conditions is noted and maintained constant throughout the comparison.

Now 1018·4 ohms have a drop of potential of 1·0184 volts, i.e. each ohm corresponds to a potential drop of 10^{-3} volts.

Therefore, when a second cell, E_1, is introduced, if R_1 has a new value, R_1' ohms at balance ($(R_1'+R_2')$ being equal to (R_1+R_2) as obtained in the first test), E_1 is $R_1' \times 10^{-3}$ volts.

The above methods of making the potentiometer direct reading are only suggested for those cases where several comparisons are to be made, for under such circumstances subsequent calculation is eliminated.

The experimental arrangements described above are most sensitive for comparison of electromotive forces of the order of one volt. If now a small difference of potential is to be determined it will be apparent that these arrangements are not sufficiently sensitive. In the case of a stretched wire, the sensitivity increases with increase in length; therefore, to measure a potential difference of the order of, say, 10^{-3} volts with fair accuracy, the potentiometer wire would require an extension of several metres of wire, or, what is more convenient, the inclusion in the circuit of a resistance several times that of the wire. For example, if a potentiometer wire of 1 m. is of 1 ohm resistance, and the accumulator E has an E.M.F. of 2 volts, a potential difference of the order of 10 millivolts will balance at a distance of 5 mm. from the end of the wire. A small error in reading 5 mm. could easily lead to an error of several per cent. Further, under such circumstances the calibration of the wire would be of great importance.

If on the other hand 99 ohms be placed in series with the wire, the drop of potential in the wire will be 1/100 of 2 volts, and the true balance point will therefore be at 50 cm. An error in estimating the balance point of 1 mm. will only be 2 per cent compared with 20 per cent in the first case.

An example of the application of this method is seen in fig. 377, p. 582.

Direct Reading Potentiometers

As described above, the potentiometer may be made 'direct reading'. Several forms of potentiometer are now on the market which are constructed on the above principles and are calibrated directly in volts.

Of the simpler forms of compact manufactured instruments, which measure potential of the order of a volt to one millivolt, we will describe the form illustrated in fig. 372 (at p. 1). The internal wiring is indicated by white lines drawn on the case, and the instrument is an application of the form shown in fig. 370. The 2-volt accumulator which is connected to the terminals, EF, sends a current through an adjustable rheostat, a series of coils provided with tappings to the studs shown, and through the slide wire. This constitutes the main circuit.

When the current is adjusted to the correct amount, the fall in potential along each of the resistances, connected to the studs, is 0·1 volt, and along the slide wire, 0·12 volt.

The wire is divided into 120 parts, each corresponding in adjustment to a fall of potential of 10^{-3} volts.

To adjust the current to this strength, a standard cell is inserted in the gap marked 'potential', and a galvanometer of low resistance in the gap marked 'galv.'. The two adjustable contacts are set at points which correspond to the true E.M.F. of the cell at the temperature of the room, e.g. if the E.M.F. of the cell is 1·018, the sliding contact at the back of the instrument is set at 1·0 and the contact on the wire made at 0·018.

The rheostat is then adjusted so that when the key is closed no current is indicated in the galvanometer.

If the rheostat is not sufficiently large to bring this about the battery should be connected to E and G, not EF. This introduces more resistance in the circuit, as indicated in the figure, and, using a 2-volt accumulator, the balance for the standard cell is attained.

To obtain the value of an unknown electromotive force, it is connected to replace the standard cell. Leaving the rheostat in the balanced position, the sliding contacts are adjusted until a balance is obtained, as indicated by zero deflection of the galvanometer. The value of the electromotive force is then directly obtained on the calibrated scales.

The width of the small divisions is sufficiently large to allow eye estimation to one-fifth of a division, but is not suitable for the measurement of potentials of the order of one-fifth of a millivolt. Thus, whereas the instrument would measure a potential of 1·0182 with a good degree of accuracy, it cannot be used to measure 0·0002 volt with any certainty, nor is it reasonable to expect it with an instrument having a range 0 to 1·5 volts.

For such measurements as that of thermo E.M.F.s an instrument having a range of 0 to 50 millivolts is more suitable.

Potentiometers with this range are manufactured by many firms, e.g. Nalder, Cambridge and Paul, Crompton, Gambrell.

Figs. 373 (at p. 1) and 374 show the general appearance and internal arrangements of a typical instrument.

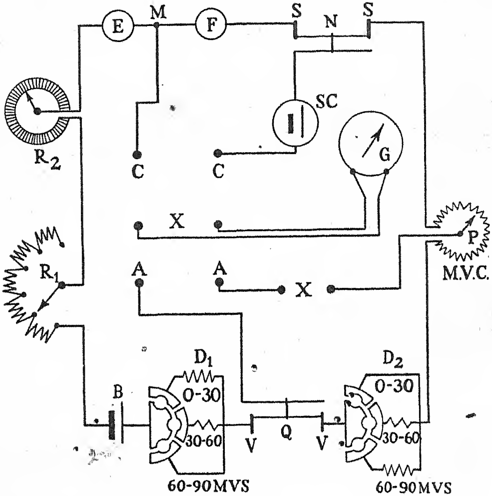

FIG. 374

The current to the main circuit is supplied by a 2-volt accumulator, B. As seen, this circuit consists of adjustable resistances, R_1 and R_2, fixed resistances, E and F, MVC, D_1 and D_2, and the two stretched wires, SS and VV.

The range of the instrument is 30 millivolts (0 to 30, 30 to 60, or 60 to 90). The resistance MVC is made up of 29 similar coils of such a resistance that when the current is adjusted as described below, the potential difference between the ends of each coil is 1 millivolt. Also the wire, VV, is of a resistance such that for this adjustment the drop of potential along its length is 1·2 millivolts. By subdivision, a value of 0·005 millivolt may be obtained.

Further, the value of the resistance, F, and that of the wire, SS, is such that the graduation along SS gives the potential between M and N when adjusted.

To standardize the potentiometer, the standard cell, SC, is connected to the galvanometer, G, by means of the double pole switch. The point, N, is chosen equal to the potential of the standard cell, R_1 and R_2 are then adjusted until the low-resistance galvanometer shows no deflection, i.e. the drop of potential along MN corresponds to the graduation value. Under such circumstances the potential difference per coil of MVC is 1 millivolt.

To measure an unknown electromotive force between 0 and 30 millivolts, plugs are inserted at the points shown in fig. 374, and the unknown potential connected to X. The commutator is thrown over so that XX are connected to the galvanometer, and a balance is obtained by varying the point of contact, P, on MVC and Q on VV. If D_2 is of zero resistance the unknown potential corresponds to that between P and Q, and is therefore obtained directly from the scale.

Suppose a bigger potential, say, 30 to 60 is to be measured, D_2 is now made of such resistance that there is a potential drop equal to 30 millivolts along its length, and therefore for the balanced position the potential is 30 + the readings of MVC and VV.

Similarly for 60 to 90 millivolts. D_2 is increased so that for the steady current in the main circuit the potential difference between the ends of D_2 = 60 millivolts.

To maintain the current in the main circuit at a fixed value, the resistance of D_1 must be decreased by the same amount as the increase in resistance of D_2.

Thermo-Electricity—Thermo-Junctions

When a circuit is composed of two dissimilar metals and the junctions of these metals are maintained at different temperatures, an E.M.F. is set up in the circuit. This electromotive force varies with the difference in temperature between the junctions, and when one junction is maintained at 0° C. is given by

$$E_t = at + bt^2,$$

where a and b are constants and t expresses the temperature of the hot junction in degrees centigrade.

The direction of the electromotive force depends on the metals.

It is customary to express E_t for any metal with respect to a standard metal which is taken as one of the pair. The usual choice of standard metal is lead.

In drawing the curve giving the relation between the E.M.F. and temperature, the E.M.F. is taken as positive when the current tends to flow from lead to the metal at the hot junction.

Thus fig. 375 shows the form of these curves for Pb/Fe and Pb/Cu. At a temperature $t°$ C., AB represents the E.M.F. developed in a Pb/Cu junction and, according to the above rule, the electromotive force is from the lead to the metal at the hot junction. Similarly, AC is the

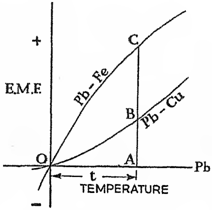

FIG. 375

magnitude of the electromotive force developed from lead to iron at the hot junction.

The law of intermediate metals may now be applied to determine the value of the E.M.F. developed at a copper-iron junction at a temperature $t°$ C., the other junction being maintained at 0° C., for,

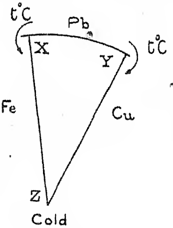

FIG. 376

according to this law the E.M.F. developed at the temperature $t°$, for a Cu/Fe junction, is expressed by

$$\begin{aligned} \text{E.M.F. Cu/Fe} &= \text{E.M.F.(Cu/Pb + Pb/Fe)} \\ &= -\text{AB} + \text{AC} \\ &= +\text{BC}, \end{aligned}$$

i.e. there is an electromotive force of magnitude BC *from copper to iron at the hot junction* since BC is positive.

Reference to fig. 376 will show this from another point of view. If junctions X and Y, are maintained at temperature, $t°$, Z being maintained at 0°C., the arrows show the direction of the E.M.F.s of magnitude AC at X counter-clockwise and AB at Y clockwise—a net result of BC counter-clockwise. This, according to the law of intermediate metals, is the value of the E.M.F. if XY are brought together, the lead being removed. The result is an E.M.F., BC, in a counter-clockwise direction, i.e. from the copper to the iron at the hot junction.

Experimental Determination of the Thermo-Electromotive Force—Temperature Diagram

The *magnitude of the thermo-electric* E.M.F. *is of the order of a* few millivolts. It is best investigated by means of a potentiometer of the form described on p. 576. An instructive result is obtained, using

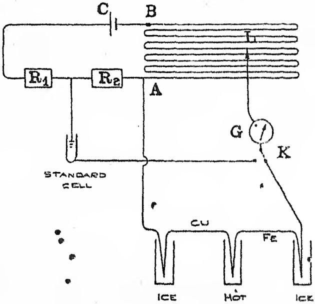

FIG. 377

the three metals, copper, iron, and lead. As will be seen from the above, it is only necessary to obtain the E.M.F. for two pairs of the metals, that of the third pair may be estimated, making use of the law of intermediate metals.

A uniform wire potentiometer of, say, ten metres is employed (one of one or two metres, if the other form is not available). If the best results are to be obtained a preliminary calibration of the wire is advised.

A steady accumulator, C, is joined in series with R_1, R_2, and the potentiometer wire (R_1 and R_2 should contain resistances up to 10,000 ohms). A preliminary experiment gives the value of r the resistance of the ten metres of wire.

A thermo-junction is constructed, using copper and iron wire. Care must, of course, be taken that the wires are in contact at the junction only. For this purpose a suitable form of junction is seen in fig. 378. One wire, B, passes down a thin glass tube, G, the other, A, is joined to it at J. To ensure good contact, J is dipped into mercury at the bottom of the test-tube, T. This does not affect the E.M.F. If one or both of the metals are affected chemically by mercury, it must be dispensed with, and special care paid to the welding of the junctions.

FIG. 378

Three of these junctions are required, one for copper and iron and two for the leads to the copper and iron as shown in fig. 377. The latter are maintained at the same temperature, 0° C., in the experiment described. The use of such junctions is to eliminate thermal electromotive forces between the metal of the connecting leads and the metals of the thermo-junction proper.

If the junctions to the connecting wires are maintained at 0° C. by surrounding with ice, and the hot junction is placed in a water bath at a temperature t° C., an E.M.F. will be developed in the direction A to L via the junctions. The positive pole of C is therefore connected to B.

The maximum E.M.F. developed in the above couple is of the order of 1500 microvolts: the rise in potential along AB should therefore be arranged not very much in excess of this. A voltmeter gives the approximate potential available from C, and r being known, the value of $R_1 + R_2$ to cause such a rise may be calculated,

$$\frac{\text{potential difference in AB}}{\text{E.M.F. of C}} = \frac{r}{r + R_1 + R_2},$$

the accumulator resistance being negligibly small.

Having fixed R_1 and R_2, the point, L, is made to coincide with A, and the standard cell, say a cadmium cell, is put in series with the galvanometer; R_2 is adjusted, keeping $R_1 + R_2$ at the value determined above, until no deflection is given in G when contact is made at A.

If the room temperature is 20° C. the E.M.F. of the cadmium cell is 1·0184. Hence the drop of potential along BA is

$$\frac{r}{R_2} \times 1{\cdot}0184,$$

whence the drop per cm. of the wire is

$$\frac{r}{R_2} \times 1{\cdot}0184 \times 10^{-3} \text{ volt.} \qquad \ldots(3)$$

The thermo-junction is now placed in circuit and the hot junction is raised to about 95° C. in a water bath: the length of wire required

for balance is obtained. The water is allowed to cool and balance points obtained for intervals of temperature of about 5° C.

The hot junction is placed in a hypsometer. When the junction is at the temperature of the steam, a steady balance point is obtained and noted.

The hot junction test-tube is next placed in a boiling tube containing mercury. This is heated slowly. Balance points are obtained at intervals and the temperature noted on a special mercury thermometer which reads to 360° C. The process is carried on until the mercury boils.

The lengths are converted to E.M.F. by multiplying by the reduction factor given in (3) above, and a graph relating temperature and E.M.F. is drawn.

The process is repeated, using a lead wire in place of the iron, and the results are plotted as in fig. 375, E.M.F. in microvolts (10^{-6} volts) against the temperature of the hot junction.

From these results obtain the corresponding curve for lead-iron.

It will be found that at a temperature of about 240° C. the Fe/Cu junction will give a maximum E.M.F. This temperature is called the neutral point.

The same results could be obtained directly, using a direct reading potentiometer. With a 10-m. instrument r will probably be just greater than 10 ohms. If a shunt, S, be placed between A and B (fig. 377) and S given a suitable value which can be calculated from a knowledge of r, the wire and shunt may be made to have exactly 10 ohms resistance.

As before, suppose the E.M.F. of the cadmium cell to be 1·0184 volts at the temperature of the experiment. Then to obtain a drop of 10^{-6} volts per mm. of bridge wire, R_2 is given the value (1018·4 —10) 1008·4 ohms. The cadmium cell is placed in series with the galvanometer, and the sliding contact, L, is made at B (fig. 377). R_1 is adjusted until a balance is obtained. Then the drop in potential from B to A $= \dfrac{10}{1018 \cdot 4}$ of 1·0184 or 10^{-2} volts, hence the drop per mm. is 10^{-6} volts.

The lengths for balance in the thermo-couple experiment now give the potential in microvolts directly.

The same process applies in the case of a single-metre potentiometer.

If no standard cell is available the value of the potential per cm. of wire may be calculated from a knowledge of r, R_1, and R_2 and E, the E.M.F. of the cell, C, as determined by means of a high-resistance voltmeter, but this method is not recommended.

Consider the curves obtained. If we take any two fixed temperatures and determine from the curve the value of the E.M.F. developed, in microvolts, we shall be provided with two equations having two

unknown constants, for we have seen that the relation between the E.M.F. developed E, and the temperature $t°$ C. is

$$\mathrm{E} = at + bt^2.$$

So by choosing two such pairs of temperatures and finding E on the lead-copper and lead-iron curve, the values of a and b for these junctions may be obtained.

Hence, since $$\frac{d\mathrm{E}}{dt} = a + 2bt,$$

we know the value of the thermo-electric power, $\frac{d\mathrm{E}}{dt}$, at any temperature.

The thermo-electric power lines for Cu and Fe against the standard, lead, may thus be drawn and the point of intersection, which gives the neutral temperature for iron-copper, may be ascertained. This should be compared with the neutral temperature as found directly from the E.M.F. temperature curve for the iron-copper junction.

The Use of a Thermo-Electric Couple as a Thermometer

It will be seen from the curves obtained (fig. 375) that a couple such as copper-iron is not a suitable one to use as a thermometer. The neutral point is too near 0° C. The ideal couple for this purpose is one with a neutral point well removed from 0° C., and which will therefore give an E.M.F. approximately proportional to the temperature difference. In many experiments in this book a couple of this sort is required. A couple of copper and eureka or constantan serves well for this purpose.

When the thermo-junction is used in this way the form of potentiometer described on p. 578, fig. 373, is convenient, but the form of direct reading potentiometer of ten metres of wire (p. 582) when once adjusted serves quite well.

In either case the electromotive force-temperature curve of the junction should be obtained as described up to, say, 240° C. The value of the temperature corresponding to any other electromotive force may then be obtained from the standardizing curve so obtained.

Lippmann's Capillary Electrometer

The essential feature of the Lippmann capillary electrometer is a very fine capillary tube drawn from a thin glass tube, mounted so that a column of mercury of variable height may be placed over the meniscus of a liquid which rises in the capillary.

Fig. 379 shows how this is usually arranged: C is the capillary attached to a vertical glass tube, AB, by means of a short length of pressure tube at B. The side tube is connected by pressure tubing to an adjustable reservoir, R, full of mercury. A scale in millimetres is

placed behind either AB or the reservoir R, so that changes in level of the mercury may be measured.

The capillary tube, which dips into sulphuric acid, should be of sufficiently small diameter to prevent the acid from being driven from the tube by the mercury above it.

At the bottom of the beaker, E, which contains the acid, is a layer of mercury, into which a wire, passing down the centre of a glass tube, D, may be placed.

A second copper wire dips into the mercury above the meniscus, either by means of a platinum wire lead fused into the tube AB, or

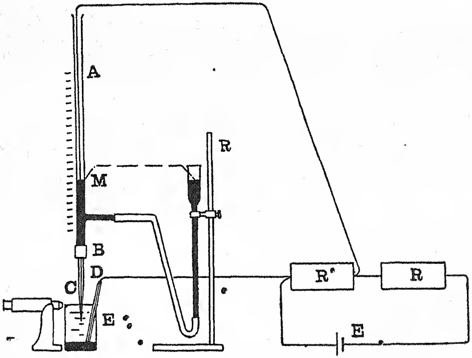

FIG. 379

by inserting a wire into the mercury in AB or R. The two leads are connected to a potentiometer which consists of two Post Office boxes in series, R_1 and R_2, so that, maintaining $R_1 + R_2$ at 10,000 ohms, and adjusting R_1 and R_2, any fraction of the potential of the cell, E, may be applied to the junction of the upper mercury and the acid.

If the meniscus is focused so that its image lies on the horizontal cross-hair in the focal plane of a microscope, it will be found that when a potential is applied in one direction, the meniscus descends, and when applied in the other direction the meniscus ascends. The effective value of the surface tension of the acid-mercury surface is altered by the applied voltage.

Suppose the reservoir, R, is lowered so that only a small pressure is applied to the surface, and the meniscus is focused on the cross-hair of the observing microscope. At the junction of two liquids a contact

difference of potential is set up, and the value of the effective surface tension depends on this contact potential. In whichever direction this contact potential acts, the result is a decrease in the effective surface tension. For two given liquids, e.g. sulphuric acid and mercury, the value is fixed in direction and magnitude.

We may regard the surface of separation of the liquids as a double layer functioning like a condenser, and having an energy per unit area of $\frac{1}{2}cV^2$, where c is the capacity and V the value of the contact potential.

Thus, if S is the effective surface energy, and S_0 the value of the surface energy if no potential exists, we may write

$$S = S_0 - \tfrac{1}{2}cV^2 \qquad \ldots(4)$$

so that whatever the sign be of V, the value of $S < S_0$.

If a potential be applied to the surface of separation in the same direction as the contact potential, then V increases and S becomes less, causing the column of acid to descend to a new equilibrium position, whereas, if the applied potential be of the opposite sign to the contact potential, we obtain a larger value for S, and consequently the acid rises in the tube.

The capillary when drawn out from the glass tube is always slightly conical, and the state of things showing two positions is seen in fig. 380.

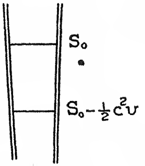

FIG. 380

If the applied E.M.F. opposing the contact potential increases, S becomes larger, until, when the applied potential is further increased, the double layer effect acts with a total potential difference of the opposite sign.

For each value of the applied opposing E.M.F. the meniscus will take up a definite position, and it is obvious that when the contact potential is just neutralized, the meniscus will be at the highest point.

The conical shape of the capillary tube, with resultant change of diameter and of focus, renders unreliable the observations of the level of the meniscus. To be sure that the diameter of the tube at which the surface of separation lies is the same, the meniscus is always observed at one point.

For example, as the value of S increases, the level is brought back to the previous one by raising the reservoir, R, and increasing the pressure on the surface. The reservoir should be readily adjustable, so that the level of the mercury in R or AB may be read for each value of the applied potential.

This process should be repeated for all values between 0 and 2 volts, or until electrolysis interferes with observation.

The values of the applied E.M.F. should be plotted as abscissae, and the pressure in cm. of mercury as ordinates.

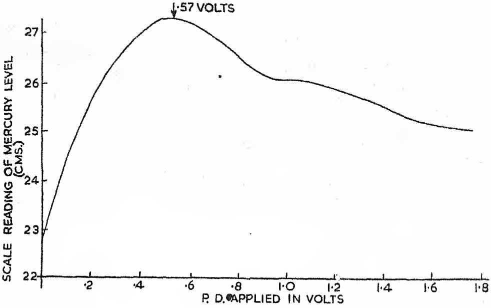

FIG. 381

The form of the curve as obtained from the results below is seen in fig. 381. The maximum of the applied pressure corresponds to a maximum, S, and corresponds to the case when the total potential at the surface is zero, i.e. it corresponds to an applied potential equal to the contact potential. From a knowledge of the direction of the applied potential, we may say at once which of the liquids is electro-positive and which electro-negative to the other.

In performing the experiment the first thing to do, having assembled the rest of the apparatus, is to draw out a suitable capillary. After one or two attempts a suitable one will be obtained, which is of sufficiently small dimensions to support the pressure.

It should be noted that the capillary tube should be drawn from a clean tube. This may be obtained by boiling the tube in nitric acid and rinsing in tap-water, or by leaving the tubes to be drawn in a solution of potassium bichromate and sulphuric acid for 12 to 24 hours and then rinsing in tap-water.

The mercury also should be cleaned, preferably redistilled.

The following is a record of an experiment where the above precautions were observed.

$$R_1 + R_2 = 10{,}000 \text{ ohms.}$$

R_1	Applied potential	Applied pressure
ohms	*volts*	*cm. Hg*
0	0·0	22·7
1000	0·2	25·3
2000	0·4	26·8
3000	0·6	27·2
4000	0·8	26·7
5000	1·0	25·9
6000	1·2	25·8
7000	1·4	25·4
8000	1·6	24·9
9000	1·8	24·7

Contact potential from curve = 0·57 volt (fig. 381). The negative terminal was connected to the upper mercury, i.e. in the experiment the mercury was electro-positive to the sulphuric acid.

CHAPTER XXII

MEASUREMENT OF CAPACITY AND INDUCTANCE

WHILE the methods of measuring inductances and capacities developed in recent years involving the use of alternating current and telephones or vibration galvanometers as detectors give the more accurate results, the older methods which use direct current and galvanometers are useful in some cases and serve to illustrate the physical principles involved.

For instructional purposes some of the earlier methods are set out in the following pages. These are followed by the newer methods which must be used when an accurate determination of the value of an inductance or capacity is the purpose of the experiment.

MEASUREMENTS USING DIRECT CURRENT

(a) Comparison of the Capacities of Condensers

(1) Deflection Method

Suppose C_1 and C_2 are the condensers, of capacity C_1 and C_2, and that to each is imparted a difference of potential equal to E volts. Let Q_1 and Q_2 be the respective charges on the plates.

Then we have
$$C_1 = \frac{Q_1}{E}, \quad C_2 = \frac{Q_2}{E},$$

or
$$\frac{C_1}{C_2} = \frac{Q_1}{Q_2}. \qquad \ldots(1)$$

In the experiment described below the quantities, Q_1 and Q_2, are compared by discharging the condensers in turn through a moving-coil ballistic galvanometer. In such a case, if θ_1 and θ_2 are the angles of the first throw of the ballistic galvanometer, corresponding to movement of the spot of light, d_1 and d_2 cm., from the zero on the scale, we have (see p. 517):

$$Q_1 = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_1}{2}\left(1 + \frac{\lambda}{2}\right),$$

$$Q_2 = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_2}{2}\left(1 + \frac{\lambda}{2}\right),$$

or
$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2}.$$

If the deflections are small:

$$\frac{\theta_1}{\theta_2} = \frac{2 \tan \theta_1}{2 \tan \theta_2} = \frac{\tan 2\theta_1}{\tan 2\theta_2} = \frac{d_1}{d_2},$$

whence $$\frac{Q_1}{Q_2} = \frac{d_1}{d_2}.$$

Thus from (1) $$\frac{C_1}{C_2} = \frac{d_1}{d_2}. \qquad \dots(2)$$

If the deflections are not small this approximation cannot be used. The values of θ_1 and θ_2 may easily be obtained, for $\tan 2\theta_1 = \frac{d_1}{L}$, where L is the distance from the scale to the mirror of the galvanometer; hence $2\theta_1$ and θ_1.

It often happens with a steady source of potential, say, a steady 2-volt accumulator, that the throws θ_1 and θ_2 are too large, i.e. E is too great. Under such circumstances, instead of using E directly we may take any fraction of E by a potentiometer method.

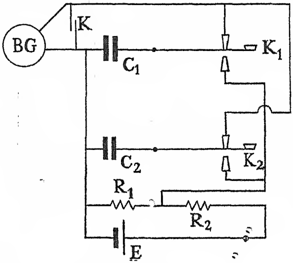

FIG. 382

Connect E in series with two resistance boxes, R_1 and R_2, making $R_1 + R_2 = 10{,}000$ ohms. By adjusting R_1, making $R_1 + R_2 = 10{,}000$, we may obtain a suitable fraction of E, for the potential drop through the resistance, R_1, is $\frac{R_1}{R_1 + R_2} \cdot E$. These resistances are then kept constant and the throws are obtained.

The scheme of connexions is shown in fig. 382. K_1 and K_2 are condenser keys. K is a single-way tapping key, useful in bringing the galvanometer to rest after observing the swings. BG a ballistic galvanometer of the moving-coil type.

K_1 is depressed and the condenser, C_1, is charged. K_1 is now raised. The charge, Q_1, is thus sent through BG, and θ_1 noted.

As soon as the galvanometer is brought to rest, the key K_2 is depressed for the same short time and then raised, θ_2 being noted. During the

interval of the two depressions of the keys, the potential difference between the poles of the accumulator will remain sensibly constant.

Of course the experiment could be done using one condenser key, and placing C_1 and C_2 in turn in the single circuit.

The observations should be repeated with C_1 and C_2 alternately. The mean value of four observations with C_1 and three with C_2 should be taken to eliminate any steady change in E.

(2) Null Method of de Sauty

For this method the two condensers, C_1 and C_2, are arranged as two arms in a Wheatstone net; R_1 and R_2, two adjustable *high* resistances

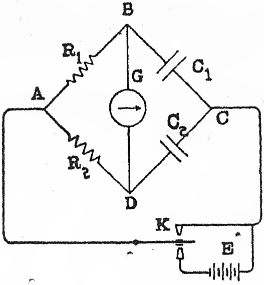

FIG. 383

making the bridge complete. Fig. 383 gives the scheme of connexions. G is a high-resistance galvanometer and E a battery of cells having an E.M.F. of several volts.

The key, K, may be an ordinary condenser key, so that when depressed the condensers are charged, and when raised the condensers are discharged. The values of R_1 and R_2 are so arranged, that when the key K is moved up and down, charging and discharging the condensers, there is no movement of the galvanometer coil, i.e. in both processes the potential at B is the same as at D.

Let E' be the potential at A, and

v_1 be the potential at B at a time, t,

v_2 be the potential at D at a time, t.

The drop in potential along R_1 is $(E' - v_1)$ and along R_2 is $(E' - v_2)$.

Thus, the currents in R_1 and R_2 are $\dfrac{E' - v_1}{R_1}$ and $\dfrac{E' - v_2}{R_2}$.

In a small interval of time, dt, the quantity of electricity passing along R_1 is $\left(\frac{E' - v_1}{R_1}\right) dt$, and along R_2 is $\left(\frac{E' - v_2}{R_2}\right) dt$. In charging the condensers the quantity, $\int \frac{E' - v_1}{R_1} . dt$, flows to C_1 and $\int \frac{E' - v_2}{R_2} . dt$ to C_2.

Now when the condensers are charged, since the potential difference is E (that of the cells) we have:

Charge on C_1 is $$C_1E = \int \frac{E' - v_1}{R_1} . dt,$$

and similarly $$C_2E = \int \frac{E' - v_2}{R_2} . dt,$$

where C_1 and C_2 are the capacitances of the two condensers;

or $$C_1R_1E = \int (E' - v_1)\ dt,$$
$$C_2R_2E = \int (E' - v_2)\ dt,$$

when no flow occurs through the galvanometer, i.e. $v_1 = v_2$ throughout the charging,

i.e. $$\int (E' - v_1)\ dt = \int (E' - v_2)\ dt;$$

or $$C_1R_1E = C_2R_2E,$$

i.e. $$\frac{C_1}{C_2} = \frac{R_2}{R_1}. \qquad \ldots(3)$$

In performing the experiment R_1 and R_2 may be obtained sufficiently high by using a Post Office box for each, and following the scheme of connexions shown.

The method can also be performed using A.C., as described on p. 612.

(3) Method of Mixtures

Comparison of the capacity of two condensers may be made in one or two ways using the 'mixture' method. In this particular method the condensers are arranged to have equal charges and the potential required for this is measured or compared by a potentiometer method. The theory of the method presents no special points which require separate treatment.

A battery of cells (two accumulators) is connected, as in fig. 384, to send a current, i, through two high resistances, R_1 and R_2, which are adjustable and connected together at a point, B.

The potentials between A and B, B and C, are proportional to R_1 and R_2.

B is connected, as shown, to the two condensers whose other plates are connected to the central cups of a cleaned ebonite Pohl commutator from which the cross-connexions have been removed (or a slab of clean paraffin wax with six holes full of mercury will serve).

A galvanometer, G, is connected to the cups, X, X^1, of the commutator.

A and C are joined to Z and Z^1.

When connexion is made (by the rocker of the Pohl commutator) between Y and Z, Y^1 and Z^1, the condensers, C_1 and C_2, are charged to the potential differences of AB and BC respectively.

If Q_1 and Q_2 are the charges, and C_1 and C_2 the capacities, we have

$$C_1 = \frac{Q_1}{R_1 i}, \quad C_2 = \frac{Q_2}{R_2 i}, \qquad \ldots(4)$$

i being the current through R_1 and R_2.

When the switch is thrown over so that Y and X, Y^1 and X^1 are connected, the condensers are discharged through G.

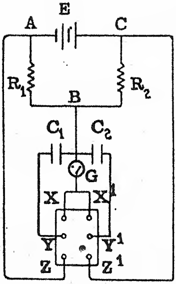

FIG. 384

Suppose the current flows in the direction, ABC, in the potentiometer circuit, then the inner plate of C_1 is at a lower potential than the outer, when charged, whereas for C_2 the inner plate is at a higher potential than the outer. When the condensers discharge through G, C_2 will send its charge in opposition to C_1.

The above process is repeated, altering R_1 and R_2 between observations, until finally a value of R_1 and R_2 is obtained, such that on discharging, no current passes through the galvanometer. Under such circumstances, $Q_1 = Q_2$.

From equation (4) above we have

$$Q_1 = C_1 R_1 i, \quad Q_2 = C_2 R_2 i.$$

In the adjusted position, since $Q_1 = Q_2$,

$$C_1 R_1 i = C_2 R_2 i,$$

or

$$\frac{C_1}{C_2} = \frac{R_2}{R_1}.$$

Determination of the Absolute Capacity of a Condenser

The value of the capacity of a condenser may be determined in any system of units if a measured potential, E, in these units, applied to the condenser, imparts Q units of charge in the same units; for the capacity, C, of the condenser is defined by $Q = CE$,

or
$$C = \frac{Q}{E}. \qquad \ldots(5)$$

For example, if Q is expressed in coulombs and E in volts, K, the capacity obtained in the equation above, is expressed in farads.

The usual method of finding C experimentally is to apply a known potential (in volts) to the condenser and measure the charge, Q, by discharging the condenser through a ballistic galvanometer.

The moving-coil ballistic galvanometer is the best form to use, as in most capacity experiments, whence if α_1 be the first observed throw of the instrument, due to the discharge, we have (p. 517):

$$Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\alpha_1}{2}\left(1 + \frac{\lambda}{2}\right),$$

alternatively, if α_1 and α_3 are the first and second displacements of the spot of light *on the same side of the zero*, we have (p. 519):

$$Q = \frac{T}{\pi} \cdot \frac{i}{\varphi} \cdot \frac{\alpha_1}{2}\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}},$$

when φ is the angular deflection produced by a steady current, i.

For small displacements of δ, d_1, and d_3 cm. on the scale corresponding to φ, α_1, and α_3, we have

$$Q = \frac{T}{\pi} \cdot \frac{i}{\delta} \cdot \frac{d_1}{2}\left(\frac{d_1}{d_3}\right)^{\frac{1}{4}}. \qquad \ldots(6)$$

Fig. 385 shows the scheme of connexions for such an experiment. C is the condenser whose capacitance is to be determined; R_1 and R_2 are resistance boxes, introducing a high resistance to the circuit, AB a resistance box of decimals of an ohm, r say, K_1 a single-way switch, K_3 a tapping key, whereby the galvanometer may be brought to rest, K_2 a condenser key, and BG a ballistic galvanometer of the moving-coil type.

The circuit, R_1, R_2, and the battery, E, serve in the first part of the experiment as a potentiometer. R_1 and R_2 have a constant sum of about 10,000 ohms. By adjusting these two values, keeping $(R_1 + R_2)$ constant, the potential difference between B and C may be made any desired fraction of the potential, E, of the battery. This adjustment is carried out, and the deflection produced in the galvanometer when the condenser is afterwards discharged through it is noted.

Since $R_1 + R_2$ is high and r is never greater than 1 ohm, the potential across $BC = \frac{R_1}{R_1 + R_2} \cdot E = V$ volts, say.

When K_2 is depressed the condenser is charged, the potential applied being V. The charge, Q, will flow through the galvanometer causing a deflection, d_1 cm., when K_2 is raised.

This process is repeated several times, and a mean value of d_1 obtained for the fixed potential, V.

For each discharge, the reading corresponding to the second deflection, d_3, on the same side as d_1, is measured. This provides the data for the damping correction for the galvanometer under the identical conditions under which α_1 is measured.

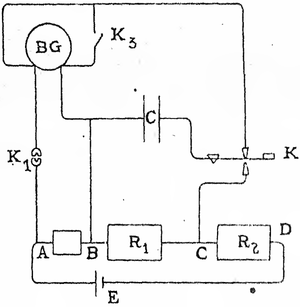

FIG. 385

The time of swing, T, of the galvanometer is found in the usual way.

To standardize the galvanometer a steady current of known magnitude must be sent through it and the scale deflection, δ, measured.

The key, K_1, is now closed. A current from the battery E passes through R_2 and R_1, and through the galvanometer, shunted by the small resistance, r.

The resistance of the galvanometer and shunt is

$$\frac{rG}{G + r},$$

where G is the resistance of the galvanometer.

Hence, if E is the potential difference in the circuit, the current in the main circuit, i, is

$$\frac{E}{R_1 + R_2 + B + \dfrac{rG}{r + G}},$$

where B is the resistance of the battery. This may be neglected when

the latter is an accumulator, and $\dfrac{rG}{r\tau G}$ is also negligible compared with $R_1 + R_2$.

Of this current, the value of the part, i, through the galvanometer is

$$i = \frac{Er}{(r + G)(R_1 + R_2)}.$$

We have also seen that the potential applied to the capacity is

$$V = \frac{R_1}{R_1 + R_2} \cdot E.$$

Hence, substituting this value of V and Q from (6),

$$C = \frac{Q}{V}$$

$$= \frac{\dfrac{T}{2\pi} \dfrac{E \cdot r}{(r + G)(R_1 + R_2)} \dfrac{d_1}{\delta} \left(\dfrac{d_1}{d_3}\right)^{\frac{1}{4}}}{\dfrac{R_1}{R_1 + R_2} \cdot E}$$

$$= \frac{T}{2\pi} \frac{d_1}{\delta} \left(\frac{d_1}{d_3}\right)^{\frac{1}{4}} \left(\frac{r}{R_1 G}\right),$$

since r is usually small enough to allow the approximation $(G + r) \to G$, E is assumed constant throughout the experiment.

Measurement of Capacity, using a Fluxmeter

A simple method of estimating the capacity of a condenser is illustrated in fig. 386.

The condenser of capacity, C, is connected to two condenser keys, S_1 and S_2. The lower studs of the keys are connected to the mains of

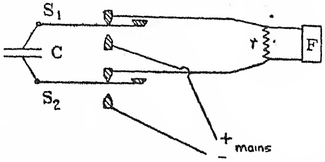

FIG. 386

the electricity supply (i.e. to 100–240 volts). When the keys have been depressed the condenser is charged to a potential, V. On releasing the keys the charge on the condenser is discharged through a low resistance, r, which is shunted across the fluxmeter, F, as shown in the diagram.

r is made small, about $\frac{1}{10}$ to $\frac{1}{100}$ ohm is suitable, and the shunt may therefore be regarded as taking the whole of the discharge, say, Q units.

The potential difference at the ends of r varies from V to zero, i.e. the current through r at any instant when the potential is E is $\frac{E}{r}$ and the total quantity of electricity passing through r is therefore

$$Q = \int \frac{E dt}{r}.$$

Now, if the resistance of the fluxmeter, usually about 30 ohms, is large compared with r, the instrument gives a deflection which is a measure of $\int E \, . \, dt$ (see p. 522). Let this deflection be x divisions or $x \times 10^4$ maxwells,

$$Q = \frac{x \times 10^4}{r}.$$

Since r is expressed in ohms, $x \times 10^4$ maxwells should now be written $\frac{x \times 10^4}{10^8}$ practical units to obtain Q in coulombs,

i.e. $$Q \text{ (in coulombs)} = \frac{x \times 10^{-4}}{r}.$$

If the potential applied to the condenser is measured by means of a voltmeter, the capacity in farads may be readily obtained.

(b) Self-inductance

(1) Rayleigh's Method

The unknown inductance, L, is placed in series with a small variable resistance, r, and arranged in the arm, AB, of the Wheatstone net, as shown in fig. 387. The remaining arms are of the same order of magnitude as the resistance, AB. G is a ballistic galvanometer preferably of the same order of resistance; K_1 and K_2 are keys. The resistance, r, is reduced to zero and the network is balanced for steady currents by closing the key, K_2, before making the galvanometer circuit through K_1.

To obtain an accurate balance, it may be necessary, when the arms are equal, to introduce a smaller resistance than is available in the Post Office box which provides the three resistances, r_2, r_3, and r_4. This may be effected by having a length of platinoid wire in series with L, and adjusting the length until an accurate balance is obtained.

If now a small E.M.F., E, is introduced in one arm, a current which depends on E will pass through each of the other resistances in the network.

Thus, if the galvanometer key is closed, and then the battery key,

an E.M.F. of magnitude $L\frac{di}{dt}$, established in the arm, AB, results in a current in each of the arms in the net, including the galvanometer. Let the current in the galvanometer branch, BD, be $k\,.\left(L\frac{di}{dt}\right)$, where k is a constant which depends on the value of the resistances. Under such circumstances the total quantity of electricity which passes through the galvanometer due to this cause is

$$Q = \int_0^t kL\frac{di}{dt}\,dt = kLi_0,$$

where i_0 is the maximum steady current flowing through AB.

This quantity, Q, may be calculated from the observed throw in the ballistic galvanometer, and thus L is obtained in terms of the constants of the galvanometer, k and i_0.

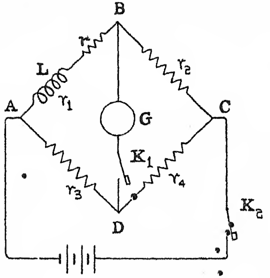

FIG. 387

Assuming that a moving-needle galvanometer is employed, we have, p. 516,

$$Q = \frac{T}{\pi}\cdot\frac{H}{G}\cdot\sin\frac{\theta}{2}\left(1+\frac{\lambda}{2}\right) = kLi_0, \qquad \ldots(7)$$

where λ is the logarithmic decrement.

To eliminate k and i_0, a measurable small potential change is introduced into the arm AB. This is brought about by adding a small resistance, r, to AB. (r should be not greater than $\frac{1}{500}r_1$, usually $\frac{1}{10}$ ohm does very well.) Assuming that the current, i_0, will not be materially affected by this small change, the potential introduced in the arm, AB, is i_0r. This causes a current, ki_0r, in the galvanometer, producing a steady deflection, θ_1.

Then G being, as above, the galvanometer constant, i.e. the field strength at the centre of the coil due to unit current, we have

$$ki_0 r G = H \tan \theta_1,$$

or

$$ki_0 = \frac{H}{G} \cdot \frac{\tan \theta_1}{r}.$$

Substituting this value in equation (7) above, we have

$$L = r \cdot \frac{T}{\pi} \cdot \frac{\sin \left\{ \frac{\theta}{2} \left(1 + \frac{\lambda}{2} \right) \right\}}{\tan \theta_1}.$$

If r is expressed in ohms and T in seconds, L is in henrys (10^9 C.G.S., E.M. units).

If a moving-coil type of ballistic galvanometer is used and r_2 and r_4 or r_1 and r_3 are small, the galvanometer may give but a small deflection as it is shunted by these resistances. This is the case when the resistance of L is small. Measurable and reliable results can be obtained if the galvanometer circuit is broken the moment the discharge has passed through it. This condition is most conveniently brought about by using a single key, as shown in fig. 331 on p. 520, for both battery and galvanometer circuits.

The three brass strips, A, B, and C, are insulated and connected to separate terminals. When C is depressed contact is made between C and B, then A and S, but not between B and A which are separated by an ebonite stop. S and A replace K_2, C and B replace K_1, when C is pressed down and a steady current flows in the circuit, the galvanometer which is in the circuit shows no deflection for a balanced bridge. On releasing C, the bottom contact is first broken (i.e. the battery circuit), and a very short interval of time afterwards C and B are separated. This interval is sufficient to allow the impulse to be given to the galvanometer coil which, with the separation of B and C, swings without excessive damping due to induced currents.

If the moving-coil instrument is used in this way, the end result is slightly modified, for in this case (see p. 517),

$$Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta}{2} = Lki_0,$$

where θ is the corrected value of the first throw for no damping, i.e. is the observed throw $\times \left(1 + \frac{\lambda}{2}\right)$.

Now for a steady current equal to $ki_0 r$ the couple on the coil is $Gki_0 r$, so that

$$Gki_0 r = \tau\theta_1,$$

τ being, as above, the restoring couple in the suspension, per unit angular displacement. Combining the last two equations, we see that

$$L = \frac{T}{\pi} \cdot \frac{r}{2} \cdot \frac{\theta}{\theta_1}.$$

(c) Mutual Inductance

(1) Direct Measurement with a Ballistic Galvanometer

The coefficient of mutual induction of two coils may be defined as the magnetic flux which is linked with one coil when unit current circulates through the other.

Thus, if a current of maximum strength, i_0, passes through one of the coils, whose mutual induction is M, Mi_0 lines of magnetic force thread the second, and whilst the current grows in the primary, the flux threading the secondary is changing. Therefore an induced E.M.F. is set up in the secondary during the time of growth of the primary current. This E.M.F. is numerically equal to the rate of change of magnetic flux in the secondary, i.e. $= \frac{d}{dt}(Mi)$,* where i is the instantaneous current in the primary during the growth of that current.

If L is the coefficient of self-inductance of the secondary coil, and i' is the current in the secondary corresponding to i in the primary, we have a further E.M.F. in the secondary due to the self-inductance numerically equal to $\frac{Ldi'}{dt}$, i.e. if R is the total resistance of the secondary coil circuit,

$$Ri' = L\frac{di'}{dt} \pm M\frac{di}{dt}.$$

The direction of the E.M.F. due to the coupling of the coils depends on the manner in which they are wound and can be in either sense. This is indicated by the sign $\pm$.

Now Q, the quantity of electricity passing through the secondary, is $\int i'dt$ where the integration is carried out over the whole time during which i rises to the steady value i_0.

$$Q = \int i'dt = \int\frac{L}{R}\,di' \pm \int\frac{M}{R}\,di.$$

The value of i' at the commencement and the end of this integration is zero, hence

$$\int\frac{L}{R}\,di' = 0$$

and $$Q = \int_0^{i_0}\frac{M}{R}\,.\,di = \frac{Mi_0}{R}.$$

If, therefore, the second coil is connected to a ballistic galvanometer and the throw is θ, due to the passage of this quantity of electricity, M may be calculated.

In this case it will be well to calibrate the galvanometer in the

* N.B.—This only applies to coils with non-magnetic cores. For if there is an iron core the value of the flux is *not* proportional to the current.

circuit since R includes the resistance of the galvanometer and the second coil.

Fig. 388 shows a convenient disposition of apparatus to carry out a direct measurement of M on these lines.

The current in the primary coil may be regulated to a suitable value by means of the resistance R_1. In series with R_1 is a small resistance r. C is a four-segment commutator, or may be a switch consisting of four mercury-filled holes in a block of paraffin wax.

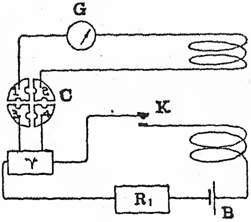

FIG. 388

If 1 and 2 are connected together the ballistic galvanometer is in direct circuit with the secondary coil, and when K is depressed a deflection, θ, is obtained for the establishing of the steady current, i_0, in the primary. If a moving-needle ballistic galvanometer is used, we have

$$Q = \frac{T}{\pi} \cdot \frac{H}{G} \sin \frac{\theta_0}{2} = \frac{Mi_0}{R},$$

where θ_0 is the value of the first deflection corrected for damping, i.e. $\theta_0 = \theta\left(1 + \frac{\lambda}{2}\right)$, where λ is the logarithmic decrement and θ is the observed throw.*

If now C is arranged so that connexion is made between 1 and 3, 2 and 4 only, and a steady current, i_0, is passed in the primary circuit, the potential drop established at the ends of r is $i_0 r$, i.e. the current through the galvanometer is $\frac{i_0 r}{R}$, since r is very small compared with the resistance of the galvanometer. If this causes a steady deflection, θ_1, we have

$$G \cdot \frac{i_0 r}{R} = H \tan \theta_1,$$

* If a moving-coil instrument is used, the form of double key, described on p. 520, should be used; K is replaced by the lower pair of contacts in such a key and the upper pair act as a key in the galvanometer circuit.

Combining (25) and (26), we obtain the following value for M:

$$M = \frac{T}{\pi} \cdot r \cdot \frac{\sin \frac{\theta_0}{2}}{\tan \theta_1}.$$

If a moving-coil galvanometer must be used, since $Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_0}{2}$,

the value for M becomes $\quad M = \frac{T}{\pi} \cdot \frac{r}{2} \cdot \frac{\theta_0}{\theta_1}.$

It will be noted that the value of R is eliminated. The resistance, r, may be a standard $\frac{1}{10}$ or $\frac{1}{100}$ ohm. T is obtained in the usual way by timing twenty or thirty swings, under the conditions of damping which obtain during the observations above, $\theta_0 = \theta\left(1 + \frac{\lambda}{2}\right)$. The value of λ may be obtained by one of the methods given on p. 152.

(2) Carey Foster's Method

A simple circuit which can be used for this method is illustrated in fig. 389. The source of supply is a direct current battery connected in

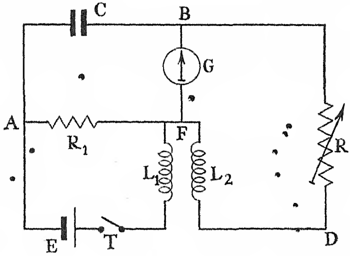

FIG. 389

series with a key, T, which is required for making and breaking the main circuit. The resistances, R and R_1, should be variable, R_1 being set at a particular value, and the balanced condition is obtained by adjustment of R. The two coils between which the mutual inductance, M, is to be measured are denoted by L_1 and L_2. The standard capacity is denoted by C, and it is convenient to use a standard variable condenser so that the experiment may be repeated for various values of C. The detector is a ballistic galvanometer, G, and the balanced condition is obtained when no charge flows through it on making or breaking

contact by T. In this case the flow of charge in G which results from charging or discharging C is equal to and of opposite sign to, that which arises from the mutual inductive effect between L_1 and L_2. This balancing can only occur if the directions of the coil windings are such that the induced charge flows in an opposite direction to that which flows from C. This must be tested by breaking in turn the circuit ABF and the circuit BFD to test that the galvanometer deflection is in opposite directions in the two cases when the key, T, is depressed. If this is not the case one of the coils, L_1 or L_2, must be connected in the opposite direction.

It has been shown (p. 601) that when the current in a primary coil, L_1, attains a maximum, I, the charge which flows in the secondary circuit $= \dfrac{MI}{\text{Total resistance}}$.

Thus, in this case, if I is the current which flows in the circuit, AEF, and the resistance of the galvanometer is G, the current through the galvanometer is

$$Q = \frac{MI}{(R + G)}.$$

The resistance, R, must be taken to include the resistance of the coil L_2, as well as the variable resistance introduced in the process of balancing.

The condenser is charged to a p.d. R_1I, and on discharge part of the total charge, CR_1I, passes through G and part through R. The amount through G is

$$Q = \frac{CR_1IR}{(R + G)},$$

these two contributions to the total charge through G being of equal magnitude.

Thus

$$M = CR_1R.$$

In carrying out this experiment fix C and R_1 at some values within the ranges provided and vary R to obtain no deflection in the galvanometer, taking care to note what variation of R on both sides of an average position is possible without appreciable deflection in the galvanometer. The less this range the greater the sensitivity of the apparatus with this setting.

Repeat the experiment with other values of R_1 in order to find a value which gives the greatest sensitivity. When the best possible conditions are obtained in this way, make changes in C in order to obtain a further improvement. Finally, repeat the readings of C, R, and R_1 under these circumstances.

A disadvantage of the method is that the resistance of L_2 is required in order to obtain the value of R. This additional resistance will be small in many cases. Let its value be S, and let the reading of the variable resistance be R_2, so that

$$R = R_2 + S.$$

Then $$M = CR_1(R_2 + S).$$

The necessity of determining S can be avoided by changing the value of R_1 about the most sensitive position and by plotting R_2 against $\frac{1}{R_1}$. The graph obtained will be linear with a slope, φ, where

$$\tan \varphi = \frac{M}{C}.$$

If S should be required it is obtained from the intercept on the axis of R_2.

The disadvantage that the resistance of L_2 occurs in the formula has been removed by a modification of the method suggested by N. F. Astbury. It will be noted that the galvanometer resistance does not appear in the formula from which M is calculated. In Astbury's modification the positions of the galvanometer and resistance R are interchanged (fig. 389 (*a*)) and the galvanometer resistance and that of

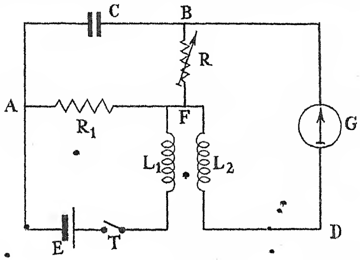

FIG. 389 (*a*)

L_2 taken together are now represented by G. As before, the quantity of charge flowing through G on account of the discharge of the condenser is

$$Q = \frac{CR_1IR}{(R + G)},$$

and for the effect of mutual inductance

$$Q = \frac{MI}{(R + G)}.$$

There is thus no difference in the theory of the two cases.

It should be added that Carey Foster's method is also useful as a determination of capacity in terms of a standard mutual inductance.

The method, using alternating current, is described again later, p. 621.

Determination of Mutual Inductance by the Fluxmeter

The mutual inductance of two coils may be very readily obtained, using a fluxmeter which is described on p. 521.

One coil, M, is connected directly to the fluxmeter, F (fig. 390). The other coil, L, is connected through a commutator, C, to a circuit consisting of an accumulator, E, an ammeter, A, and a variable resistance, R.

The current in L is adjusted to some convenient amount, c amperes, and then reversed, causing a deflection of x divisions in the fluxmeter.

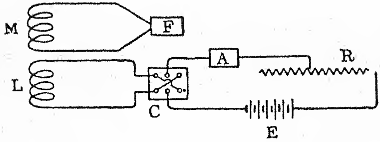

FIG. 390

If each division of the fluxmeter (as usual) corresponds to 10,000 maxwells, the change in the number of maxwells in the secondary, M, due to a reversal of c amperes in L is $10{,}000x$, i.e. for $\frac{c}{10}$ E.M. units of current in L there are $5000x$ lines of magnetic force threading M.

Hence, since the coefficient of mutual inductance is defined as the number of lines in one coil per unit current in the other, it has a value,

$$\frac{50{,}000x}{c},$$

for the coils used.

The experiment is repeated, using various values of c, and the mean value of $\frac{x}{c}$ is obtained, and hence the coefficient of mutual inductance in E.M. units may be calculated.

The mutual inductance in henrys is 10^{-9} times the above value, for two coils are said to have a coefficient of mutual inductance of 1 henry when a current change of 1 ampere per second in the primary sets up an E.M.F. of 1 volt in the secondary, or a flux of 10^8 lines for a current change of 10^{-1} E.M.U. per second, i.e. the henry is 10^9 E.M. units.

Measurements with Alternating Current Bridges

The experiments to be described in this group are concerned with the measurement of self- and mutual-inductance and of capacity. The basis of the calculation of these quantities rests upon the application

of Kirchhoff's laws to circuits in which an alternating current is flowing. The term alternating current is applied to a current the value of which changes over a certain period of time during which it takes both positive and negative values. For the present set of experiments a sinusoidal variation both of current and E.M.F. is assumed. The current, for example, is in this case represented by

$$I = I_0 \sin(\omega t + \theta),$$

where $\omega = 2\pi f$, f denoting the frequency. The period $T = \frac{1}{f}$ and I_0 is the amplitude. It must be borne in mind that the assumption is that the current is of a single frequency, no harmonics being present. This is not generally the case, and some of the difficulties met with in these experiments are due to the presence of harmonics.

In accurate work a non-harmonic generator of alternating current must be used, but in order to study the principles illustrated in these experiments such a generator is not absolutely necessary.

It is convenient to use the exponential expression $\exp(j\omega t)$ or $e^{j\omega t}$ in the representation of sinusoidally varying quantities.

Thus, instead of the above expression for I, we prefer to write:

$$I = I_0 e^{j(\omega t + \theta)}.$$

The real part of this expression is $I = I_0 \cos(\omega t + \theta)$ and the imaginary part $I = I_0 \sin(\omega t + \theta)$. When alternating currents are equal both their magnitudes, I_0, and phases, θ, are equal. This applies also to other quantities similarly represented.

When an alternating current flows through an ordinary ohmic resistance it is opposed as in the case of direct currents and heat is produced. An alternating p.d. applied at the ends of such a resistance produces an alternating current $I = \frac{V}{R}$, where V denotes the p.d. and R the resistance. This equation is

$$I = \frac{V_0 e^{j\omega t}}{R}$$

if we suppose that the E.M.F. is $V = V_0 e^{j\omega t}$. The amplitude of the current is $\frac{V_0}{R}$, and it is in phase with the p.d.

There is a difference in the behaviour of the resistance towards an alternating current and a direct current. In the former case the current tends to keep to the outer parts of the conductor and the effect is the more pronounced the more rapidly the current varies. Thus a conducting wire presents a higher resistance to a rapidly oscillating current than to a direct or slowly oscillating one. This is spoken of as the skin effect and the resistance depends on the frequency.

If an oscillating current flows through a helical coil even though the ohmic resistance is small enough to be neglected, there is still opposition to the flow of current. This is due to the self-inductance of the coil, the magnitude of which is measured by a quantity, L, which denotes the number of lines of magnetic flux through the coil per unit current. The opposing E.M.F. is $L\frac{dI}{dt}$, which for a sinusoidal current is $j\omega LI$. This can be interpreted by saying that the coil presents a reactance $j\omega L$ to the current I. This reactance is analogous to ohmic resistance, and it is described as inductive reactance. If a condenser is placed in the path of the current a charge, Idt, collects on one plate in the small time interval, dt, and an equal charge of opposite sign on the other. The p.d. between the plates due to this is

$$dV = \frac{Idt}{C},$$

where C is the capacity of the condenser.

If the E.M.F. is alternating,

$$\frac{dV}{dt} = j\omega V,$$

and the relation becomes $$V = \frac{I}{j\omega C}.$$

In this case resistance is replaced by $\frac{1}{j\omega C}$ and the condenser is said to present a capacitative reactance of magnitude $\frac{1}{j\omega C}$.

Ohm's law can be applied to a series circuit by treating these reactances as if they were resistances in the following way.

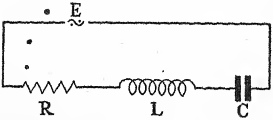

FIG. 391

Suppose a generator develops an E.M.F. $E = E_0 e^{j\omega t}$ and that it is in series with a resistance, R, an inductance, L, and a condenser of capacity, C.

The current $$I = \frac{E}{\left(R + j\omega L + \frac{1}{j\omega C}\right)}.$$

The quantity in the denominator is described as the vector impedance

of the circuit. This is a complex quantity and it can be expressed in the exponential form:

$$R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right).$$

If $X = \omega L - \frac{1}{\omega C}$ we can write this in the form $R + jX$, and X is described as the reactance of the circuit.

$(R + jX)$ can be expressed in terms of an amplitude and a phase angle by writing

$$R + jX = Z = Z_0 e^{j\alpha} = Z_0 (\cos \alpha + j \sin \alpha).$$

Thus $$Z_0^2 = R^2 + X^2.$$

Z can be described as the vector impedance and Z_0 is the magnitude or length of this vector. The impedance can be represented on a diagram as a vector and the phase angle by the angle α made by the

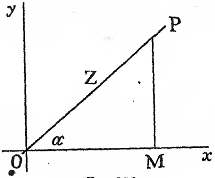

FIG. 392

vector (OP) with the x-axis. The two components, OM and MP, represent the real and imaginary parts of Z, and they differ in phase by the difference in the angles made with the x-axis, i.e. by 90°.

Thus the current in the series circuit can be expressed in the form:

$$I = \frac{E_0 e^{j\omega t}}{Z_0 e^{j\alpha}} = \frac{E_0 e^{j(\omega t - \alpha)}}{Z_0}.$$

Compare this with the expression $E_0 e^{j\omega t}$ for the E.M.F. and it appears that the current amplitude is $\frac{E_0}{Z_0}$, and that it differs in phase from the applied E.M.F. by α, where

$$Z_0^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2, \qquad \tan \alpha = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}.$$

Suppose that the applied E.M.F. is the real part of $E_0 e^{j\omega t}$, i.e. $E_0 \cos \omega t$. The current is obtained from the real part of I, i.e.

$$\frac{E_0}{Z_0} \cos (\omega t - \alpha).$$

In the case of a branched circuit, Kirchhoff's laws can be applied in the same way as in direct current circuits, provided that inductances and capacities are treated as offering reactances in the way that has just been described.

In the following group of experiments we are concerned with the Wheatstone net, to one pair of opposite corners of which an alternating E.M.F. is applied.

The arms of the bridge may now contain, besides ohmic resistances, inductances and condensers. If the vector impedances are denoted by

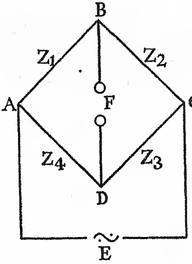

Fig. 393

Z_1, etc., the condition in which B and D are always at the same potential is

$$\frac{Z_1}{Z_2} = \frac{Z_4}{Z_3}.$$

This relation implies two conditions, one of which requires an equality of magnitudes and the other an equality of phases. The two conditions are obtained by equating the real and imaginary parts of this equation. To illustrate this point, take the case of Owen's bridge.

r_1 denotes an ohmic resistance, r_2 is also an ohmic resistance consisting partly of the resistance of the inductive coil, L, and partly of an added resistance. C_1 and C_2 are condensers and R an ohmic resistance.

The condition of balance is

$$\frac{r_1}{r_2 + j\omega L} = \frac{\dfrac{1}{j\omega C_1}}{R + \dfrac{1}{j\omega C_2}}.$$

This gives the following two relations to be satisfied:

$$L = C_1 r_1 R \quad \text{and} \quad C_1 r_1 = C_2 r_2.$$

In this introduction to bridge methods it has been taken for granted that the quantities represented by L and C, the inductance and capacitance, are localized in a coil or condenser placed in the circuit. But it must be remembered that lines of electric intensity may pass to objects not forming part of the bridge. Thus there may be lines of force passing to the hand or other part of the body of the experimenter. If this is the case a displacement current flows from the bridge to

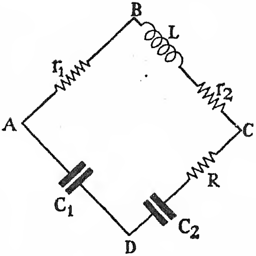

FIG. 394

earth and the current entering one branch of the bridge is not the same as that leaving the other end. Impedances of this sort are often capacitative in character, and are described as stray capacitances. It is important to avoid them whenever possible, and the possibility of their occurrence must not be overlooked. Transmission through a capacity is easier at high frequencies than at low ones, for we have seen that the capacitative reactance is inversely proportional to the frequency. In some of the experiments to be described where the frequencies are within the audio-frequency range and the capacities are comparatively large, the difficulties and errors which occur are not great and they will not require any special precautions.

Stray capacities such as are introduced by the experimenter are of the order of 10 micromicrofarads (10 $\mu\mu$F.), or in modern terminology, 10 picofards (10 pF.). They cannot be neglected when the capacities subject to measurement are of this order of magnitude, and in order to learn the principles of bridge methods it is advisable in the first instance to avoid the complication resulting from stray capacities by making measurements on capacities of a higher order of magnitude, certainly not less than 0·001 microfarad (μF.).

A method for avoiding stray capacities by means of the use of the device of the Wagner earth will be described later. The student is recommended at this stage to refer to other works for a study of the question of screening (e.g. *Radio-Frequency Measurements*, by L. Hartshorn).

In these experiments two arms have often to be occupied by non-inductive resistances, and it must be regarded as an essential to satisfactory work that such standard resistances are available. One arm often contains a condenser and this should be without loss. A well-made mica condenser is usually satisfactory in this respect, and can be used as a standard.

Comparison of Capacities

The Method of de Sauty

A simple method of comparing capacities by a bridge method at audio-frequencies is that of de Sauty. In the method already described (p. 592) the current was supplied from a direct current battery. It is preferable to use a generator giving an alternating E.M.F. in the audio-frequency range and to place it in the arm, AC (fig. 349). The galvanometer is replaced by ear-phones, or some other form of alternating current detector, and the condition of balance obtained by the method described. The capacities, which are assumed to be perfect, i.e. without conduction, offer impedances $\dfrac{1}{j\omega C_1}$ and $\dfrac{1}{j\omega C_2}$.

Thus, at balance
$$\frac{R_1}{R_2}=\frac{C_2}{C_1}.$$

Wien's Series Resistance Bridge

In this bridge one of the arms, AB (fig. 395), contains a variable standard condenser, C, in series with a variable standard resistance, r. The condenser should be as nearly perfect as possible, and the resistance purely ohmic.

The two arms, BC and CD, contain two pure resistances, R_1 and R_2 respectively, which are of the same order of magnitude and may be made equal.

The bridge is used to measure the capacity and resistance of a condenser which is placed in the remaining arm and which will be regarded as equivalent to a perfect condenser of capacity C_1 with a resistance r_1 in series with it. Ear-phones placed in the arm, BD, are used as detectors. The condition of balance of the bridge when there is silence in the phones is given by

$$\frac{r_1+\dfrac{1}{j\omega C_1}}{R_2}=\frac{r+\dfrac{1}{j\omega C}}{R_1}.$$

Thus $$r_1 = \frac{R_2 r}{R_1}, \quad C_1 = \frac{CR_1}{R_2}.$$

In practice the condition of balance is obtained by adjusting the capacity, C, until a minimum of sound is heard in the phones. The resistance, r, is then adjusted until silence is obtained as nearly as possible.

It may happen that a variable condenser is not at hand. In this case the bridge should be balanced by varying the resistances, R_1 and R_2. A fixed condenser as nearly equal to the unknown condenser as possible is taken as a standard. If the order of magnitude of the

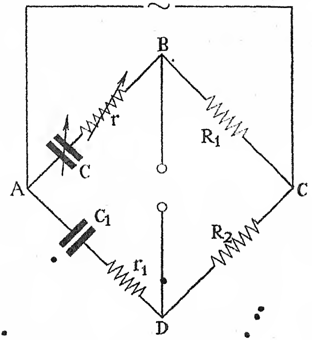

FIG. 395

condenser to be measured is unknown, a preliminary calculation with any standard condenser should be made. After this a standard should be chosen nearly equal to the value obtained. The reason for this is that the bridge is most sensitive when all the arms are equal. In obtaining the position of balance when a minimum of sound has to be relied upon for the final setting, a number of separate attempts of finding the balance should be made. These will vary from one another by slight amounts, and the arithmetic means of the results obtained should be taken.

It is important to record the frequency of oscillation at which the measurements are made, since both the capacity and the resistance of the condenser are dependent on frequency. This is obtained directly from the scale of the generator or by reference to a calibration graph of the instrument.

A common way of recording the condenser resistance is by means of the power factor. Suppose that a leaky condenser is represented, as in the present case, by a pure capacity and a series resistance, and suppose a p.d. $V = V_0 e^{j\omega t}$ is placed across it.

The fraction of V across r_1 is

$$\frac{r_1}{r_1 + \dfrac{1}{j\omega C_1}} = \frac{r_1}{Z_0 e^{-j\alpha}},$$

where $$Z_0^2 = r_1^2 + \frac{1}{\omega^2 C_1^2} \quad \text{and} \quad \tan\alpha = \frac{1}{r_1\omega C_1}.$$

r_1 is much less than $\dfrac{1}{\omega C_1}$ in all cases in practice, so that $Z_0 = \dfrac{1}{\omega C_1}$ to a sufficient degree of accuracy.

Thus the amplitude of V across r_1 is $r_1\omega C_1 V_0$ and $r_1\omega C_1$ is described as the power factor of the condenser.

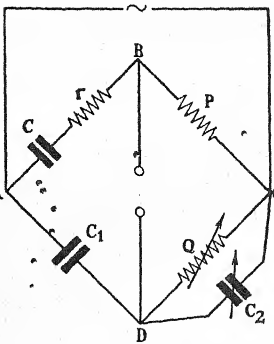

FIG. 396

The Schering Bridge

In the bridge two non-inductive resistances are again required. These are denoted by P and Q in fig. 396. The condenser to be tested is placed in the arm, AB, and it is represented as having a capacity, C, and a series resistance, r. The standard condenser is placed in the arm, AD, and it is supposed to be purely capacitative with a value, C_1. A variable air condenser is placed in parallel with Q and its capacity is denoted by C_2.

The bridge is balanced by varying Q and C_2.

The two conditions of balance are contained in the equation:

$$\frac{r+\dfrac{1}{j\omega C}}{P}=\frac{\dfrac{1}{j\omega C_1}}{\dfrac{Q}{(1+j\omega C_2 Q)}},$$

which gives $$C=\frac{QC_1}{P},\quad r=\frac{PC_2}{C_1}.$$

The power factor of the condenser, $r\omega C = Q\omega C_2$.

A suitable capacity for the test is provided by an oiled paper condenser of about 0·003 μF.

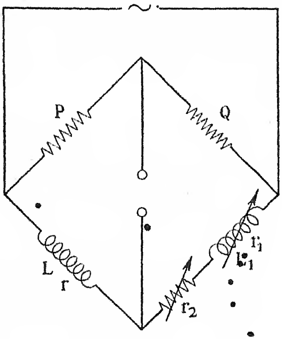

Fig. 397

Measurement of Self-Inductance

Maxwell's Inductance Bridge

In this case an unknown self-inductance is compared with a standard. The latter is provided by a variable inductometer, the self-inductance, L_1, being variable, but the resistance, r_1, associated with it being constant. Two non-inductive resistances are again necessary.

The resistance, r_2, is variable and must be non-inductive.

The resistances, P and Q, should be of the same order of magnitude and equal to the order of magnitude of the reactance to be measured. This is to ensure that the bridge is in the sensitive condition and can thus be accurately balanced.

The condition of balance,

$$\frac{r + j\omega L}{P} = \frac{r_1 + r_2 + j\omega L_1}{Q},$$

gives

$$L_1 = \frac{QL}{P},$$

$$r = \frac{P(r_1 + r_2)}{Q}.$$

The latter relation cannot always be satisfied, for if $r < \frac{Pr_1}{Q}$, r_2 would have to be negative. In this case r_2 must be placed in the same arm as L, when the equation for the determination of r becomes

$$r = \frac{Pr_1}{Q} - r_2.$$

As an example of the magnitudes used in laboratory experiments, suppose that the frequency of oscillation is 1000 cycles per sec. ($\omega = 2000\,\pi$), and that the value of L is 50 millihenrys (mH.).

$L\omega$ is then about 300 ohms. Thus P and Q and the inductometer must be set at about this value.

The values of the resistances of the inductive coils are usually small.

The balance is obtained by varying the inductometer and then by altering r_2. The final setting should be made by alternating these adjustments.

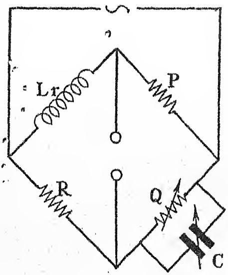

Fig. 398

Maxwell's Inductance-Capacity Bridge

In this case an inductance is measured by comparison with a variable standard capacity, assumed to have no leakage. Three non-inductive resistances are required and the bridge is made up as illustrated in the diagram.

The resistance, Q, and the capacity, C, are variable.

The relations which apply when the bridge is balanced are

$$L = PRC, \quad r = \frac{PR}{Q}.$$

The resistances, P and R, are made equal, e.g. a few hundred ohms. If the capacity is about $\frac{1}{4}$ μF. and the resistance, Q, about 10,000 ohms, the bridge will be sensitive for inductances of about 100 mH.

Anderson's Method

This bridge method is rather more complicated than those which have been described so far, but it is interesting and is a good practical method. The inductance is measured in terms of a capacity and resistances, as in the last experiment.

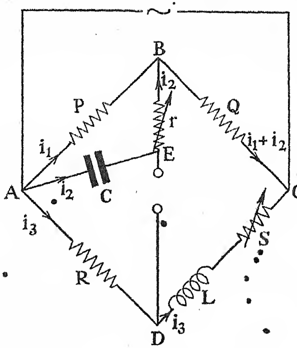

Fig. 399

The condition of balance in this case is that the potentials at D and E are the same. Under these conditions the currents can be represented as in fig. 399, since no current flows along the path, DE.

The potential drop along ABC is equal to that along ADC.

This is expressed by the following equation:

$$i_1 P + (i_1 + i_2) Q = i_3 (R + S + j\omega L).$$

In the circuit, ABEA, there is no E.M.F.

Thus $$i_1 P - i_2 \left(r + \frac{1}{j\omega C}\right) = 0.$$

The p.d. from A to E is equal to that from A to D

Thus $$\frac{i_2}{j\omega C} = i_3 R.$$

On substituting i_3 from the last of these equations into the first, we obtain

$$i_1(P + Q) = i_2\left(\frac{R + S + j\omega L}{j\omega CR} - Q\right).$$

Eliminating i_1 and i_2, with the help of the second equation, we obtain by equating imaginary parts,

$$S = \frac{RQ}{P},$$

which is the relation satisfied by the bridge for steady currents and by equating real parts,

$$L = \frac{CR(Pr + Qr + PQ)}{P}.$$

The equation for S gives the total resistance in the arm, CD. Part of this is taken from a standard variable resistance box, and the remainder is the resistance of the inductive coil.

Substituting $S = \frac{RQ}{P}$ in the expression for L, we obtain

$$L = C\{RQ + r(R + S)\}.$$

The impedances of the arms should be of the same order of magnitude. If L is completely unknown a preliminary estimation, which need not be very accurate, should be made by this or some other method to determine the approximate value of L. Q and R should then be given values of the order ωL. If the experiment is then carried out an accurate value for L should result.

It should be noticed that it may be found impossible sometimes to obtain a balance by varying r and S. The formula for L with a value of RQ which satisfies the formula for S may require a negative value of r. It is thus desirable to know an approximate value of L and to take care that the product CRQ is less than L. A balance is then possible with a positive value of r.

The bridge is adjusted to the balanced condition by alternately varying r and S until there is no sound in the telephones. In the method originally described for this bridge a battery was used as the source of supply. A balance was obtained for a steady current by adjusting S. This requires the relation $S = \frac{RQ}{P}$ characteristic of the ordinary Wheatstone bridge. A ballistic galvanometer was used as the detector, and with a tapping key in series with the battery it was, in general, observed that the bridge was no longer balanced at make or break of the current. This further condition of balance could be obtained by adjusting r without disturbing the balance for steady currents.

This method of using the bridge can be used as an alternative to that in which an oscillatory E.M.F. is used, but the accuracy is considerably less.

Owen's Bridge

In this case the self-inductance is measured in terms of a standard capacity and of resistances.

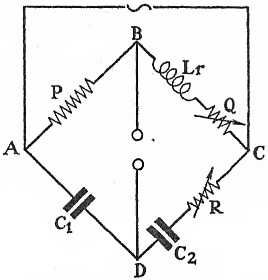

FIG. 400

P, Q, and R denote standard non-inductive resistances; C_1 and C_2 are standard condensers.

In the condition of balance,

$$Pj\omega C_1 = \frac{(Q + r + j\omega L)}{R + \dfrac{1}{j\omega C_2}},$$

and this leads to the two conditions:

$$r = \frac{PC_1}{C_2} - Q$$

$$L = PRC_1.$$

In order to satisfy the first of these it may be necessary to vary P.

The minimum of sound should be obtained by varying Q. This should then be reduced by varying R, and so on alternately until no sound is heard in the telephones.

Owen has shown that the bridge has a wide range. Thus, with the condensers each having a capacity of $\frac{1}{3}$ μF. and with values of P ranging from 1 to 200 ohms, a range of inductance between 2 μH. and 0·5 H. can be measured. An accuracy of about 1 per cent for inductances of low values (a few μH.) and to one part in ten thousand for values of the order of 0·1 H. can be attained.

It should be noted that the orders of impedance in the arms are not the same in some applications of this method. This is the case for low values of the inductance. The reason is that correspondingly large mica condensers, which are most suitable, are not available. This, however, does not detract seriously from the accuracy of the method.

In the bridge experiments described it has been recommended that the ratio arms of the bridge should have impedances of the same order of magnitude. While this is a good working rule, it need not be slavishly followed. The question of the sensitivity of bridges is rather a complicated one and the impedances in the branches of the generator and of the detector are concerned in it.

One of the following experiments on sensitivity is intended as an illustration of this question, and is concerned with the case when these two impedances are small. It will appear in this case that the sensitivity is increased as the sum of the impedances in the ratio arms diminishes. In considering the question of sensitivity each case should be considered by itself with regard to the particular conditions imposed by the apparatus.

Measurement of Mutual Inductance by means of a Self-inductance Bridge

Let two coils, AB and CD, have self-inductances, L_1 and L_2, and let them be set up so that their mutual inductance is M. If they are

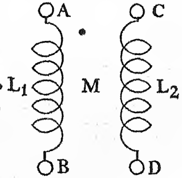

FIG. 401

joined together by connecting A and C to form a single coil, the self-inductance will become $(L_1 + L_2 \pm 2M)$, since the total flux per unit current is increased by $\pm 2M$ due to the mutual inductance. If the connexion is made by joining A and D the effect of mutual inductance will be of the same magnitude, but opposite in sign. These connexions can be made alternately by the use of a double pole switch. With the switch in one position the value of $(L_1 + L_2 + 2M)$ is determined and $(L_1 + L_2 - 2M)$ is determined in the other position. The difference gives 4M. Thus one of the methods described for the determination

of self-inductance can be applied to this case. It is necessary to take particular care in the measurements in order to secure an accurate result. Suppose that the values of the self-inductances in the two cases are L and L′, so that

$$L - L' = 4M.$$

L and L′ may have been measured to a high degree of accuracy, but the difference will not be known to the same degree.

If the values of L and L′ are close together, i.e. M is small, the result will be inaccurate. Suppose, for example, that L and L′ are determined to an accuracy of 1 per cent, and suppose that L is 100 units and L′ 80 units according to the measurements. Then, in the worst case L could be actually 101 and L′ 79. Thus, the difference would actually be 22, while the measurements give 20, i.e. an error of 10 per cent. This danger of inaccuracy is always associated with measurements depending upon the difference of two quantities.

Carey Foster's Method of Measuring Mutual Inductance

The battery, E, and the key, T, can be replaced by an A.C. supply, the galvanometer being replaced by telephones (p. 605).

In this case the resistances should be non-inductive.

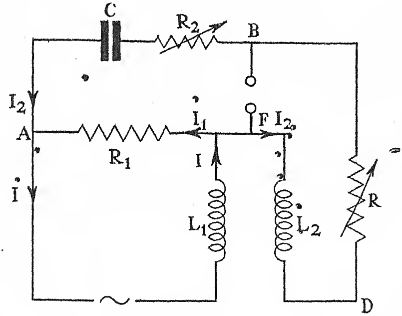

FIG. 402

An important modification is that a variable resistance must be included in the circuit in series with the condenser. The reason for this appears from a consideration of the conditions of balance. The circuit is now illustrated in fig. 402.

The condition to be satisfied in this case is that the p.d. between B and F should be zero at every instant.

Since no current flows along the arm, BF, the distribution of currents in the various parts of the circuit is that represented in the figure with

$$I = I_1 + I_2.$$

In the circuit, BFD, the E.M.F. arises from the mutual inductance and is of magnitude $M\frac{dI}{dt}$. It is assumed that I is sinusoidal at a frequency, f, and writing $\omega = 2\pi f$, we have for this E.M.F., $j\omega MI$.

Thus, since the points B and F are at the same potential,

$$j\omega MI = j\omega L_2 I_2 + RI_2$$

i.e.

$$j\omega MI_1 = j\omega (L_2 - M) I_2 + RI_2.$$

Since the fall of potential along FA is equal to that along BA,

$$R_1 I_1 = \left(R_2 + \frac{1}{j\omega C}\right) I_2.$$

From these two equations it follows that

$$j\omega M\left(R_2 + \frac{1}{j\omega C}\right) = R_1 \{j\omega (L_2 - M) + R\}.$$

The two conditions to be satisfied are therefore

$$M = CR_1R$$

and

$$\frac{L_2}{M} = \frac{(R_1 + R_2)}{R_1}.$$

As in the previous case, M must be inserted in the correct way in order to obtain a balance at all. If the first adjustments indicate the impossibility of obtaining a balance, this point should be examined by interchanging the connexions to one of the two coils, L_1 or L_2. Another point to notice is that the second condition requires that L_2 should be greater than M. Approximate values of L_2 and M should be known before the experiment is begun.

In the process of balancing, R and R_2 are adjusted alternately until there is no sound in the telephones.

As in the previous case, R includes the resistance of the coil, L_2, and the value of the resistance may be eliminated or determined as before by varying R_1 and plotting R_2 against $\frac{1}{R_1}$.

The same suggestions with regard to the determination of the conditions of greatest sensitivity should be followed in this case as in the previous one (p. 621).

If the condenser is a leaky one its series resistance is to be added to R_2 and the R_2 of the formula is to be regarded as the sum of this series resistance and that of the variable standard. The method is not to be recommended as a determination of the ratio of self- and mutual inductances, and there is no necessity to follow up the suggestion of

the second condition. This condition must, however, be satisfied in order to obtain a balance, and it is evident that for this purpose a resistance, R_2, has to be placed in series with the condenser.

A suggestion for this experiment is that it should be used to determine how the mutual inductance of two coils depends on their separation. L_1 and L_2 should be mounted co-axially with a variable distance between their centres. A graph should be prepared showing the relation between the mutual inductance and this distance.

As an alternative, the value of M should be measured for different inclinations of the planes of the coils. For this purpose the coils should be flat and circular made so that one can lie concentrically within the other and mounted so that one can turn about a common diameter over a scale of degrees measured from zero when the coils are parallel. Plot the values of M against the inclinations, θ.

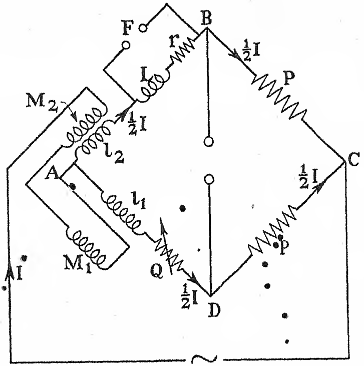

Fig. 403

FURTHER EXPERIMENTS ON INDUCTANCE AND CAPACITY MEASUREMENTS

The Campbell-Heaviside Equal Ratio Bridge. Measurement of the Inductance and Resistance of a Coil

In this bridge the generator is connected in series with the primary of a Campbell mutual inductometer, shown in fig. 403, as M_1 and M_2. The secondary consists of two parts, l_1 and l_2, the latter being situated in one ratio arm in series with the coil under test and the former in

a second arm in series with a variable non-inductive resistance, Q. The other ratio arms contain equal non-inductive resistances of magnitude, P. When the bridge is balanced the current given by the generator can be represented by I and that in the arms by $\frac{1}{2}$I.

If M_1 and M_2 denote the mutual inductances between M_1 and l_1 and between M_2 and l_2 respectively, and if l_1 and l_2 denote the self-inductances of the inductometer coils in the two arms, the equality of the falls of potential along AD and AB gives the equation

$$j\omega M_1 I + j\omega l_1 \tfrac{1}{2} I + \tfrac{1}{2} QI = -j\omega M_2 I + j\omega (l_2 + L) \tfrac{1}{2} I + \tfrac{1}{2} rI,$$

$j\omega M_1 I$ is the E.M.F. induced in the arm, AB, by the mutual inductance. The windings of the coils are such that when this is positive the induced E.M.F. in AD is negative and equal to $-j\omega M_2 I$. The two relations which result from equating real and imaginary parts are

$$r = Q,$$
$$2(M_1 + M_2) = l_2 - l_1 + L.$$

or, denoting the total mutual inductance provided by the inductometer by M_1,

$$2M = (l_2 - l_1) + L.$$

The inductometer is provided with a scale giving M at various settings and the bridge is balanced by adjusting M and Q alternately.

In order to determine $(l_2 - l_1)$, the inductance, L, is short-circuited by the key, F, and the balance is obtained by setting the inductometer at a value, M_0, and Q at zero.

In this case

$$2M_0 = (l_2 - l_1),$$

and thus

$$L = 2(M - M_0).$$

M_0 should be much smaller than M to avoid the inaccurate result which is likely to arise in the difference of two numbers not greatly different even when each is measured to a high degree of accuracy.

In a typical case P was of the order of 100 ohms and the value of L was about 1 mH.

An interesting experiment consists in measuring the effect on the self-inductance of a coil wound in the form of a solenoid when bars of non-magnetic substances are placed along its axis. A suitable metal for this purpose is copper or brass, of which rods of varying diameters may be inserted.

It is important to record in this experiment, as in all bridge experiments, the frequency of the E.M.F. given by the generator.

The Sensitivity of an A.C. Bridge. Use of Vibration Galvanometer

Let the branches of a bridge network contain impedances as shown in the diagram.

The current, i, in the detector, i.e. in the arm containing Z_5 is given by

$$i = \frac{(Z_1 Z_3 - Z_2 Z_4)\,E}{\Delta},$$

where E is the E.M.F. in the arm containing the impedance, Z_6.

The determinant, Δ, is given by

$$\Delta = \begin{vmatrix} -Z_3 & -(Z_3 + Z_4) & (Z_3 + Z_4 + Z_6) \\ (Z_2 + Z_3 + Z_5) & (Z_2 + Z_3) & -Z_3 \\ -Z_5 & (Z_1 + Z_4) & -Z_4 \end{vmatrix}.$$

This result is obtained by applying Kirchhoff's laws to the network and requires the solution of simultaneous equations. The calculation is long but straightforward.

In experiments with bridges we are interested in the conditions near those of balance, i.e. when $Z_1 Z_3 = Z_2 Z_4$. If a small deviation from the condition of balance results in a large change in the current through the detector, the bridge is sensitive.

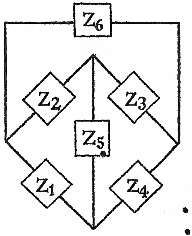

FIG. 404

The best conditions for sensitivity are reached when the impedances in the six arms are all equal. The theoretical discussion of this point is complicated. A simple case will be taken for the purpose of the present experiment.

The values of Z_5 and Z_6 will be taken to be small, Z_6 being the impedance of the secondary of a transformer which in a particular case gave 12 volts at 50 cycles per sec. The self-inductance of this secondary was 0·056 H., which gives an impedance of about 18 ohms. The detector was a vibration galvanometer with a resistance of 3 ohms. The bridge used was a de Sauty bridge and Z_1 was provided by a condenser 1 μF., i.e. Z_1 had a value of about 3200 ohms. The impedance Z_2, was a variable condenser and Z_3 and Z_4 were resistances always kept at values over 20 ohms.

The bridge is balanced by varying Z_4, i.e. by varying the resistance of one of the arms.

We can examine the sensitivity near the balance-point by considering $\dfrac{di}{dZ_4}$ at this point.

From the above equation for i,

$$\frac{di}{dZ_4} = -\frac{Z_2E}{\Delta} - \frac{(Z_1Z_3 - Z_2Z_4)\,E}{\Delta^2}\frac{d\Delta}{dZ_4},$$

so that at balance
$$\frac{di}{dZ_4} = -\frac{Z_2E}{\Delta_0},$$

where Δ_0 is the value of the determinant when $Z_1Z_3 = Z_2Z_4$.

If Z_5 and Z_6 are negligible, as we have arranged in this experiment,

$$\Delta_0 = Z_4(Z_1 + Z_2)(Z_2 + Z_3),$$

whence
$$\frac{di}{dZ_4} = -\frac{E}{Z_4(Z_1 + Z_2 + Z_3 + Z_4)},$$

or
$$\delta i = -\frac{E}{(Z_1 + Z_2 + Z_3 + Z_4)}\cdot\frac{\delta Z_4}{Z_4}.$$

The factor $\dfrac{1}{(Z_1 + Z_2 + Z_3 + Z_4)}$ is a measure of the sensitivity of the bridge which is usually expressed in milliamperes per percentage variation in Z_4.

The sensitivity increases when the sum of the impedances in the arms of the bridge decreases.

Let the ratio of the capacities in the arms containing Z_1 and Z_2 be a.

$$Z_1 = \frac{1}{j\omega C_1},\quad Z_2 = \frac{1}{j\omega C_2}.$$

$$\therefore\ C_1 = aC_2.$$

Thus $Z_2 = aZ_1$; and at balance $Z_3 = aZ_4$. Since Z_3 and Z_4 are ohmic resistances we may write $Z_4 = R$, $Z_3 = aR$.

Thus
$$s = \frac{1}{(Z_1 + Z_2 + Z_3 + Z_4)} = \frac{1}{(1 + a)(R + Z_1)}$$
$$= \frac{1}{\left(1 + a\right)\left(R - \dfrac{j}{\omega C_1}\right)}.$$

Let
$$R - \frac{j}{\omega C_1} = Ae^{-j\theta}$$

so that
$$s = \frac{e^{j\theta}}{(1 + a)\,A},$$

where
$$A = \sqrt{R^2 + \frac{1}{\omega^2C_1^2}}\quad\text{and}\quad \tan\theta = \frac{1}{R\omega C_1}.$$

We are concerned only with the amplitude of s, not with its phase angle. In the experiment which has been quoted $\frac{1}{\omega C_1}$ was approximately 3200 ohms.

The current amplitude in the detector is thus

$$\delta i = \frac{E}{\left(1+a\right)\left(R^2 + \frac{1}{\omega^2 C_1{}^2}\right)^{\frac{1}{2}}} \frac{\delta R}{R}.$$

This formula indicates that the sensitivity, s, increases as R decreases, until R^2 gets small compared with $\frac{1}{\omega^2 C_1{}^2}$, when it tends to a constant value. A further increase in s is apparently obtained by making a small, but this is only true within the limitations of the approximations we have made, i.e. Z_5 and Z_6 are supposed to be small.

The increase obtained in making a small means that Z_2 and Z_3 are small, and this is in the direction of the general result that all the impedances should be equal. In the present case this means that they should all be small. The experiment consists in measuring δi and the corresponding value of $\frac{\delta R}{R}$. To do this let $\frac{\delta R}{R} = 5$ per cent both ways, i.e. with δR positive and with δR negative.

Take the average value of δi and determine $\dfrac{\delta i}{\frac{\delta R}{R}}$.

Denote this by y and plot $\frac{1}{y^2}$ against R^2.

The relation is

$$Ey^2 = \left(1+a\right)^2\left(R^2 + \frac{1}{\omega^2 C_1{}^2}\right),$$

so that a linear graph should result with a slope $(1+a)^2$. Let a be given the values $\frac{1}{2}$, 1, 2, 5, by adjusting the variable condenser, and verify that the slopes are in agreement with these values.

The intercept on the axis of R^2 is negative and of the numerical value $\frac{1}{\omega^2 C_1{}^2}$. This also should be verified.

A suitable detector for this purpose is a moving-coil vibration galvanometer (Duddell oscillograph, p. 526), which can be tuned to the frequency of the 50 cycles mains A.C. by adjustment of the length and tension of the suspension.

The amplitude of the oscillations of the current in the galvanometer coil is measured by the width of the band of light which is reflected on to a scale by the mirror of the instrument. If the band has a breadth

when the balance condition is obtained this must be subtracted from the bands obtained when a definite deflection is observed.

It is important to place a switch in series with the galvanometer and to leave it open until the boxes are set near to the condition of balance. The necessary resistances can be calculated for this case.

The Effects of Inductive and Capacitative Errors in Resistances and of Losses in Condensers

The usual method of winding resistance coils results in residuals of self-inductance in series and of capacity in parallel as illustrated in fig. 405.

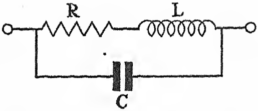

FIG. 405

The magnitudes of C and L are small enough to allow their product to be neglected.

Thus the impedance associated with the resistance, R, is

$$\frac{(R + j\omega L)}{1 + j\omega C\,(R + j\omega L)} \simeq R + j\omega\,(L - CR^2).$$

For coils below 100 ohms $(L - CR^2)$ is usually positive, and for higher resistances usually negative.

In resistance boxes of the low-inductance type compensation is made for this effect.

If a condenser is faulty it can be represented as a pure capacity, C, in series with a resistance, r. It then has a power factor, $\omega r C$. This factor varies very little with the frequency, r varying inversely as ω.

In order to examine these effects, let a de Sauty bridge be used with an alternating source of supply in the audio-frequency range.

As a standard capacity, use a mica condenser of about $\frac{1}{3}$ μF. and let the resistances be of the low-inductance type. Let the test condenser be also of high insulation with a value not far from $\frac{1}{3}$ μF.

With various values of the standard resistance, R, e.g. 500, 1000, and 2000 ohms, adjust R_0 to produce silence in the phones of the arm, BE. Make a careful record of the results of repeated settings of the value of R_0 in each case. With condensers of good insulation and low-inductance resistance it should be possible to obtain agreement to one-thousandth for R_0.

Now replace the standard R by an ordinary resistance box where compensation has not been made for the inductive and capacitative

effect. Let the condensers and the resistance, R_0, be perfect and non-inductive respectively. In order to obtain a balance the following condition must be satisfied:

$$\frac{R + j\omega (L - CR^2)}{R_0} = \frac{C_0}{C}.$$

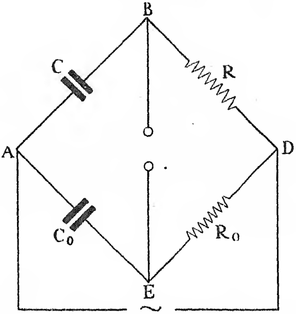

FIG. 406

The appearance of the imaginary term makes this impossible and the practical result is that silence cannot be obtained in the detector by adjustment of R_0.

It is necessary to place a resistance, r, in series with one of the condensers.

It is convenient to arrange terminals A and A′ at the ends of the resistance, r, so that the supply may be connected to one terminal or the other, thus placing r either in series with C or C_0. r should be variable and of the magnitude of a few ohms.

In the diagram λ denotes that R is to be regarded as having a reactive component, $\lambda = L - CR^2$. With r in series with C, the condition of balance is

$$\frac{\left(r + \frac{1}{j\omega C}\right)}{\frac{1}{j\omega C_0}} = \frac{(R + j\omega\lambda)}{R_0}.$$

This leads to the two conditions:

$$CR = C_0R_0, \quad \lambda = C_0R_0r.$$

For resistances, R, less than 100 ohms, λ is positive, and in the above arrangement of the bridge a balance is possible. When R is

greater than 100 ohms λ is negative and r must be placed in series with C_0 and the supply connected to A′.

In this case the two conditions are

$$CR = C_0R_0 + \omega^2\lambda rCC_0 \quad \text{and} \quad \lambda = -rRC_0.$$

The product λr is small, so that the first condition is in practice

$$CR = C_0R_0.$$

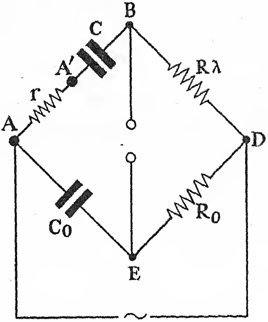

Fig. 407

Thus, in the comparison of the two capacities the additional resistance r has no effect.

In the experiment determine the value $\frac{\lambda}{R}$ for values of $R = 10$, 100, and 1000 ohms.

To examine the case of a faulty condenser, retain a mica condenser as a standard, but choose as a test condenser one with paraffined paper insulation with a capacity of about 2 μF.

The resistances, R and R_0, should be of the low-inductance type.

If C is the faulty condenser, it can be represented as a capacity with a series resistance, S, and an exact balance with a perfect condenser, C_0, is not possible.

A resistance, S_0, must be placed in series with C_0 and both S_0 and R should be adjustable to 0·1 ohm. R and R_0 should be of low inductance, but S_0 may be taken from any box.

The experiment should be carried out with $R_0 = 500$ and 1000 ohms.

The condition of balance requires

$$\frac{S}{S_0} = \frac{R}{R_0} = \frac{C_0}{C}.$$

Thus S_0 should be the same for the two values of R_0 and $S = \dfrac{C_0 S_0}{C}$.

Finally, suppose that the condenser, C, is imperfect and is represented by a pure capacity, C, with a resistance, S, in series, and that the resistance, R, is not of the low-inductance type, but has reactance $\omega\lambda$. The resistance, R_0, is supposed to be a low-inductance resistance, and C_0 a perfect capacity. Let a resistance, S_0, be placed in series with C_0.

The bridge is represented in fig. 408.

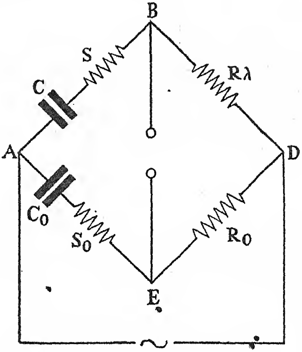

Fig. 408

A balance is obtained when the condition

$$\frac{(R + j\omega\lambda)}{R_0} = \frac{\left(r + \dfrac{1}{j\omega C}\right)}{\left(r_0 + \dfrac{1}{j\omega C_0}\right)}$$

is satisfied. This requires

$$\frac{R r_0 + \lambda}{C_0} = R_0 r \quad \text{and} \quad \frac{R}{C_0 - \omega^2 \lambda r_0} = \frac{R_0}{C}.$$

Neglecting λr_0, the value of the capacity, C, results from the usual formula:

$$\frac{R_0}{C} = \frac{R}{C_0}.$$

But the resistance, r, and consequently the power factor, $\omega r C$, cannot

be determined accurately because of the unknown quantity, $\frac{\lambda}{C_0}$. This illustrates the importance of the use of pure resistances, as in the case of Wien's series resistance bridge already described (p. 612).

The Determination of the Effective Inductance and Effective Resistance of an Air-cored Transformer

Suppose that a primary circuit consists of a generator, E, connected to an inductance, L_1, and a resistance, R_1, in series.

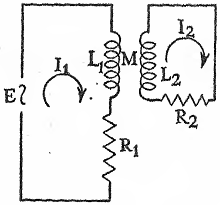

Fig. 409

Let this primary circuit be coupled to a secondary circuit consisting of an inductance, L_2, and a resistance, R_2, the mutual inductance being M.

The action of the secondary upon the primary is in effect to alter its self-inductance and resistance by amounts which depend upon the constants of the secondary, and upon M.

In the first place, these changes will be calculated by the application of the theory of circuits and the results of the calculation will be tested experimentally.

Suppose that the generator, E, produces a current of frequency, f, and of magnitude, I_1, in the primary. Let the current induced in the secondary be denoted by I_2, and suppose that the effect of mutual inductance acts in such a way as to oppose E.

Then, for the primary circuit

$$E = (R_1 + j\omega L_1)\, I_1 - j\omega M I_2,$$

and for the secondary

$$j\omega M I_1 = (R_2 + j\omega L_2)\, I_2.$$

It thus follows by substitution for I_2 that

$$\begin{aligned} E &= \left(R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2}\right) I_1 \\ &= \left\{\left(R_1 + \frac{\omega^2 M^2 R_2}{R_2^2 + \omega^2 L_2^2}\right) + j\omega\left(L_1 - \frac{\omega^2 M^2 L_2}{R_2^2 + \omega^2 L_2^2}\right)\right\} I_1. \end{aligned}$$

It thus follows that the primary circuit behaves as if its resistance had changed to

$$R_1' = R_1 + \frac{\omega^2 M^2 R_2}{R_2^2 + \omega^2 L_2^2},$$

and its self-inductance to

$$L_1' = L_1 - \frac{\omega^2 M^2 L_2}{R_2^2 + \omega^2 L_2^2}.$$

R_1 and R_2 denote the total resistances in the primary and secondary circuits respectively.

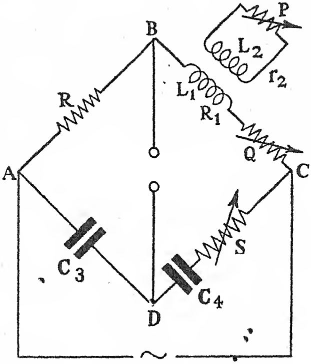

FIG. 410

The test of these formulae can be made by means of Owen's bridge. The primary circuit consists of a coil, L_1, of self-inductance, L_1, and resistance, R_1. A self-inductance of about $\frac{1}{3}$ H., or rather higher, is suitable, and it is placed in series with a variable standard resistance, Q, which must be of low inductance. The secondary circuit consists of a self-inductance, L_2, also about $\frac{1}{3}$ H., and a variable standard resistance, P, of low inductance. If the resistance of the coil L_2 is r_2, the total resistance of the secondary is $R_2 = P + r_2$. The effective resistance in the arm, BC, when coupled to the secondary circuit is $(R_1' + Q)$, where R_1' is given by the above formula. The resistance, R, may be set at 100 ohms and S should be a variable standard. Both should be of low inductance. Similarly, the resistance, Q, is a variable standard and may consist of resistance boxes in series, one with a range 0–1 ohm, the other with a range 0–10 000 ohms. The condensers

C_3 and C_4, should be mica condensers. Capacities of $\frac{1}{3}$ μF. have been found suitable.

The frequency should lie in the audio-frequency range about 1000 c/s., and it is advisable to use a pair of telephones of good quality.

The two conditions of balance are:

$$C_3R = C_4(R_1' + Q),$$

and

$$L_1' = C_3RS.$$

L_1' is the effective self-inductance of L_1 when coupled to the secondary, and its value is given above in terms of L_1 and the constants of the secondary circuit.

Begin by determining the values of L_1 and L_2 and of their resistances, R_1 and r_2. This can be done by using the bridge in the way already described, inserting first L_1 and then L_2 into the arm, BC.

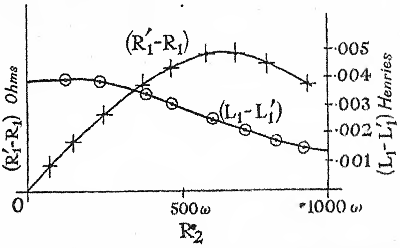

FIG. 411

When this has been done place the coil, L_2, in a fixed position with respect to L_1, e.g. lying on the top of it, and connect L_1 into the arm, BC. Complete the secondary circuit through a resistance, P, which can be set at a series of values. This resistance can conveniently be a low-inductance resistance box ranging from 0 to 10,000 ohms, and it should be set at various values, e.g. 0, 50, 100, 150, 200, 300 . . . 1000 ohms, and finally at infinity corresponding to the removal of the secondary.

According to the formulae derived from values of $(R_1' - R_1)$ and $(L_1 - L_1')$ will be obtained as a function of R_2, i.e. $(P + r_2)$. Thus, tables of values of $(R_1' - R_1)$ and $(L_1' - L_1)$ for the corresponding values of R_2 can be obtained and graphs drawn to show the dependence of the quantity on R_2.

Fig. 411 shows the results of an experiment.

In order to test the formulae from the curves the values at the maxima should be examined.

The expression for $(R_1' - R_1)$ indicates that this should occur on the curve of variation of resistance at the value

$$R_1' - R_1 = \frac{\omega M^2}{2L^2},$$

where $\omega = 2\pi \times$ frequency of the electric oscillations.

In the case of $(L_1 - L_1')$ the maximum lies at the value

$$L_1 - L_1' = \frac{M^2}{L_2}.$$

The ratio of the two values should thus be $\frac{\omega}{2}$.

The two formulae each give a value for M^2 and a good agreement in a case where the mutual inductance is obtained may be regarded as a test of the validity of the formulae deduced.

As a third test, the slope of the resistance curve at the value $R_2 = 0$ may be examined.

According to the formula this should be

$$\frac{d}{dR_2}(R_1' - R) = \frac{M^2}{L_2^2} \text{ at } R_2 = 0.$$

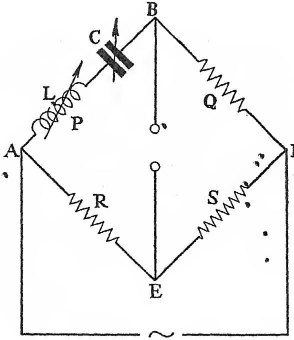

FIG. 412

Determination of the Frequency of an Oscillator by a Bridge Method, using the Resonance Condition

In this bridge a variable standard condenser and a variable standard self-inductance are placed in series in one arm of a bridge with non-inductive resistances on the remaining arms. The generator is within the audio-frequency range and the object is to determine its frequency

When the bridge is balanced with Q and S placed equal the impedance in the arm, AB, containing the inductance and capacity, must be purely resistive in character. Thus the resonance condition

$$L\omega = \frac{1}{C\omega}$$

must be satisfied.

C is set at various values and L and R adjusted to produce the condition of balance. This is repeated by altering C and readjusting L. It may be advisable to add a variable non-inductive resistance in the arm, AB, in case the resistance of L is small, and it is difficult to adjust R to a sufficiently low value. L should be plotted against $\frac{1}{C}$ when a linear graph through the origin results with a slope of $\frac{1}{\omega^2}$. In a particular case suitable values for Q and S were found to be 500 ohms, and the range of C was from 0·25 μF. to 0·75 μF., corresponding to values of L about 0·2 H.

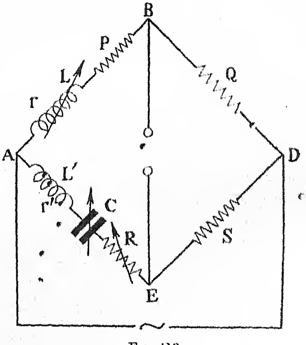

FIG. 413

Determination of the Frequency of an Oscillator by means of the Frequency Bridge

The chief object of this experiment is the determination of the frequency of an oscillator, but at the same time the self-inductance and resistance of a coil can be found.

For this purpose a variable standard self-inductance and a variable standard capacity are required. The resistances placed in the arms of

the bridge, P, Q, R, and S, should be non-inductive, and it will be assumed that the condenser is without loss. Q and S should be made equal. In a particular experiment they were 500 ohms, and the unknown self-inductance was about 1 H.

Let the condenser be set at a value, C, and L and R varied to produce silence in the telephones used as the detector. With $Q = S$, the condition of balance is

$$r + P + j\omega L = R + r' + j\omega L' + \frac{1}{j\omega C},$$

where r' is the resistance of the coil, L′, and r that of L.

Thus $$L = L' - \frac{1}{\omega^2 C},$$

and $$r' = P - R + r.$$

The first of these relations applies when $L > L'$.

If L′ lies beyond the upper limit of the standard inductance, L, they must be interchanged in the arms of the bridge, the equations for this case being

$$L' = L - \frac{1}{\omega^2 C},$$

and $$r = P - R + r'.$$

The determination of the resistance, r', is not important for our present purpose. It will be noticed that it requires a knowledge of the resistance, r, of the standard inductance. The use of both resistances, P and R, avoids difficulties that may arise if the relative magnitudes of r' and r are unknown. Thus, the first relation for r' cannot be satisfied, if R is not inserted, unless $r' > r$, nor the alternative relation unless $r' < r$.

When a difficulty occurs in obtaining silence in the telephones after varying L and R alternately, proceed to vary P. Finally, try interchanging the coils L and L′.

It is advisable to know approximately the value of L′, and it may be determined by one of the methods described.

When the balance is obtained record L and C and repeat the experiment for other values. Thus it is possible to tabulate L and corresponding values of $\frac{1}{C}$ and to draw a graph which, according to the equation in L, L′, and C, should be linear. The intercept on the axis of L gives L′ and the slope $\frac{1}{\omega^2}$.

For use in the laboratory a suitable frequency has been found to be 500 c/s. and for the capacity a range from 0·10 to 0·15 μF

The Use of the Wagner Earthing Device

When the impedances to be measured are small, in the case of capacities, for example, when they have values of the order of 0·001 μF., it becomes necessary to pay attention to methods of avoiding stray capacities in bridge measurements at high frequencies.

An important method of eliminating their effects is provided by the Wagner earthing device. It can be used in connexion with any bridge, and it will be illustrated here by means of Wien's series resistance capacity bridge.

Let ABCD denote a Wheatstone network in which the arms, AD and DC, contain equal resistances, R.

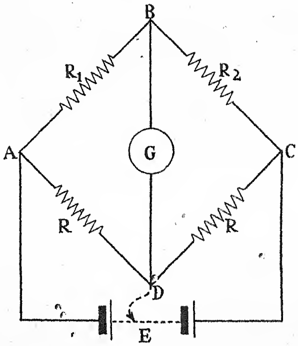

FIG. 414

If this is a direct current bridge and no current can leak away from it, the condition of balance is $R_1 = R_2$. If current escapes from D due to imperfect insulation, the current in AD will have a value, i, and in DC a different value, i', in the case when no current flows through the galvanometer. In this case, assuming no loss at B, we have

$$\frac{Ri}{Ri'} = \frac{R_1}{R_2}.$$

The escape of current from D can be represented as a flow to earth, i.e. as a flow through a resistance to a point of the battery which happens to be at zero potential.

For direct currents in the simple method of measuring ohmic resistances such a leakage is very small and produces no appreciable result, but for varying currents in bridges the position is very different.

When an operator uses the telephone, T, or touches any part of the bridge (fig. 415) he puts an impedance between the arms of the bridge and the earth. This impedance is distributed along the arms, but it can be represented with a sufficient degree of accuracy as if it were lumped at the corners of the bridge. The impedance is usually a capacitance and is represented diagrammatically, as at D in the figure, but such capacitances may occur at the other corners of the bridge. Unless the reactances occurring in this way are small in comparison

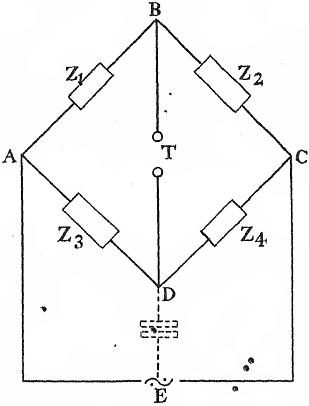

FIG. 415

with the impedances of the arms, serious inaccuracies may occur. Thus, if Z_3 and Z_4 are capacitances of the order of 100 $\mu\mu$F., the errors are liable to be great.

If impedances, Z_1, Z_2, Z_3, and Z_4 are placed in the arms of the bridge the condition in general will be that represented in fig. 416.

The point, E, denotes a point at earth potential and the capacities, X, Y, U, and V, are stray capacities. These lie in parallel with the arms of the bridge and will affect the condition of balance. If the points B and D are brought to earth potential in the process of balancing the effects of the capacities U and V are eliminated.

It is important to consider how the points, B and D, are brought to the earth potential. The simplest way of doing this would be to earth D directly and thus eliminate the effect of V. By adjustments in the

arms of the bridge, silence would finally be attained in the telephones and B would also be at earth potential, so that the capacity, U, would have no effect. In this case the bridge has been balanced for the condition in which X and Y are in parallel with Z_3 and Z_4 respectively. The condition of balance is therefore:

$$Z_1 : Z_2 = \frac{Z_3 X}{Z_3 + X} : \frac{Z_4 Y}{Z_4 + Y}.$$

Since X and Y are unknown and are dependent on the condition of the bridge and the method of using it, it is impossible to make use of this relation.

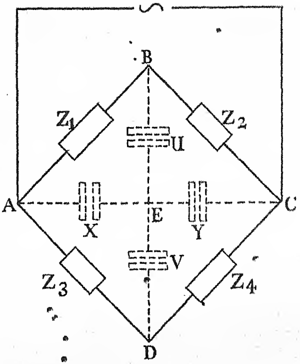

FIG. 416

The use of the Wagner earthing device overcomes this difficulty. Two additional arms containing impedances Z_5 and Z_6 are provided, as illustrated in fig. 417. These arms connect the points, A and C, which are connected to the generator, to an earthed point, E. The point is connected to one side, F′, of a two-way switch, LFF′, the other side, F, being connected to D. A balance is required in the two cases when L is in contact with F′ and when in contact with F. It will be noted that Z_5 is in parallel with X and Z_6 with Y. If a final balance in both conditions is obtained by making contacts at F′ and F alternately, B and D are finally at zero potential, and we have:

$$Z_1 : Z_2 = \frac{Z_5 X}{Z_5 + X} : \frac{Z_6 Y}{Z_6 + Y} = Z_3 : Z_4.$$

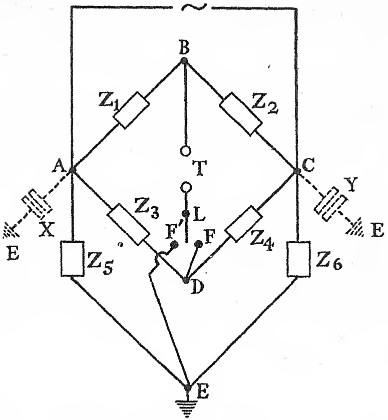

FIG. 417

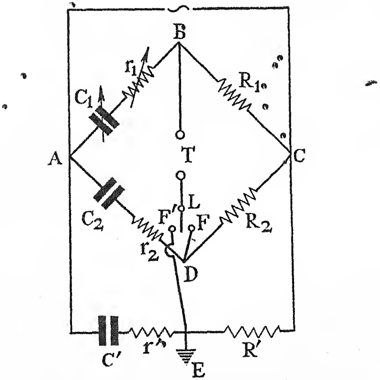

FIG. 418

Thus one of the impedances, Z_1, Z_2, Z_3, and Z_4, can be obtained if the others are known.

In practice the ratio $\frac{Z_5}{Z_6}$ approximates closely to $\frac{Z_1}{Z_2}$ and $\frac{Z_3}{Z_5}$, so that the Wagner arms imitate the arms of the bridge. For a sensitive bridge the arms should be roughly of the same order of magnitude.

For the purpose of the experiment the capacity, C_2, to be measured should not be greater than 0·001 μF. Use the bridge illustrated in fig. 395, the theory of which has already been discussed (p. 612).

The resistance, r_2, is that of the condenser C_2 and is determined by the experiment.

First connect the detector across BD by making contact between L and F, and obtain a balance by altering the arm, AB, say. Then alter the switch so that contact is made between L and F′ and balance the bridge, ABCE, by altering the arms AE or CE. Repeat this process alternately until a condition of balance occurs for both positions of the switch. In this case B, D, and E are all at the same potential, and by applying the usual relation for balance to the bridge, ABCD, the following relations are obtained:

$$r_2 = \frac{R_2 r_1}{R_1}, \quad C_2 = \frac{R_1 C_1}{R_2},$$

which give the resistance and capacity of the condenser under test.

CHAPTER XXIII

THE QUADRANT ELECTROMETER

A DEVELOPMENT of the Kelvin quadrant electrometer is seen in fig. 419, and is due to Dr. F. Dolezalek. The four quadrants, QQ, etc., are supported on ambroid pillars, AA. As shown in the figure, two quadrants are mounted on a pivot, and may be swung aside to allow of the

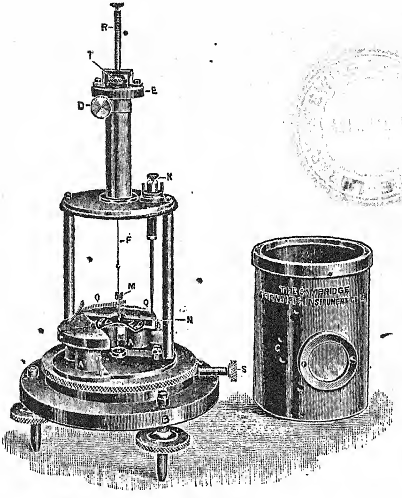

FIG. 419

introduction of the 'needle'. Alternate quadrants are joined together to terminals under the base plate of the instrument, care being taken that such terminals are very well insulated from the case.

The needle may be either a light paper frame (fig. 420), coated with a metal to make it conducting, or, as in fig. 421, a thin mica sheet sputtered with silver. In some of the more recent instruments a very

thin aluminium needle of the shape of the second form is used. The needle is attached to a light rod which carries a small mirror, M, and is supported by means of a thin quartz or phosphor-bronze strip from the torsion head, R.

The method of observing the deflection produced by a difference of potential on the quadrants consists of the use of the usual lamp and scale, as with galvanometers.

A beam of light from a lamp is directed on the mirror, M, which reflects the light on to a scale placed one metre away. If the mirror

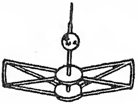

FIG. 420

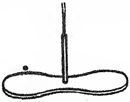

FIG. 421

is concave, and of the correct focal length, a clearly defined image of the source will be obtained on the scale. If M is a plane mirror, a lens is required to produce a sharp image. Such a lens is placed in the path of the incident beam from a source of light below the scale; the reflected beam does not, therefore, pass through the lens.

The Suspension

(1) *Quartz Fibre*

If a quartz fibre is used as a suspension, the needle may be charged by means of the charging device shown at K. K is connected to the source of potential and the metal rod turned until contact is made between it and the rod which supports the mirror on the needle. K is then turned to its original position, leaving the needle charged. Owing to the very good insulating property of the quartz, this charge will be maintained. However, there is a danger of breaking the quartz suspensions during this process.

For most purposes it is more convenient to avoid using the charging device. To make this possible the suspension is coated lightly with a calcium chloride solution. The hygroscopic property of the latter ensures a conductivity which is sufficiently good to maintain the needle at the potential of the source which is permanently connected to R.

An alternative method is to sputter the quartz with silver, but this cannot as a rule be carried out conveniently in the laboratory.

Suitable quartz fibres are very simply made by the aid of a coal gas-oxygen flame. A burner for this mixture is shown in fig. 422. A piece of quartz rod is heated in the flame until thoroughly soft; when at this stage the ends are drawn apart. If this is done rapidly a fine quartz thread will result. If a still finer fibre is required draw out the

quartz rod to about 1 mm. diameter, and then reintroduce into the flame. When the quartz becomes very soft the pressure in the flame itself will blow the quartz outwards into very fine fibres. It is advisable to have a black velvet cloth on the bench to receive these threads.

Small hooks may be fastened to the ends of the fibre, when cut to the correct length, by means of a small globule of shellac solution in methylated spirits. A hot iron is held over the globule, which is placed on the hook over the end of the quartz. The spirit evaporates and the fibre adheres.

Another method of fastening the hook to the fibre is to use Indian ink. The end of the hook is dipped into Indian ink which is allowed

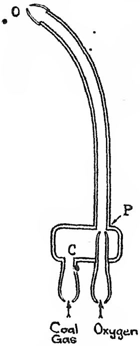

FIG. 422

to become 'thick' by evaporating. The fibre is then placed on this plastic drop. When dry the two will be found to be very firmly held.

(2) *Phosphor-bronze Strip*

Of the metal suspensions phosphor-bronze strip is most satisfactory. Platinum and tungsten wires may be obtained with a smaller diameter, but usually have the disadvantage that the zero of the needle is not stable when very fine suspensions of these metals are used.

For the experiments described in this chapter, the finer phosphor-bronze strip obtainable commercially gives a sensitivity which satisfies the needs of the experiment. If, however, the phosphor-bronze strip available is too thick for the purpose, it may be treated as described below. This method enables a fair sensitivity to be attained.

A solution of one part nitric acid to four parts of water is taken, and the suspension immersed in it. The action of the acid causes an evolution of gas which adheres to the suspension, causing it in time to rise to the surface. The chemical action should be slow. If violent action takes place the solution should be further diluted. Of course a preliminary test should be made with a small sample of the phosphor-bronze to be used in order to avoid undue waste.

The strip should be very well washed and dried, and its sensitivity tested in the instrument. The process is repeated until the required sensitivity is obtained. When this process has once been performed with the specimen of strip available, the time of immersion in the acid required to reduce to a suitable cross-section may be very readily estimated, and the time spent in testing will be reduced, e.g. the strip may be sufficiently reduced by immersing four times in the acid, allowing it to come to the surface each time.

In this way a sensitivity of 700 to 1000 cm. per volt may be obtained with the instrument shown, but for the experiment in this chapter a sensitivity of about 25 cm. per volt will be found to be quite sufficient.

The above process is carried out, using the wire without hooks. If the hooks are soldered on to the ends, the action of the acid will sever them. The small hooks are then soldered to the ends of the strip, using a very small pointed iron. Care should be taken that the ends and hooks are clean and the iron hot. Use soft solder and a trace of fluxite.

Adjustment

To prepare the instrument for use, the following adjustments must be made. Both pairs of quadrants are connected to earth, and are therefore at the same potential; R is also earth connected. Thus, the only forces acting on the needle are those due to the torsion of the suspension. The needle is raised or lowered by means of the screw, T, until it swings about the mid-plane within the quadrants. By means of the levelling screws, B, the whole instrument is then levelled, so that the suspension hangs centrally within the quadrants. To ensure this, the rod which supports the mirror, M, is sighted along the two diagonal spaces between the four quadrants and a slight adjustment made if required to bring the rod truly central. The torsion head, E, is then turned until the needle appears symmetrical with respect to the quadrant. A suitable high potential of, say, 100 volts is applied to the needle (in the manner described later (fig. 423)). If the needle position is precisely symmetrical no movement should result on making this change. Any error in the adjustment by eye for symmetry may now be corrected, by noting the direction of deflection of the needle. The potential is removed and the needle earthed. The torsion head is given a very slight turn in the direction of the movement previously observed.

The needle is again charged and the process repeated, until on charging and earthing the needle alternately the deflection produced is of the order of a few cm. only. To reduce this small deflection to zero a slight adjustment of a levelling screw will be found sufficient—a portion of a turn of one of the levelling screws will be found to increase or decrease the deflection produced. When the potential is applied to the needle, the screw is turned in the direction required to reduce the deflection until it is finally eliminated.

With a fine suspension the above process is apt to be somewhat tedious until the student becomes familiar with the instrument; but it is essential to the success of any observation.

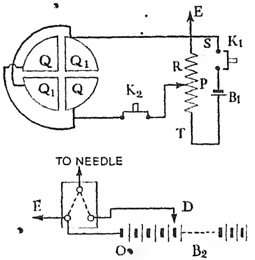

FIG. 423

When this adjustment has been satisfactorily made, the instrument should be tested for leak. To do this the arrangement of apparatus shown in fig. 423 is a convenient one, for by means of this the adjustment already described may be also performed. Thus, when the two-way switch is to the left, the needle is at zero potential: and if K_1 is open and K_2 closed, the quadrants are at zero potential. The needle is charged by closing the two-way switch to the right.

These switches are best made by boring small holes in a clean slab of paraffin wax. The holes are filled with clean mercury and connexion is made by using a length of copper wire, bent twice at right angles and mounted on a sealing-wax handle, as in fig. 424.

B_1 is a steady 2-volt accumulator; R a high resistance, provided with a slide contact, or two resistance boxes in series. Either arrangement serves as a potentiometer by means of which any fraction of the E.M.F. of B_1 may be obtained between S and P when K_1 is closed. When K_2 is closed this potential difference is applied to the quadrants.

The needle, being maintained at a fixed potential, say 100 volts, by the battery, B_2, will be deflected, causing a movement of the spot of light of, say, d cm. K_2 is then opened. If the quadrants are fully insulated no movement will be shown by the needle. If, on the other hand, a decrease in the deflection occurs, it indicates that the quadrants are losing charge, i.e. the insulating supports are faulty. The usual cause for this is dust or grease on the surface of the pillars,

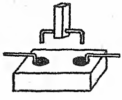

FIG. 424

which should therefore be cleaned. When this has been carried out as thoroughly as possible the test is again made. In general, one cannot remove the leak entirely, but it may be reduced to a very small amount. The rate of movement of the needle, when one pair of quadrants is charged and then insulated, is measured by timing, with a stop-clock, the movement of the spot of light. The number of divisions per second is called the *natural leak* of the instrument.

Note

The high potential is connected to the needle of the electrometer through a water resistance. This may be a small U-tube filled with water; the wire from the high potential is inserted in one limb, and the lead from the needle into the other. In the case of an accidental contact between the quadrant and the needle, the fibre would be ruined, in the absence of a high resistance. The potential on the needle is unaffected by the resistance, but unless measured by an electrostatic voltmeter there will appear to be a fall in potential. This is due to the fact that an ordinary voltmeter is not of a very high resistance compared with the water resistance. If only a moving-coil voltmeter is available to test this potential, it should be applied to the point at which the high potential enters the water resistance.

A simple theory of the instrument (see, for example, Whetham, *Electricity and Magnetism*) shows that if one pair of quadrants is maintained at a potential, V_1, and the other at a potential, V_2, the needle, being maintained at potential V_n, will move, through an angle, θ, which depends on the potential difference, $V_1 - V_2$, and

$$\theta = c\left(V_1 - V_2\right)\left(V_n + \frac{V_1 + V_2}{2}\right),$$

where c is a constant.

In general, V_1 and V_2 are small compared with V_n, and we may write:

$$\theta = c\,(V_1 - V_2)\,V_n. \qquad \ldots(1)$$

Such a simple theory therefore leads to the conclusion that the deflection for a given potential difference on the quadrants is proportional to the potential of the needle. One would conclude, therefore, that the sensitivity is proportional to this potential. This, in fact, is not so, as may be seen from the following experimental test, using the arrangement of fig. 423.

Adjust the potential difference between P and S to such a value that the electrometer gives about 20 or 30 cm. deflection (K_1 and K_2 being closed) when a potential of about 100 volts is applied to the needle.

Note the deflection produced. Throw the switch over to the left and so earth the needle and reduce the deflection to zero; when the spot of light is steady take the zero reading. Move D to some other point in the battery, and then put the switch back to the right-hand side; again read the value of the deflection. Repeat this for a wide range of values of V_n. The value of V_n is, of course, taken for each setting, by means of a voltmeter which is connected to D and earth.

The zero readings between each deflection should be the same. Plot a curve showing the relation between the deflection produced for the fixed potential difference on the quadrants, and the potential on the needle. It will be found that the deflection produced increases with V_n, but not indefinitely, i.e. there is a potential above which there is no gain in sensitivity, this is usually between 70 and 120 volts for the type of instrument described.

Maintain the potential of the needle at this value and verify the fact that the deflection is proportional to $(V_1 - V_2)$ by applying different values to the quadrants. By adjusting P on the resistance R, the ratio, $\frac{\text{resistance PS}}{\text{resistance TS}}$, may be chosen to give, say, 0·01, 0·05, 0·01, 0·15 volt. For these values the deflections will be found to be as 1 : 5 : 10 : 15.

Thus, for a fixed potential on the needle the deflection is proportional to the difference of potential of the quadrants.

Thus, the equation (1) holds so long as the needle is maintained at fixed potential, and the instrument may be used to compare potentials.

A more complete theoretical treatment of the instrument may be found in *Phil. Mag.*, 1903, by G. W. Walker, or later (*Phil. Mag.*, 1912, p. 380) by Prof. A. Anderson. Here the variation of θ with V_n is more correctly stated. This leads to the expression:

$$\theta = \frac{2\alpha V_2\left(V_n - \dfrac{V_2}{2}\right) - 2\gamma K V_n^2}{F + K V_n^2},$$

where F is the torsional couple due to the fibre per unit angular displacement; α, K, and γ are constants depending on the particular instrument for their value.

Sensitivity

By the sensitivity we understand the number of cm. or mm. deflection obtained on a scale 1 m. away, when one volt potential difference is applied to opposite pairs of quadrants.

With the same instrument this factor may have vastly different values depending upon the dimensions of the suspending fibre. In most cases, especially in laboratory work, the fibre is conducting, and very conveniently made of phosphor-bronze strip. The type of instrument with a fine phosphor-bronze strip may give, say, 60 cm. deflection per volt.

This factor may be found by use of the arrangement of fig. 423, as already set up for the previous experiments.

The needle being raised to the high potential already chosen, K_1 and K_2 are closed and P is adjusted so that a deflection of about 20 or 30 cm. is obtained. The value of the resistance of SP, r ohms say, is noted as is the total resistance, R, of ST.

A standard cell is then connected to S and P through a galvanometer, and the resistance, SP, is adjusted, keeping R the same as before, until no deflection is given in the galvanometer, i.e. the usual method of standardizing the potential is followed. If r_1 is the resistance, SP, under such circumstances, then the potential previously applied to the needle is obviously $\frac{r}{r_1}$ E, when E is the E.M.F. of the standard cell.

From this the number of cm. deflection per volt may be calculated.

As already mentioned, a suitable sensitivity is about 25 to 50 divisions per volt. The suspension should be made of suitable size to give this value. The method has already been described.

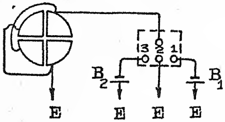

FIG. 425

Comparison of the E.M.F. of Two Cells

The adjusted quadrant electrometer is connected as shown in fig. 425. The needle is raised to the potential already found to be most satisfactory. The three-way paraffin wax switch is made and connected to

the batteries B_1 and B_2 by mercury cups, 1 and 3. Cup No. 2 is connected to earth as are the negative poles of B_1 and B_2.

By connecting to cup 2, all the quadrants are at zero potential. The battery, B_1, is then placed with its positive pole in contact with one pair of quadrants (via 1), the other pair being earthed. The deflection, d_1, is noted. The quadrants are then earthed, and B_2 is connected (via 3); the deflection, d_2, due to this also being observed.

Then $$\frac{E_1}{E_2} = \frac{d_1}{d_2}.$$

For large deflections d_1 and d_2 cm. must be replaced by the corresponding angles, θ_1 and θ_2; this may be readily done, for

$$\tan 2\theta_1 = \frac{d_1}{D}, \quad \tan 2\theta_2 = \frac{d_2}{D},$$

whence $$\frac{E_1}{E_2} = \frac{\theta_1}{\theta_2}.$$

D denoting the distance between mirror and scale.

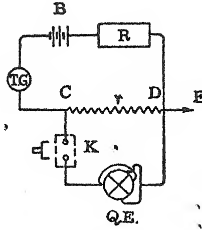

FIG. 426

Verification of Ohm's Law

For this purpose a battery, B, an adjustable resistance, R, and a tangent galvanometer are connected in series with a resistance of fixed amount, r ohms, which may be a length of manganin or platinoid wire (fig. 426).

The end, C, is connected through a paraffin wax switch, K, to one pair of quadrants of an adjusted electrometer, and the end, D, is connected to earth and the other pair of quadrants.

The current flowing in the wire, CD, may be measured by means of the tangent galvanometer, for if θ is the deflection produced in this instrument as measured by means of the lamp and scale method,

we have $$i = \frac{Ha}{2\pi n} . \tan \theta,$$

or
$$i \propto \frac{\tan\theta}{n},$$

where n is the number of turns used.

Further, the potential difference between the ends, C, D, is proportional to φ, the deflection produced in the quadrant electrometer.

If φ is small, the corresponding scale deflections, d, may replace φ below.

For a range of values of i it may become necessary to alter the number of turns, n, used in the galvanometer. Vary R and tabulate the results as below.

$i \propto \frac{\tan\theta}{n}$	$E \propto \varphi$	$\frac{E}{i} \propto \frac{n\varphi}{\tan\theta}$

It will be found, if care has been taken to avoid large currents which would cause appreciable heating in the wire, that the value of $\frac{\varphi n}{\tan\theta}$ is constant.

If the sensitivity of the Q.E. has been found, the value of the relation, $\frac{\text{potential}}{\text{current}}$, may be found absolutely, i.e. for a sensitivity of d cm. per volt, deflections, φ_1, φ_2, etc., being small, the potential drop along r is $\frac{d_1}{d}, \frac{d_2}{d}, \ldots$ volts.

i may be calculated in amperes from the usual tangent galvanometer formula:

$$i = \frac{5Ha}{\pi n} . \tan\theta,$$

where a is the mean radius of the coils, H the horizontal component of the earth's magnetic field (about 0·185 unit in London).

Whence, tabulating absolute values, the third column gives the constant value of the resistance, r, at room temperature in ohms. This should be tested independently.

Measurement of High Resistance

The following is a method of measuring a high resistance by finding the rate of leak of a charged condenser through the resistance.

A quadrant electrometer is set up and adjusted as previously

described, care being taken to reduce the natural leak to a minimum. One pair of quadrants is connected to earth; the other pair, as shown in fig. 427, is connected to a condenser, K, and through a switch, K_1 (see p. 648), to a 2-volt cell, B, the other pole of which is earth-connected. Connexion is also made to a two-way switch, K_2, by means of which this pair of quadrants may be connected to earth or to an earth-connected resistance whose value, R, is to be determined.

With K_2 open, close K_1; in this way the condenser will be charged to a potential difference equal to that of the cell, B. Note the deflection, d_1, corresponding to an angular deflection, φ_1. This is due to a potential difference of V_1 volts, say. Now close K_2 by joining 1 and 3; open

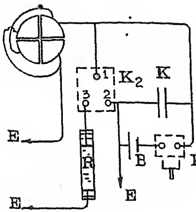

FIG. 427

K_1 and start a stop-watch. The charge on the condenser will slowly leak to earth through R. The potential of the quadrants will in consequence be reduced, i.e. d or φ will gradually decrease. If this process be allowed to continue for about two minutes or until the deflection is reduced to about 60 per cent of its original value (t seconds), the final deflection, d_2 (angular deflection φ_2), will be shown to be such that

$$R = \frac{t}{C \log \frac{\varphi_1}{\varphi_2}}.$$

For let Q and V be the charge and potential on the quadrants at any time,

Q_1, V_1 the corresponding values at the moment K_1 is opened,

Q_2, V_2 the values after t seconds,

C the capacity of the condenser,

i, the current at any instant, is equal to the rate of decrease of the charge on the condenser,

or $$i = -\frac{dQ}{dt}.$$

Also by Ohm's law $i = \frac{V}{R} = \frac{Q}{CR}$, since $V = \frac{Q}{C}$,

i.e.
$$-\frac{dQ}{dt} = \frac{Q}{CR},$$

or
$$-\frac{dQ}{Q} = \frac{dt}{CR}.$$

Integrating,
$$-\int_{Q_1}^{Q_2} \frac{dQ}{Q} = \int_0^t \frac{dt}{CR},$$

$$\log \frac{Q_1}{Q_2} = \frac{t}{CR},$$

$$R = \frac{t}{C \log \frac{Q_1}{Q_2}}; \qquad \ldots(2)$$

and since
$$\frac{Q_1}{Q_2} = \frac{V_1}{V_2} = \frac{\varphi_1}{\varphi_2},$$

$$R = \frac{t}{C \log \frac{\varphi_1}{\varphi_2}}.$$

If φ_1 and φ_2 are small we may approximate and write:

$$R = \frac{t}{C \log \left(\frac{d_1}{d_2}\right)} = \frac{t}{2{\cdot}303\, C \log_{10} \left(\frac{d_1}{d_2}\right)}. \qquad \ldots(3)$$

In making the determination of R several values of t should be taken—the range of values will be largely fixed by the value of R and the condenser used. When several values are obtained, the mean value of $\frac{t}{\log \frac{d_1}{d_2}}$ is taken and substituted in equation (3) above. When C is expressed in farads, t in seconds, R is in ohms.

The method outlined above may be very conveniently used for a determination of the specific resistance of, say, cadmium iodide solution in xylol. Such a solution is very useful when a high resistance is required for any purpose. The value of the specific resistance of solutions of different concentrations should be measured. Pure xylol will be found to be practically without effect on the leak of the condenser.

A suitable container of liquids is made by selecting a length of glass tubing of uniform bore about 2 cm. diameter and about 80 cm. long. This is closed at both ends by corks and the tube clamped vertically (fig. 428). A disk of brass which just fits the tube rests on the upper

surface of the lower cork and can be connected to an outside circuit by means of a thin brass rod soldered to the under-surface of the disk and passing through the cork. This disk serves as one electrode.

A second brass disk of the same area is supported vertically above this by means of a brass rod which passes through the upper cork, as seen in fig. 428.

Using such a container a definite length of liquid, l, with an area of cross-section practically equal to that of the area of the disks may be employed and hence the specific resistance may be calculated from the value of R measured in the experiment, l, the distance between the disks, and a, the radius of the disks.

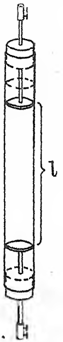

FIG. 428

Note

This experiment could also be performed using a ballistic galvanometer. The value of Q_1 at the commencement could be obtained by discharging the condenser through the ballistic galvanometer. The condenser is then recharged, allowed to leak through the resistance for a measured time, and then discharged through the galvanometer once more, thus Q_1 and Q_2 are measured and R is obtained from (2).

Measurement of the Capacity of the Quadrant Electrometer

Fig. 429 shows a simple arrangement of apparatus which will enable an estimation of the capacity of the quadrant electrometer to be made.

K_1 is a small capacity of known size. When switch, S_1, is closed from 1 to 2, and S_2 is closed, the condenser becomes charged, and the

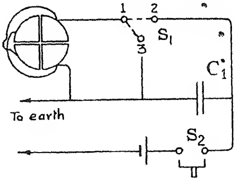

FIG. 429

electrometer is deflected an amount, φ_1 say, corresponding to a scale deflection of the spot of light of d_1 cm. If now switch, S_1, is closed from 1 to 3, the electrometer becomes discharged, and when the connecting strip is replaced to the position 1 to 2, S_2 now being open, the

condenser, C_1, shares its charge with the electrometer. If C_1 is chosen of the same order of magnitude as the capacity, C, of the electrometer, the latter will show a marked drop in deflection compared with the former, φ_1. Let this new deflection be φ_2.

The charge lost by the condenser, C_1, is equal to that gained by the electrometer, CV_2, where V_2 is the final potential corresponding to the deflection φ_2. This loss is also $C_1 (V_1 - V_2)$, where V_1 is the original potential,

i.e. $$CV_2 = C_1 (V_1 - V_2),$$

or $$C = C_1 \left(\frac{V_1 - V_2}{V_2}\right) = C_1 \frac{\varphi_1 - \varphi_2}{\varphi_2},$$

whence, if C_1 is known, C may be calculated.

If the known small condenser has a capacity which is large compared with the capacity of the electrometer, the value of φ_2 will not be very different from φ_1, and the value of C, as calculated from the above equation, will be inaccurate.

If no smaller capacity is available, the following slight modification of the method may be made.

The key, S_1, is moved from position 1 and 2 to 1 and 3, a definite number, n, times, i.e. the electrometer is charged and discharged n times, until the deflection is finally reduced appreciably.

At the end of the first sharing of charge we saw above that

$$CV_2 = C_1 (V_1 - V_2),$$

i.e. $$V_2 = \frac{C_1}{C_1 + C} \cdot V_1.$$

When the condenser, C_1, which is now at potential, V_2, is again connected to the discharged electrometer, a new common potential, V_3, will be the result where

$$CV_3 = C_1 (V_2 - V_3),$$

$$V_3 = \frac{C_1}{C_1 + C} \cdot V_2$$

$$= \left(\frac{C_1}{C_1 + C}\right)^2 V_1;$$

so, after n repetitions $$V_{n+1} = \left(\frac{C_1}{C_1 + C}\right)^n V_1,$$

or $$\frac{V_{n+1}}{V_1} = \left(\frac{C_1}{C_1 + C}\right)^n;$$

or if φ_1 and φ_{n+1} are the corresponding deflections,

$$\left(\frac{C_1}{C_1 + C}\right)^n = \left(\frac{\varphi_{n+1}}{\varphi_1}\right),$$

whence $$C = C_1 \left\{\left(\frac{\varphi_1}{\varphi_{n+1}}\right)^{\frac{1}{n}} - 1\right\}.$$

A suitable form of measurable capacity to use in this and the following experiments, which require a known capacity, is a circular, parallel plate air condenser having a guard ring.

The guard ring, GG (fig. 430), surrounds a central plate, A. Plate B may be adjusted by a micrometer screw attachment, M, moving past a fixed scale, S, so that the distance, d, between the plates may be measured, provided the zero reading on the scale is known, whence, if plate A has an area A sq. cm., the capacity, C_1, in the above experiment is

$$\frac{A}{4\pi d} \text{ E.S. units}$$

or

$$\frac{A}{9 \times 10^{20}\, 4\pi d} \text{ E.M. units} = \frac{A}{9 \times 10^{11}\, 4\pi d} \text{ farads.}$$

To obtain the zero reading of the instrument, the plate, A, is connected to a quadrant electrometer, and the system is given a charge, and then insulated. The plate, B, is moved towards the lower plate until the electrometer deflection is reduced to zero.

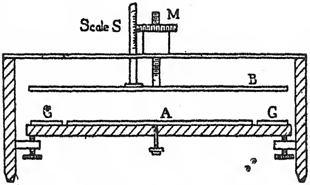

FIG. 430

To allow for possible irregularity of plate surface or a slight inclination of one of the plates, the effective zero of the condenser, may be obtained by a method which consists in finding the value of the deflection of the electrometer (φ) for each scale reading of the condenser (d). Several values of d and φ are taken as the plates approach each other. Plot φ against d. The point where this curve produced cuts the axis of d corresponds to the zero of the scale.

Another form of condenser suitable for the above experiments, and which is simply made, is described on p. 663.

Comparison of Small Capacities

Capacities such as small air condensers may be compared, using the method indicated in fig. 431. S_1 and S_2 are keys made as already described, and B is a 2-volt cell.

With S_2 closed and S_1 open, C_1 and the quadrant electrometer are charged to a potential, V_1, equal to the E.M.F. of B.

If now S_2 is opened and S_1 closed, the charge on C_1 and the Q.E. is shared with C_2.

Let φ_1 be the deflection of the Q.E. before S_1 is closed, and φ_2 the deflection after the charge is shared, corresponding to a potential, V_2.

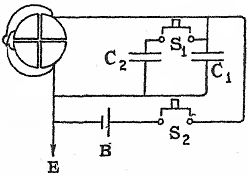

FIG. 431

We have C_2V_2 = charge acquired by C_2, and $(C_1 + C)(V_1 - V_2)$ is the charge lost by the Q.E. and C_1, where C is the capacity of the electrometer,

i.e.
$$C_2V_2 = (C_1 + C)(V_1 - V_2),$$

or
$$\frac{V_2}{V_1 - V_2} = \frac{C_1 + C}{C_2} = \frac{\varphi_2}{\varphi_1 - \varphi_2},$$

or
$$C_1 = C_2\left(\frac{\varphi_2}{\varphi_1 - \varphi_2}\right) - C.$$

Whence by observing the deflections before and after the sharing of the charge, and knowing C, the relation between C_1 and C_2 may be obtained.

When φ_1 and φ_2 are small the corresponding scale deflections, d_1 and d_2 cm., may be used in the equation.

Determination of the Dielectric Constant

The value of k, the dielectric constant of a medium, may be obtained by measuring the change in capacity of a parallel plate air condenser, when a parallel plane-faced slab of the medium is introduced between the condenser plates.

The capacity of the condenser when the plates are entirely separated by air is

$$\frac{A}{4\pi d},$$

where A is the area of the plates and d the distance between them.

If the slab of thickness, t, and dielectric constant, k, is introduced between the plates of the air condenser, the capacity becomes

$$\frac{A}{4\pi\left\{d - t\left(1 - \frac{1}{k}\right)\right\}}. \qquad \ldots(4)$$

Thus the change produced in the capacity is numerically equal to the change produced when the distance between the plates is reduced by $t\left(1 - \frac{1}{k}\right)$, when air is the dielectric.

Suppose the capacity be determined when the plates are a definite distance apart. Then let the slab be introduced; the capacity increases. If now the plates are separated until the capacity is restored to its original value, the distance, D, through which the plates are moved is obviously equal to the equivalent movement produced by the introduction of the slab. Thus

$$D = t\left(1 - \frac{1}{k}\right),$$

or

$$k = \frac{t}{t - D}.$$

To make a determination of k experimentally a convenient form of condenser is a circular parallel plate condenser, one plate of which is surrounded by a guard ring, as described on p. 657 (fig. 430).

Such a condenser is connected as shown at C_1 in fig. 429, the large plate, B, being earth-connected and the central plate, A, inside the guard ring, connected to switches S_1 and S_2. The quadrant electrometer is adjusted, and the condenser, C_1, and one pair of quadrants are raised to the potential, V_1, of the cell, by closing S_1 and S_2, producing a deflection, φ_1 or d_1 divisions. The key, S_2, is then opened, and the slab of dielectric is introduced between the plates of C_1. The slab should be of uniform thickness and have an area not less than the plate, A, of the condenser.

Since the system has a fixed charge, the effect of the increase in the capacity of the condenser is evidenced by a drop in potential to V_2 (deflection φ_2). The reading of the micrometer adjustment of the condenser is taken and the movable plate is then withdrawn until the deflection of the quadrant electrometer is again φ_1, i.e. the whole system has once more the original capacity. The movement of the condenser plate (D cm.) is known from the micrometer readings.

Hence, if t is measured in the usual way, and a mean value taken, k may be calculated.

Comparison of a Large and a Small Capacity

The method of comparing capacities given on p. 657 is applicable

when the capacities are of the same order. To compare a large capacity with a small one, the method of repeated sharing of charges may be employed. This method was outlined for the case where the difference in the capacities was not very great on p. 655. Now, when the order of the capacity of the two condensers is very different, as in the case of, say, $\frac{1}{3}$ μF., and a simple parallel plate air condenser, n, the number of times the charge is shared, must be a large number to cause an appreciable change in the potential, or in the deflection, φ. Some mechanical means must therefore be introduced to carry out rapidly the n steps. In fig. 432, which shows the arrangement of apparatus

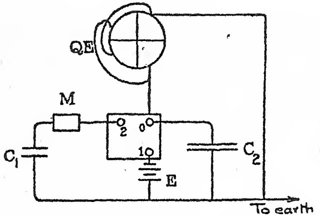

FIG. 432

for this determination, such a mechanical device, which is described later, is placed at M.

C_1 is the condenser of small capacity, C_1. A guard ring parallel plate air condenser, as described on p. 657 or p. 663, will do very well. The distance between the plates is fixed and measured (d cm.); hence

$$C_1 = \frac{A}{4\pi d}.$$

The large capacity, C_2, of, say, $\frac{1}{3}$ μF. is connected to a paraffin wax key at 0.

Mercury cups 0 and 1 are connected together, thereby charging C_2 to the potential of the cell, E, and the deflection, φ_1, of the quadrant electrometer corresponding to this potential, V_1, is noted.

The mechanical sharing device is then set in vibration, and when regular, connexion between 0 and 2 is made (1 and 0 being separated). At the same time a stop-watch is started. The sharing of the charges is carried on for an exact number of seconds, t, until the deflection of the electrometer, φ_{n+1}, is from one-half to two-thirds of φ_1.

If the sharing is performed n times per second, N, the total number of times the charge is shared, is nt.

Now the capacity of the electrometer is small compared with that of C_2, and may be neglected in comparison with C_2, hence, as on p. 656,

$$\left(\frac{C_2}{C_1 + C_2}\right)^N = \frac{\varphi_{N+1}}{\varphi_1},$$

$$\frac{C_1}{C_2} = \left(\frac{\varphi_1}{\varphi_{N+1}}\right)^{\frac{1}{N}} - 1,$$

or

$$C_2 = \frac{C_1}{\left\{\left(\frac{\varphi_1}{\varphi_{N+1}}\right)^{\frac{1}{N}} - 1\right\}}. \quad \ldots (5)$$

The mechanical device for sharing the charge of C_2 with C_1 is seen in fig. 433. (*a*) shows the end view and (*b*) shows one form of vibrator which carries at one end a rectangular piece of copper wire, AB.

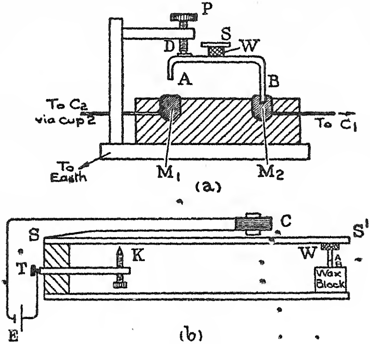

FIG. 433

SS′ is a flat steel strip (e.g. a steel foot-rule) fastened on an ebonite support at the end, S. A coil, C, is arranged over the strip, and a current from a cell, E, is arranged to pass round the circuit TKSC, causing a maintained vibration in SS′. This apparatus could be very simply made from an old electric bell.

Attached to the end of SS′ by means of clean sealing-wax, is the wire frame, AB; A is shorter than B, as shown in the end view in (*a*). This is moved up and down by the vibrating spring, SS′. The ends of A and B dip into mercury cups, M_1 and M_2, in a block of wax. B is of sufficient length to be always in the mercury, whilst A is first

in M_1 and then lifted into contact with the small disk, D, at the end of the earth-connected screw, P.

Thus, if M_1 is connected to C_2, the large capacity, and C_1 to M_2, the charge on C_2 is shared with C_1 when the wire, AB, is depressed. C_1 is discharged when AB rises and makes contact with D. The continued vibration, thus shares the charges and reduces the value of φ, the deflection produced in the quadrant electrometer.

A method to be followed is, therefore: charge C_2 by joining the cups, 0 and 1, by means of a copper wire mounted on clean sealing-wax. Note the deflection, φ_0, produced in the quadrant electrometer. Place the copper wire now between 0 and 2, having first adjusted C to a steady state of vibrations. Make a careful time estimate from the instant 0 and 2 are joined until the deflection, φ_1, is reduced to φ_{N+1} about one-half its original value.

Now if the value of the capacity of C_1 is calculated from the dimensions of the condenser, the capacity, C_2, of condenser, C_2, may be obtained if n is known.

If the vibrating bar is of such frequency that its vibrations may be counted, n may be estimated by counting the number of vibrations in several seconds. For quicker vibrations the frequency may be obtained either by matching the note emitted or preferably by attaching a light style to the end of the strip, SS′, and finding n by examination of the trace produced on a chronograph.

Such an experiment gives all the terms on the right-hand side of equation (5), and therefore the value of C_2 in E.S. units may be obtained.

The exact value of C_2 in E.M. units may be determined by the method given on p. 595.

The capacity of the same condenser is therefore measurable in both systems of units.

Now

$$\frac{\text{Capacity of } C_2 \text{ in E.S. units}}{\text{Capacity of } C_2 \text{ in E.M. units}} = \frac{\text{E.M. unit}}{\text{E.S. unit}} = v^2,$$

whence v may be calculated (see also the following pages).

MISCELLANEOUS ELECTRICAL EXPERIMENTS

Measurement of a Small Capacity in Electromagnetic Units. Comparison of the E.S. and E.M. units

To compare the electrostatic (E.S.) and electromagnetic (E.M.) units of capacity, we may find the capacity of a small air condenser in the E.S. units by calculation from a knowledge of the dimensions of the condenser, and determine the value of the capacity in E.M. units by experiment. An alternative method is given on p. 661. We have then:

$$\frac{\text{E.S. unit}}{\text{E.M. unit}} = \frac{\text{Capacity of the condenser in the E.M. system of units}}{\text{Capacity of the same condenser in the E.S. system}} = \frac{1}{v^2},$$

when v is equal to the velocity of light (3×10^{10} cm. per sec.) (see J. J. Thomson's *Elements of Magnetism and Electricity*, Chapter XII).

The object of this experiment is to find v from the above equation.

A suitable condenser to use for this purpose is a parallel plate air condenser, such as the one shown in fig. 430, p. 657. The zero having been found by the method suggested in the account given, the plates are set at a distance, d cm., apart.

If the area of the plate is A sq. cm., the capacity in E.S. units is $C_s = \frac{A}{4\pi d}$; when air is the dielectric. The same formula gives the capacity for another simple form of parallel plate air condenser, which may be easily made as a substitute for the more elaborate condenser described.

This air condenser is made taking two large sheets of plane glass, fig. 434. One sheet is coated with a circular sheet of tinfoil of about

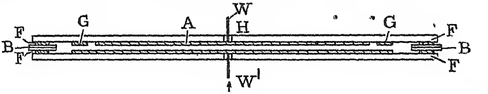

FIG. 434

50 cm. diameter. This may be attached to the glass by means of Seccotine or shellac. The tinfoil is connected to a wire, W′, which passes through a small hole in the glass; on the second sheet a smaller circular strip of tinfoil, A, is fastened and connected to a wire, W, through the hole, H. An annular circular strip of tinfoil is fastened outside A, shown at GG in the figure, to act as a guard ring. The

radius of A is carefully measured. The two sheets of tinfoil are then set up as shown, at a fixed distance apart. This may be done by placing three or four stops, BB, between the plates. If BB are small pieces of ebonite or microscope slide glass cut from the same piece, the distance between the plates will be uniform. The thickness of the stops is measured before insertion, and the stops are then covered above and below with pieces from the same tinfoil which was used in the construction of the condenser plates. In this way the distance, d, is equal to the thickness of the stops (0·2 to 1 mm.).

Using either form of air condenser, C_s is calculated.

To find C_m, the capacity in E.M. units, the Wheatstone bridge, shown in fig. 435, is set up with resistances, r_1, r_3, r_4; the condenser,

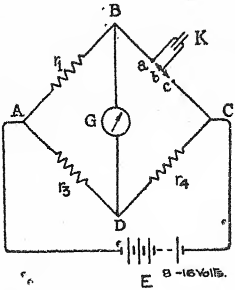

Fig. 435

K, is connected in the arm, BC, as shown. One plate of the condenser is connected to b which moves rapidly between a and c, so that when at c the condenser is charged, and when at a the condenser is discharged. By a mechanical device, described below, this process is carried out regularly n times per second. Under such circumstances the capacity may be balanced against r_1, r_3, r_4, in the ordinary way. When balanced, it may be shown that

$$\frac{1}{nC_m} = \frac{r_1(r_3 + r_4 + B) + Br_3}{r_3(r_4 + B) + (G + r_1)(r_3 + r_4 + B)} \left\{ G + r_4 + \frac{r_4}{r_3}(G + r_1) \right\},$$

where G is the resistance of the galvanometer and B the resistance of the battery (see, for example, J. J. Thomson's *Elements of Magnetism and Electricity*).

The value of B is negligible compared with r_1, r_2, or r_3, and Br_3 negligible in comparison with r_1r_3 or r_1r_4.

Thus $\dfrac{1}{nC_m} = \dfrac{r_1(r_3+r_4)}{r_3r_4+(G+r_1)(r_3+r_4)} \cdot \left\{ G + r_4 + \dfrac{r_4}{r_3}(G+r_1) \right\}$,

which reduces to

$$\frac{1}{nC_m} = \frac{r_1r_4}{r_3}\left\{ \frac{\left(1+\dfrac{r_3G}{r_4(r_1+r_3+G)}\right)}{\left(1-\dfrac{r_3^2}{(r_3+r_4)(r_1+r_3+G)}\right)} \right\}.$$

Now when C_m is small, as in the case taken, r_3 will be small compared with r_4 or r_1, and the above expression reduces to

$$\frac{1}{nC_m} = \frac{r_1r_4}{r_3}, \qquad \ldots(1)$$

i.e. the condenser acts as though it were a resistance of $\dfrac{1}{nC_m}$ ohms.

To produce a rapid charge and discharge n times per second, the apparatus shown in fig. 436 may be used. R and R′ are similar steel

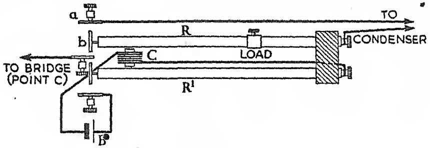

FIG. 436

rods fastened in an ebonite block and provided with terminals, as shown. A small soft iron rod is surrounded by a coil of wire, making a small electromagnet, C; from the end of R′ a light metal style is arranged just to touch a metal plate which is connected to a separate accumulator, B. The connexions shown in the figure are made and the vibrations of the rod are maintained electrically.

The rod, R, is loaded, as shown, until it also vibrates with the same frequency, which is maintained by means of C. To the end of R is a second light metal style (b) which just touches the two plates, a and c, at the ends of the vibration.

The connexions to the condenser and the bridge are indicated, and the charge and discharge is thus brought about n times per second.

To find the value of n, the frequency of the note emitted by the fork is obtained by tuning with the note of a variable monochord. The frequency may also be measured by use of the dropping smoked plate, and the chronograph (see p. 431). The mean value obtained is taken as n in equation (1).

The bridge is then balanced and the values of r_1, r_3, r_4 found. It

will be advisable to make these values as high as possible, since C_m is very small and $\frac{1}{nC_m}$ great.

r_1 and r_4 may each be 1 megohm (10^6 ohms) and r_3 adjusted until a balance is obtained.

The value of the resistances must be expressed in E.M. units, i.e. if r_1, r_3, r_4 are in ohms, equation (1) becomes

$$\frac{1}{nC_m} = \frac{r_1 r_4}{r_3} \times 10^9 \text{ E.M.U.}$$

Having obtained C_s and C_m by these methods, we have

$$\frac{C_s}{C_m} = v^2.$$

It is advisable to determine n by at least two methods. A simple comparison with a monochord is inadequate, as mistaken tuning of overtones is liable to take place.

Lecher's Experiment. The Determination of the Velocity of Electric Waves in Wires (*Ann. d. Physik*, Band 41, 1890, p. 850)

The apparatus necessary for this experiment is illustrated in fig. 437. It consists of two square metal plates with sides about 40 cm. long, connected by wires about 50 cm. long to two spherical metal knobs, FF′, of diameter 3 cm., separated by an air gap of 0·75 cm.

They are connected by flexible leads, W and W′, to a large induction coil, the primary current for which is supplied by accumulators or the supply mains.

The plates B and B′ are plates similar to A and A′ placed at about 4 cm. from them and thus forming two condensers with air as the dielectric. From B and B′ are led away two long wires of length about 10 m. For the greater part of their length they are parallel, and their distance apart may lie between 10 and 50 cm. This separation has to be arranged conveniently, as will be seen later. The ends of these wires are connected to the condenser, K, which consists of two parallel circular plates of radius, R, separated by air and at a distance, d cm., apart.

In Lecher's experiment R had the value, 8·96 cm., and d 0·99 cm. It will be a good plan to keep to these dimensions.

When the induction coil is working electric vibrations are set up in the wires, the ends of which undergo rapid variations of potential. By analogy with the case of a closed organ pipe, in which at the closed end there occur variations of pressure and at other points of the length places where the pressure remains unchanged, we anticipate that at some point between B and K there will be no change of potential.

A discharge tube is placed close to the ends of the wires, TT′, as illustrated by GG.

It is not necessary that there should be metallic connexion between the wires and tube. Lecher recommends that this should not be the case, and in his experiment glowing occurred in the tube when it was 10 cm. from the ends.

The tube may either be held a few centimetres from the ends of the wire, or may rest with the glass lying on the edges of the condenser plates, as the figure illustrates.

If the wire be bridged by means of a wire carried on a wooden handle, as, for example, at CC′, supposing for the present that DD′

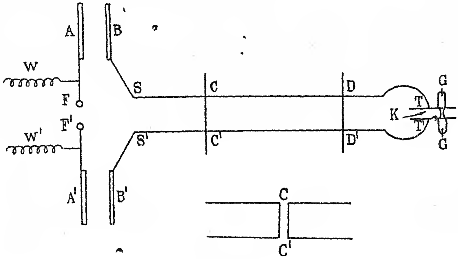

FIG. 437

is absent, we expect that there will be a flow of current across CC′ unless it happens that C and C′ are always at the same potential. Thus, generally, GG′ will glow less brightly when the wire is bridged.

But the circuit, BCC′B′, has a definite frequency, and on moving CC′ about the changes in frequency can be observed by the sounds from the sparking at FF′.

Likewise, CTT′C′ has a definite frequency and we may obtain resonance between the two circuits, in which case the glowing continues although the wire is bridged.

It was found that the wires could be cut in two in this case at C and C′, and joined separately, as shown in the lower figure, and the glowing did not cease.

The inductive action arises chiefly from the bridge, CC′, and if this is long the effect is great, while if it is short the effect is small and may even vanish altogether.

If CC′ is too long it is not easy to place CC′ in the position in which the glowing ceases because the energy from it, even when only a small disturbance passes through it, is large enough to excite the tube. The

tube remains bright for all positions of CC′. When CC′ is very short the tube remains dark for all positions of CC′.

With the above dimensions it will be found possible to find two positions, CC′ and DD′, at which the wires may be simultaneously bridged and maximum brightness produced in the discharge tube. It must be ascertained that there are no intermediate positions between BB′ and CC′, between CC′ and DD′, or between DD′ and TT′ at which bridges may be also placed and leave the tube glowing. We may then regard the wires and condensers as forming three resonating circuits.

Since T and T′ are always in opposite phases the fundamental vibration in DTT′D′D is that in which a half-wave occupies the circuit while in DCC′D′D we have a whole wave.

If L denotes the inductance of the former circuit, and K the capacity of the condenser, the period of the oscillation is

$$T = 2\pi\sqrt{LK}.$$

In this case we have neglected the resistance of the wire, as we may, since fairly thick copper wires are employed.

The value of K should be large, but not large enough to make the circuit non-oscillatory. Lecher used a condenser of capacity, 20 electrostatic units.

The value of L may be deduced from Neumann's formula for self-induction. If l denotes the length of wire in the circuit, TDD′T′T, including, of course, the bridge, and its radius of cross-section is r, we have

$$L = 2l\left(\log_e \frac{2l}{r} - 1\right) \text{ electromagnetic units.}$$

The value of K expressed in this system of units is

$$\frac{R^2}{4d} \times \frac{1}{9} \times 10^{-20}.$$

Thus $$T = \frac{\pi R}{3} \times 10^{-10} \times \sqrt{\frac{2l}{d}\left(\log_e \frac{2l}{r} - 1\right)}.$$

The wave-length is obtained from the circuit, CDD′C′C, and if CD is of length, a, and the bridge of length, b, we have

$$\lambda = 2\,(a + b).$$

Hence the velocity, c, is given by

$$c = \frac{\lambda}{T},$$

or $$c = \frac{6 \times 10^{10}\,(a + b)}{\pi R}\sqrt{\frac{d}{2l\left(\log_e \frac{2l}{r} - 1\right)}}.$$

The experiment is performed by stretching two long wires not less than 10 m. in extent along the laboratory and clear of metal pipes. The distance apart should be about 40 cm., and the capacities may all be formed of sheets of tin of the dimensions given above. The capacity, K, must be set up so that the plates are parallel, and its dimensions should be very carefully measured. Likewise the diameter of the wires must be accurately determined by means of a screw gauge.

If the wire is sufficiently long, a third bridge may be inserted simultaneously with the other two, and a second determination of wave-length is then possible.

Lecher gives the following values observed in one of his experiments which are useful for comparison:

Total length of each wire, 1122 cm.
Distance between C and D, 939·6 cm.
Length of bridge, 42 cm.
Length of wire in DTT'D'D, 303·2 cm.
Diameter of wire, 0·1 cm.

Interest in this form of the experiment is now chiefly historical. In its modern form the parallel wires should take the form of thin brass rods two to three metres long. A loop of wire is used to connect the rods at one end and, by means of a valve generator, oscillations should be generated in the system, the loop being used as a pick-up. The tube detector should be replaced by a thermal detector and the distance, CD, measured between two positions in which the detector shows maximum current. As before, $\lambda = 2(a + b)$. The generator should give simple oscillations of 50 to 100 cm., so that more than one length, CD, may be obtained corresponding to the maximum reading in the detector. In this way $\frac{\lambda}{2}$, λ, and possibly $\frac{3\lambda}{2}$ may be measured. The experiment illustrates a simple method of determining the frequency of oscillation of a generator from the formula $v = \frac{c}{\lambda}$.

Measurement of $\frac{e}{m}$ and V for Cathode Rays

The measurement of $\frac{e}{m}$, and of v, the velocity of the electrons in a cathode stream, may be carried out, using the apparatus described later, by applying a magnetic and an electrostatic field of measurable magnitude and observing the deflections produced in the cathode stream.

We will first briefly consider the effect of such fields on a moving electron.

In fig. 438 let OX be the undeviated path of the electron, of mass, m, charge, e (E.M.U.), and velocity, v.

If now a uniform magnetic field, H, be applied parallel to OZ, at right angles to the plane of the paper, the electrons will be deflected in the same manner as a current-conveying conductor in the magnetic field.

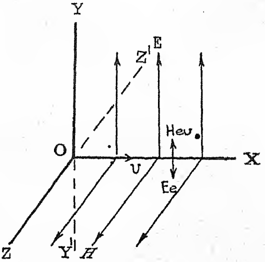

FIG. 438

In this case the electron stream is equivalent to a current, ev, in a direction, X → O, since the electron has a negative charge; the force causing the deflection upwards is therefore Hev per electron, and is always normal to the direction of motion, and to the direction of the field, H. The path of the electron will, therefore, be circular. If ρ be the radius of curvature of the deflected path, we have

$$\frac{mv^2}{\rho} = \mathrm{H}ev.$$

In the case of a uniform electrostatic field the deflection produced is in the opposite direction to the field, for such a negative charge. The electron passing through the field of strength, E, and parallel to OY, is subjected to a force, Ee, in a direction parallel to YO (E in E.M.U.).

If both fields are applied simultaneously the deflection will be the result of both, and, if suitably adjusted, the fields may be made to produce deflections which neutralize each other, i.e. the force on the electron Hev upwards and Ee downwards become equal and opposite:

$$\mathrm{H}ev - \mathrm{E}e = 0, \quad \text{or} \quad v = \frac{\mathrm{E}}{\mathrm{H}}. \qquad \ldots(2)$$

Consider the above expression for the radius of curvature and refer to fig. 438. By adjusting H, $\frac{dy}{dx}$ can be made small and $\left(\frac{dy}{dx}\right)^2$ negligible compared with unity,

hence

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} = \frac{H}{v} \cdot \frac{e}{m}.$$

Whence
$$y = \int \left[\int \frac{H}{v} \cdot \frac{e}{m} dx \right] dx, \qquad \dots(3)$$

where y is the displacement produced by the field H, the integration being carried from the point where the electron enters the field up to the point at which y is measured:

$$\frac{e}{m} = \frac{vy}{\int [\int H dx] dx}. \qquad \dots(4)$$

It is seen, therefore, that for known electric and magnetic fields, $\frac{e}{m}$ and v may be obtained.

Further discussion of the above results are postponed until the general experimental method has been described.

The form of apparatus first used by Sir J. J. Thomson is frequently used for this experiment. Fig. 439 shows its essential features. The

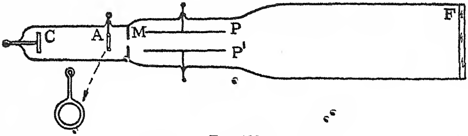

FIG. 439

long glass tube contains a plane-faced cathode, C, a ring anode, A, and is provided with a mica screen pierced with a small hole, M, at the centre. P, P′ are two parallel plates which are long compared with the distance between them and which may be connected to an external source of potential. F is a fluorescent screen, very lightly graduated in millimetres. C and A are connected to a good induction coil, the commutator of which is arranged so that C is the negative terminal. The electron stream, leaving the cathode normally, passes through A, and a narrow pencil passes through M and strikes the screen, F, making a small luminous patch on the scale. The interrupter of the coil is adjusted so that a steady output is obtained.

The tube from M to F is covered on the outside by thin sheets of tinfoil, connected to the earth, to shield it from stray electrostatic effects.

It is first fixed in some definite direction relative to the permanent magnetic field, i.e. the earth's field. If placed normal to the horizontal component of the earth's field, the electron will suffer a slight initial deflection. This deflected position may be observed; and later the magnetic field, when applied, is measured independently of the earth's field. The change in deflection due to this is noted. For many purposes this method is advisable. Another method is to place the tube *in the direction of the horizontal component* of the earth's field.

The path of the electron thus coincides with the direction of the horizontal component and will consequently suffer no deflection due to this. The vertical component of the earth's field will only cause a lateral displacement of the luminous spot, and is thus of no account, as we are measuring vertical displacements due to an applied horizontal field.

Two large bar magnets are now arranged normally to the tube, parallel to the face of the plates, P, P′, and at such a distance from M that only a very slight effect is produced on the electrons before passing through the hole. For if the magnetic field between A and M is large, the electron stream will be deflected before passing through M. Those electrons which pass through M do not do so along the axis of the tube and therefore the deflection measured on the screen, which we consider due to the field between M and F, will be too great. A trial experiment will show the order of deflection produced at F.

To introduce an electrostatic field the plates, P, P′, are connected to a constant source of potential. A nest of small accumulators or a battery of small dry cells serve this purpose well. If these are not available the plates may be connected to the mains of the electricity supply, through a water-resistance, but usually this is not sufficiently steady.

If the plates are not too far apart we may take the field, E, so produced as a uniform field and of an extent coinciding with the limit of the plates, i.e. neglect the end corrections if the plates are close together.

Apply a constant potential difference to P, P′ and note the deflection; then introduce the bar magnets in the direction required to reduce this deflection and arrange the poles N and S (fig. 440) symmetrically and at such a distance apart that the field is of the sufficient strength to reduce the deflection to zero, i.e. fix E and adjust H to balance.

It is important that the leads from the coil to A and C should be well insulated from the rest of the apparatus, and that one plate, P, say, be connected to earth, otherwise troublesome leakages may occur.

Having obtained a balance, measure the potential difference between P and P′ by means of a high-resistance voltmeter.

Remove the source of potential and note the deflection due to the magnetic field alone. Let this be y cm.

It will be apparent that the magnetic field used is not a uniform one, therefore the value $\int[\int \mathrm{H}dx]\,dx$ of equation (4) must be found graphically.

In order to compute the integral directly in this way two graphs

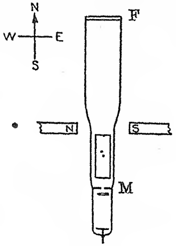

FIG. 440

are required and the procedure is laborious. This double process can be avoided by expressing the integral in another way.

Let $$\mathrm{F} = \int_0^x \mathrm{H}dx \quad \text{so that} \quad \frac{d\mathrm{F}}{dx} = \mathrm{H}.$$

The value of $\int_0^l \mathrm{F}dx$ is required.

Integrating by parts:

$$\int_0^l \mathrm{F}dx = \Big[\mathrm{F}x\Big]_0^l - \int_0^l x\frac{d\mathrm{F}}{dx}\,dx = l\int_0^l \mathrm{H}dx - \int_0^l x\mathrm{H}dx$$
$$= \int_0^l (l - x)\,\mathrm{H}dx.$$

Thus, a table of values of H and $(l - x)$ is required from which the product $(l - x)\,\mathrm{H}$ is plotted against x and the value of the integral is derived in the usual way. The accuracy of the evaluation may be improved by writing

$$\mathrm{H} = \overline{\mathrm{H}} + h,$$

where $\overline{\mathrm{H}}$ is the average value of H and h will be found in practice not to be very large.

Thus, the above integral may be transformed into the form

$$\int_0^l (l - x)\,\mathrm{H}dx = \tfrac{1}{2}l^2\overline{\mathrm{H}} + \int_0^l (l - x)\,hdx.$$

To the table of values add a column of the values of h, and instead of plotting $(l - x)$ H plot $(l - x)\,h$ against x. The integral, $\int_0^l (l - x)$ [illegible]ial comes out as a relatively small correction, so that any error in the measurement of the area under the graph is not important.*

The positions of M and F are marked on the bench (fig. 441), and the tube is removed, care being taken that the magnets are not disturbed.

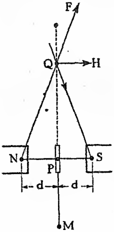

Fig. 441

A central line, MF, is drawn and subdivided by points, 1 or 2 cm. apart. The value of the field, H, at each point is then obtained. This may be done by means of the fluxmeter, or by means of a magnet.

Determination of the Field by the Fluxmeter

The fluxmeter is set up and the mirror arranged to reflect a beam of light from a lamp on to a scale one metre from it. The search coil of area, A, and containing n turns is placed at P (fig. 441) at right angles to the field at its strongest point. The deflection is noted. The search coil is then reversed and the deflection again noted; the total deflection so obtained is halved, to give the value corresponding to the field, H_m. The mirror deflections are then standardized against the fluxmeter scale and the value of H_m is obtained by dividing the flux, corresponding to the throw observed, by the product, nA.

For points such as Q, away from the point, P, the magnetic field will be comparatively small. The deflection produced in the fluxmeter will be correspondingly reduced, and the value of the field, H_Q, at Q will probably be inaccurate. To increase the deflection produced, a search coil with a larger number of turns should be used.

* This method of making the calculation was used by Sir J. J. Thomson. We are grateful to Prof. H. R. Robinson for calling our attention to it.

Using a fluxmeter with a search coil placed with its plane normal to NS and parallel to FM, the throw observed corresponds to the field normal to the axis of the tube (FM) due to both poles of each magnet; that is, the fluxmeter values are those effective in producing the deflection. The earth's field, which produces no vertical deflection, also produces no effect on the fluxmeter since the plane of the search coil is in the magnetic meridian.

If the tube is placed in an east and west direction the value of the deflecting field will be equal to the observed field, plus or minus the earth's field, for in this case the fluxmeter readings include the horizontal component of the earth's field.

If the field magnets are very long, compared with the distance, MF, the values of the field, H, at points, Q, along MF may be calculated from the fluxmeter readings at P; and the distances, $PQ = l$ and $NS = 2d$. Under such conditions the effects of the distant poles are neglected.

The field at Q normal to FM is

$$H = \frac{(m + m')\, d}{(l^2 + d^2)^{\frac{3}{2}}}, \qquad \ldots(5)$$

where m and m' are the pole strengths of N and S.

Also the field H_m at P is

$$\frac{m + m'}{d^2},$$

therefore (5) becomes

$$H = \frac{H_m d^3}{(l^2 + d^2)^{\frac{3}{2}}}. \qquad \ldots(6)$$

When such a method is justifiable, H at all points along FM is calculated from the observed maximum value at P.

Determination of H by means of an Oscillating Magnet

A small needle is allowed to oscillate freely at each point marked on the line, MF, the axis of the needle being arranged at a height above the table corresponding to the axis of the tube when in position. The time of oscillation of such a needle is

$$T = 2\pi \sqrt{\frac{I}{MH}},$$

where I is the moment of inertia of the needle about the axis of suspension, M the magnetic moment of the magnet, and H the field in which it is placed.

As a preliminary standardizing experiment the needle is allowed to

oscillate freely in a position well removed from the magnets, i.e. in the earth's field only. If H_0 is the known value of the earth's field

$$T_0 = 2\pi\sqrt{\frac{I}{MH_0}},$$

or
$$T_0^2 = \frac{k}{H_0}, \quad \text{where} \quad k = \frac{4\pi^2 I}{M}.$$

The value of T for each point is obtained, hence H at each point being $\frac{k}{T^2}$, we have

$$H = \frac{T_0^2}{T^2} \cdot H_0. \qquad \ldots(7)$$

This gives the value of the deflecting magnetic field *so long as the needle when at rest sets normal to* MF; for points near the screen and aperture this condition will not hold as nearly as for points close to the magnets, owing to the presence of the earth's field.

The value of the field actually measured is R, the resultant of H and H_0.

If, therefore, the magnet is brought to rest and the inclination of its axis with FM, θ, be measured at each position, we have

$$H = R \sin \theta.$$

Equation (7) should be written

$$R = \frac{T_0^2}{T^2} H_0,$$

and therefore

$$H = \frac{T_0^2}{T_0} H_0 \sin \theta. \qquad \ldots(8)$$

Tabulate results as below:

$H_0 =$ $\qquad T_0^2 =$

Distance from aperture (in cm.)	θ	$\sin \theta$	Time for 20 swings	T	T^2	$H = \frac{T_0^2}{T^2} \cdot H_0 \sin \theta$
0						
2						
4						
etc.						

Use this table in the method described for computation of the double integral by means of the single graph. The integration is carried out from the aperture, M, to the point, F, on the screen.

In the case of the electrostatic field the limits of the field are assumed to be the limits of the plate.

The field, $E = \frac{V}{D}$, where V is the potential in electromagnetic units (10^{-8} volts) and D is the distance between the plates in cm.

Now consider the deflection produced by the electric field.

We will assume that the electrostatic field set up by the potential on the plates is E electromagnetic units. The force on the electron is therefore Ee parallel to the field and not normal to the path as in the magnetic field. The equation of motion in this direction in the notation of fig. 438 is therefore

$$m\frac{d^2y}{dt^2} = Ee,$$

and since v, the velocity of the electron in the path, is $\frac{dx}{dt}$, we have, by integration of the above equation,

$$\frac{dy}{dt} = E\frac{e}{m}t + B,$$

$$v\frac{dy}{dx} = E\frac{e}{m}t + B. \qquad \ldots(9)$$

Now the value of $\frac{dy}{dx}$ at E (fig. 442) is obtained by putting t equal to

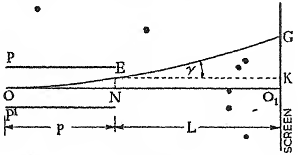

FIG. 442

the time, t_0, the electron takes to travel the length of the plates, and since when $t = 0$ and $\frac{dy}{dx} = 0$, $B = 0$ and $vt_0 = p$, the length of the plates,

$$\frac{dy}{dx} = E\frac{e}{m}\frac{p}{v^2},$$

which is the value of the tangent at E of the path OE. On leaving the electrostatic field the electron continues along the tangent EG, striking the screen at G. Putting angle GEK $= \gamma$,

$$\tan\gamma = E\frac{e}{m}\frac{p}{v^2}.$$

Now from (9) we have, for any point, x cm. from the beginning of the electrostatic field, since $x = vt$,

$$\frac{dy}{dx} = \mathrm{E}\frac{e}{m}\frac{x}{v^2},$$

or $$\mathrm{EN} = \int_0^p \mathrm{E}\frac{e}{m}\frac{x}{v^2}\cdot dx = \frac{1}{2}\left[\mathrm{E}\frac{e}{m}\cdot\frac{x^2}{v^2}\right]_0^p = \frac{1}{2}\mathrm{E}\frac{e}{m}\frac{p^2}{v^2}.$$

Thus the total deflection on the screen is $\mathrm{O_1G}$.

$$\begin{aligned}\mathrm{O_1G} &= \mathrm{O_1K} + \mathrm{GH}\\ &= \frac{\mathrm{E}ep^2}{2mv^2} + \mathrm{L}\tan\gamma\\ &= \frac{\mathrm{E}ep}{mv^2}\left(\frac{p}{2} + \mathrm{L}\right).\end{aligned} \qquad \ldots(10)$$

Since $\mathrm{O_1G} = y$, for when the two fields are acting no deflection is produced, we have numerically:

$$\frac{\mathrm{E}ep}{mv^2}\left(\frac{p}{2} + \mathrm{L}\right) = \frac{e}{m}\frac{1}{v}\int[\int \mathrm{H}dx]\,dx;$$

or $$v = \frac{\left(\frac{p}{2} + \mathrm{L}\right)p\mathrm{E}}{\int[\int \mathrm{H}dx]\,dx}, \qquad \ldots(11)$$

$$\frac{e}{m} = \frac{vy}{\int[\int \mathrm{H}dx]\,dx}. \qquad \ldots(12)$$

Thus every term is obtained for substitution in (11) and (12).

Hence $\frac{e}{m}$ in E.M.U. and v in cm. per sec. are obtainable.

Repeat, using a higher potential difference between A and C; note the increased v but the same value for $\frac{e}{m}$.

Another way of finding $\frac{e}{m}$ and v which involves a greater uncertainty than the above method eliminates the electrostatic field.

The induction coil is adjusted as in the last method, so that the output is uniform, i.e. when the magnetic field is applied the spot of luminescence is not drawn out into a very long patch, showing that the velocity of the electrons is approximately constant.

Suppose the potential applied to the terminals A and C be V.E.M.U., we have

$$\mathrm{V}e = \tfrac{1}{2}mv^2. \qquad \ldots(13)$$

Also by a magnetic deflection, as already described, we have from equation (4)

$$y = \frac{e}{m}\frac{1}{v}\int[\int \mathrm{H}dx]\,dx. \qquad \ldots(4a)$$

Hence, dividing (13) by (4*a*)

$$v = \frac{2Vy}{\int[\int Hdx]\,dx}. \qquad \ldots(14)$$

Hence

$$\frac{e}{m} = \frac{2Vy^2}{\{\int[\int Hdx]\,dx\}^2}. \qquad \ldots(15)$$

To find y and $\int[\int Hdx]\,dx$, proceed as already described. The determination of V may be made approximately by arranging an alternative spark gap in parallel with the terminals of the induction coil and A and C.

The spark gap should consist of two equal-sized *smooth* and *polished* spheres of known diameter, and their positions so adjusted that a spark will only occasionally pass when the tube is connected. The value of the potential is then obtained from tables, knowing the diameter and distance apart of the spheres.

For such tables, see Kaye and Laby, *Physical Constants*.

Thus for balls of 2·0 cm. diameter in air we have:

Spark gap in cm.	0·5	0·8	1·0	1·5	2·0
Potential $\times 10^{-3}v$ in volts	17·5	26·0	30·8	39·0	47·0

RADIOACTIVITY AND CONDUCTION OF ELECTRICITY THROUGH GASES

Saturation Curve

The object of this experiment is to determine the relation between the potential difference applied to an ionized gas, and the resulting current. Two metal plates are supported horizontally by ebonite rods, a fixed distance apart. The lower plate is coated thinly with a radioactive substance, say uranium nitrate, and is connected to a battery of cells. The point of contact of this connecting wire is variable and the other end of the battery is earth-connected, so that the applied potential may be any multiple of that of the single cell. The upper plate is connected to one pair of quadrants of an adjusted quadrant electrometer, and to a mercury cup of a key of the type shown in fig. 424. The second pair of quadrants is joined to the second mercury cup, which is earthed. When the mercury key is closed the four quadrants are all at the same potential (zero).

To carry out the experiment, the potential of one cell, say two volts, is applied to the lower plate, the mercury key is opened, and the resulting charge acquired per second by the quadrant is observed by timing the rate of movement of the reflected spot of light of the electrometer. The charging up of the electrometer should be regular, i.e. the

time taken to traverse one centimetre of the scale should be constant. This is timed with a stop-watch over, say, 20 cm., and the number of divisions per second calculated. The key is then closed and all quadrants earthed. The process is then repeated two or three times and a mean value obtained.

Similar observations are then made when the potential of two, three, four, etc., cells is applied in turn to the coated plate, until further potential produces no increase in the rate of movement of the electrometer needle.

Having by this method obtained the rate of charging of the electrometer, the corresponding currents may be calculated from a knowledge of the sensitivity of the instrument and its capacity, as obtained by the methods given on pp. 650 and 655. For example, suppose the sensitivity is s divisions per volt, the capacity k, and for one potential on the plate the number of divisions per second is D. The increase of potential per second is $\frac{D}{s}$ volts, and the quantity of electricity given to the quadrants per second is, therefore, $\frac{kD}{s}$, where k, D, and s are in the same units. As described above, k will be in electrostatic units, and is therefore equal to $\frac{k}{9 \times 10^{20}}$ E.M. units, or $\frac{10^{-11}k}{9}$ farads; the current is therefore $\frac{10^{-11}kD}{9s}$ amperes. The results obtained are then represented graphically, the ionization current being plotted as ordinate, and the applied potential as abscissa. It will be found that the general form of the curve is similar to that of fig. 439.

Repeat the above series of observations for one or two values of the distance between the plates, and note the effect on the saturation current and the potential required to produce saturation.

Note

It may be found that, when measuring the movement of the needle for the higher values of the potential, some uncertainty arises due to its rapidity. In such a case insert an air condenser in parallel with the electrometer and adjust the distance between the condenser plates to produce a slower movement of the needle. When calculating the current from such observations, k in the above formula is the sum of the capacity of the condenser and the electrometer.

The Gold-leaf Electroscope

For many measurements of the ionization produced by X-rays, α-, β-, or γ-rays,t he gold-leaf electroscope is often used. There are many forms of this instrument; but a simple one, which will be found to

be very useful for many laboratory experiments, is illustrated in fig. 443.

A cubical box of about 12 cm. side is cast in aluminium, or otherwise built up. The inside faces are made smooth and the corners rounded. The box is finally closed by a 'lid' of thin aluminium foil. In opposite faces are two windows, as shown by the broken lines. These windows are closed by pieces of glass or of split mica, attached with sealing-wax.

From an ebonite collar is suspended a brass rod, A, about 2 cm. long. This rod carries a sulphur plug, S. The sulphur plug is cast by

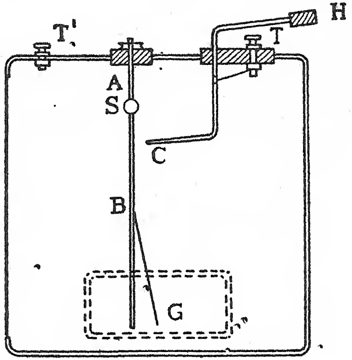

FIG. 443

melting pure sulphur, care being taken that it does not turn dark brown during this process, as this impairs its insulating properties. Insulated from A by the sulphur plug, S, is a thin strip of brass, B, a few mm. broad and about 1 mm. thick. A thin strip of gold leaf, G, is attached to B by means of shellac. The gold leaf is cut by taking a sheet of the leaf between two sheets of the paper pages in the book of gold leaves, and cutting with a sharp razor on a paper pad. T′ is a terminal in contact with the case, attached so that the latter may be readily earthed.

A charging rod, CH, with an insulating handle, H, is capable of rotation in an ebonite collar, and is of such a length that when turned it makes contact with B, thereby charging the gold-leaf system up to the potential to which T or CH is raised by a battery of cells or a

rubbed ebonite rod. When BG is charged, C must be insulated and then earthed by touching the case of the electroscope.

The leaf is viewed by means of a low-power microscope, through the windows. To aid this, daylight or artificial light is reflected, by means of a small piece of plane mirror, through the windows towards the microscope. In the eyepiece of the microscope is a graduated glass scale. When the leaf is charged, it moves across the field of view of the microscope and its image is arranged to coincide with an extreme graduation. When CH is turned away from B, the leaf should remain charged for a considerable time.

The *natural leak* of the electroscope is first determined. When the room is dry and free from radioactive contamination this should be small.

The leaf is charged and viewed by the microscope (C being turned away from B). Care is taken that the microscope and the electroscope are not moved relative to each other during, or subsequent to, the observations; and the time is taken for the gold leaf to move over a definite range of scale readings. Let the number of divisions per second be n_0. n_0 is the natural leak of the instrument.

If an instrument is to be serviceable this leak must be small.

β-Ray Absorption

An electroscope is set up as described above and its natural leak is measured. The radioactive source is set up at a fixed distance from the thin side of the electroscope, with a thick piece of lead sheet shielding the latter from the rays. A convenient source of β-rays may be made from old broken emanation tubes. The Ra E present gives out β-rays.

The leaf is charged, causing a deflection beyond a certain mark and the lead sheet is removed. The ionization produced causes the electroscope to discharge. When the image of the leaf coincides with the mark a stop-watch is started, and after a convenient fall the stop-watch is stopped. This is repeated several times, taking care that there is no relative movement of the source, the electroscope, and the microscope.

The mean value of the time is obtained. From it the number of divisions traversed per second by the leaf, n_1, is obtained.

The rate of movement due to the ionization produced by the radioactive source is $(n_1 - n_0)$.

A sheet of aluminium of about 0·015 mm. thickness is then placed between the source and the electroscope (near to the latter) and the rate of movement n_2 is obtained as above. The ionization due to the β-rays passing through the aluminium is $(n_2 - n_0)$. This is repeated with an increasing number of sheets of aluminium of known thickness, and $(n - n_0)$ is plotted against t, the total thickness of aluminium

interposed; $\log(n - n_0)$ is also plotted against t. For many sources this latter curve is a straight line.

The relation, $I = I_0 e^{-\mu t}$, may be taken as representing the relation between I_0, the intensity of the unintercepted beam and I the intercepted beam, μ being the mean absorption coefficient of the rays in aluminium.

Now I is proportional to the ionization produced, i.e. to $(n - n_0)$,

hence
$$\frac{n - n_0}{n_1 - n_0} = e^{-\mu t},$$

or
$$\mu = \frac{2{\cdot}303}{t} \cdot \log_{10} \frac{(n_1 - n_0)}{(n - n_0)},$$

where n is the observed rate of movement when the rays have passed through a thickness, t, of aluminium. For the β-particles from Ra E, $\mu = 44$ per cm.

Absorption Coefficient of γ-Rays

The determination of the absorption coefficient of γ-rays may be carried out in a similar manner to the above, but in this case a suitable absorbing substance is lead foil.

If the radioactive source also emits β-rays, these should be first of all absorbed by a thin sheet of lead foil and the emerging radiation may then be examined by interposing successive sheets of lead foil.

On plotting the logarithmic curve of ionization against the thickness of lead, a straight line indicates a homogeneous beam. If the curve is not straight it may, as a rule, be analysed into two or more straight lines, from the slopes of which the coefficient of absorption in lead of the individual radiations may be obtained.

In all the above cases the electroscope is connected by means of T′ to the earth, but if any stray electrostatic fields are present in the room (e.g. Wimshurst machines, induction coils) it may be necessary to enclose the whole apparatus in a large earth-connected tin box, e.g. a biscuit tin, to shield it from stray effects which cause erratic movements of the gold leaf.

The Decay of a Radioactive Substance

The rate of decay of a radioactive body may be expressed by the relation

$$N = N_0 e^{-\lambda t}, \qquad \ldots(16)$$

where N_0 is the original number of atoms present,
N is the number after a time, t seconds,
λ is the transformation constant.

The constant λ is characteristic of the radioactive body and, as may

be seen below, is equal to the fraction of the atoms which break up per second; for from (16) above,

$$\frac{dN}{dt} = -N_0e^{-\lambda t} = -N\lambda,$$

or

$$\lambda = -\frac{dN}{dt}\bigg/N.$$

Another constant, called the *half value period*, T, is also characteristic of the substance. T may be defined as the time in seconds during which the activity of the substance is reduced to half the original value. It is related to λ in the following way:*

$$T = \frac{1}{\lambda}\log_e 2 = \frac{0{\cdot}693}{\lambda}.$$

To find the constants λ and T for radium emanation a radium emanation tube, not more than three or four hours old, is used as source.

The source is placed at a fixed distance from the electroscope as before, and the rate of movement is obtained, say, n_1. The net rate of movement is $n_1 - n_0$. n_0 should be determined before and after each value of n_1, as it may vary under these circumstances. The time at which the observation is made is noted. After an interval of twelve hours the observation is repeated. In the interval the source is screened by a sheet of lead foil, the electroscope is charged up, the lead is removed, and the gold leaf is timed *always over the same range of scale readings.* This is repeated for ten or twelve days.

The value of $(n - n_0)$ is plotted against the mean time of the observation, and a graphical representation is then available of the decay of the γ-ray activity of the source.

* The student should consult Rutherford, *Radioactive Transformations*; Makower and Geiger, *Practical Measurements in Radioactivity*; and a small summary, *Radioactivity and Radioactive Transformations*, Chadwick.

CHAPTER XXV

THERMIONIC EMISSION AND VALVE CHARACTERISTICS

Thermionic Emission

O. W. Richardson has shown that the electron current per square centimetre from a heated metal surface when the electrons are in equilibrium with the metal at an absolute temperature, T, can be expressed as a function of T according to the following formula:

$$i = AT^2e^{-\frac{b_0}{T}}.$$

i denotes the current density, b_0 is a constant dependent upon the nature of the surface, and A, according to theory, is a constant for all pure metals, its value being 60·2 amp. per sq. cm. per (degree)^2.

In actual experiments it is found that A varies over the range 3 to 100.

For an emitting surface of any area, S, the total current, I, is obtained from $I = iS$, and writing the equation in a form convenient for use in the experiment to be described,

$$\log_{10} \frac{I}{T^2} = \log_{10} B - \frac{b_0}{2{\cdot}303T}.$$

The constant, B, will contain the area of the filament emitting the electrons as a factor and will be a constant for the particular material supplying the electron current in the experiment. This will usually be thoriated tungsten.

The object is now to verify this formula and to determine the two constants. From b_0 the work function at absolute zero, φ_0, for the substance may be obtained, for in the course of the derivation of the equation b_0 is substituted for $\frac{e\varphi_0}{k}$, where e is the electronic charge ($1{\cdot}59 \times 10^{-20}$ E.M.U.) and k is Boltzmann's constant ($1{\cdot}37 \times 10^{-16}$ ergs per deg.). φ_0 measures the potential difference through which the electron can be considered to move in escaping from the surface at absolute zero. In other words, $e\varphi_0$ measures the work necessary to move the electron from the surface. Thus φ_0 is measured in units of potential and should be recorded in volts, e.g. for platinum its value is 6·27 volts.

In the original experiments a wire was situated inside a metal cylinder in a highly evacuated tube. It is now convenient to make use of a diode or a triode in which the grid and plate are connected together.

The filament is supplied with a current and is thus heated and emits electrons. If a potential difference is maintained between the filament and plate a current is detected in the anode circuit, and may be measured either by a calibrated galvanometer or a micro-ammeter.

In order to measure the temperature of the filament, its resistance is determined and the temperature is deduced from the resistance-temperature relation.

The arrangement of the circuit is illustrated in the diagram (fig. 444). In this figure r denotes a small resistance capable of carrying the

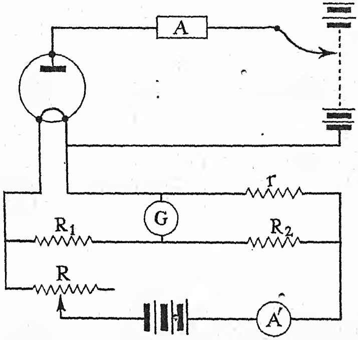

Fig. 444

heating current to the filament. R_1 and R_2 denote high resistances. R is a variable resistance by means of which the current to the filament and consequently its temperature can be varied. The ammeter, A′, serves to control the magnitude of the filament current. The high tension on the plate must be sufficient to produce saturation, and the condition of saturation should be verified by the fact that, as this tension is varied, the current in A remains constant.

There are two parts to the experiment: (*a*) the measurement of temperature, and (*b*) the measurement of the anode electron current.

Measurement of Temperature

Langmuir has determined many electrical constants for tungsten (*Gen. Elec. Rev.*, XXX, p. 354), and in particular has obtained values of the specific resistance, σ, over a wide range of temperature. The results can be summarized in the following formula:

$$\sigma = \sigma_0 (1 + \alpha t + \beta t^2 + \gamma t^3),$$

where σ denotes the specific resistance at the centigrade temperature, t, and σ_0 is the value at 0° C. The values of the constants are

$$\alpha = 5{\cdot}238 \times 10^{-3}, \quad \beta = 0{\cdot}700 \times 10^{-6}, \quad \gamma = 0{\cdot}062 \times 10^{-9}.$$

Thus, for a tungsten wire, the dimensions of which do not vary appreciably, the resistance-temperature relation may be taken as

$$R_t = R_0\,(1 + \alpha t + \beta t^2 + \gamma t^3).$$

The value of the resistance at 0° C., R_0, is required for the purpose of the calculation. In order to determine this value by experiment, it would be necessary to cool down the wire to 0° C. It would be difficult to do this in the case of the diode or triode, and the resistance should be measured at room temperature, t, R_0 being deduced by means of the formula.

A graph should then be plotted of $\frac{R_t}{R_0}$ against t, from which temperatures can be determined as a result of measurement of R_t.

Measurement of Current

After ascertaining that the condition of saturation has been reached, the value of the anode current, I, is measured for various filament currents. The latter are obtained by varying the rheostat, R. In each case the filament resistance is measured and its temperature determined. From these observations a series of values of $\frac{I}{T^2}$, $\log \frac{I}{T^2}$, and $\frac{I}{T}$ are tabulated and a graph drawn of $\log \frac{I}{T^2}$ against $\frac{I}{T}$. From the intercept on the axis of $\log \frac{I}{T^2}$ the value of log B is determined and from the slope the value of b_0, and hence of φ_0, is obtained.

The object of the experiment is attained in the demonstration of the linearity of the graph, which is a verification of the law of electron emission and in recording the values of B and φ_0.

The Characteristic of the Diode Valve

The diode is the simplest form of a thermionic or electron valve by means of which it is convenient to begin the study of the practical applications of thermionic emission.

It consists of an evacuated vessel containing two electrodes. One of these is a filament of wire which can be heated so that it emits a stream of electrons and the other is placed opposite to it, and consists of a metal plate known as the anode, or simply as the plate. The filament is also described as the cathode. These names, taken from the nomenclature applied to voltameters, imply that the current flows into

the apparatus at the anode and out at the cathode. The current actually consists of a stream of negative charges flowing within the tube from cathode to anode, giving externally the effect suggested by the nomenclature.

The cathode may be heated directly by passing a current through it from a battery of cells, but it is preferable to use an indirectly heated cathode in which the emitting surface is heated by a filament insulated from it.

The electrical properties of the diode are recorded by drawing a graph of the relation between the current flowing between the electrodes, described as the anode current, and the difference of potential between them. The curve obtained is described as the characteristic curve of the diode.

A theory has been developed which gives the relation between these two quantities, but certain assumptions made would suggest that the result can only be an approximation to the characteristic found experimentally. The relation is

$$I = AV^{\frac{3}{2}},$$

where I is the current and V the potential difference.

A is a constant depending on the design of the tube and independent of the temperature of the emitting filament. The characteristic curves drawn for different filament temperatures suggest this independence, for they coincide over a considerable extent of their length; almost until conditions of saturation are reached, and the relation is no longer valid.

In order to construct the characteristic the filament should be supplied with current from a battery of a few volts. A 2-volt accumulator is often sufficient. The potential difference between the electrodes should be varied to a maximum of about 110 volts, and is obtained from a constant D.C. supply.

A simple way of carrying out the experiment to avoid the use of a voltmeter is to place a resistance mat, divided into equally spaced tappings, about ten in number, across the supply. A total resistance of 400 ohms is suitable. The mat then forms a potentiometer and ten points are provided for the characteristic.

The actual magnitudes are determined by the fact that the curve should be drawn well into the region of saturation. The current can be measured by means of a milliammeter or shunted galvanometer, but, if the latter is used, the experimenter should record the actual current in milliamps, and not be content merely to record a deflection proportional to the current. The reason is that in work with valves actual values of the quantities are important. In the present case the actual values are not required for drawing the characteristic, but the opportunity should not be lost at this early stage of valve experiments to become acquainted with actual magnitudes.

A simple way of obtaining the variations required in this case is to shunt a galvanometer, using a small shunt resistance at first. Then arrange that, with the maximum potential difference and maximum heating, the galvanometer deflection is as large as possible.

The heating current may be varied by placing a resistance in series with the filament battery, and may take the form of a piece of Eureka wire (22 gauge), the length in the circuit being varied.

For different filament currents, and consequently for different temperatures (T) of the emitting surface, characteristics will be obtained similar to those in fig. 445.

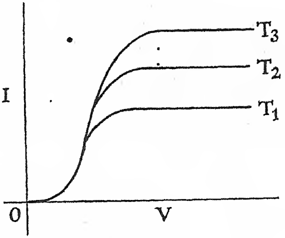

FIG. 445

The curves may be divided into two parts: (1) a part common at all temperatures in which the current is limited by the space charge, and (2) the part corresponding to conditions of saturation.

Filaments which are oxide-coated, and to a less degree thoriated filaments, do not show a well-marked saturation value.

The experiment may be carried out with a triode valve if a diode is not available. In this case the anode and grid are connected together by a wire to make virtually a single plate.

The Static Characteristics of a Triode

In the experiment on the diode characteristic the relation between two variables was investigated. These were the anode current and anode voltage. The relation obtained for the space charge limited region was independent of the temperature; the equation of this characteristic can thus be represented in the general form

$$F(I_a, V_a) = 0.$$

The approximate special form applicable to the diode is

$$I_a = AV_a^{\frac{3}{2}},$$

where A is a constant.

In the case of the triode, the potential on the grid is an additional variable. This potential is negative in the normal use of the valve, and it affects the electrostatic field near the cathode, with the result that it controls the number of electrons arriving at the anode.

To the variable anode current, I_a, and anode potential, V_a, occurring in the case of the diode, a third, V_g, is added, denoting the potential of the grid. Both these potentials are measured with respect to the cathode.

There is a relation between these which can be expressed in the general form

$$I_a = F\,(V_a, V_g).$$

The meaning of this is simply that the plate current depends both on the anode and the grid potentials.

When conditions are steady with definite potentials applied there is a definite current flowing through the valve. A negative stream of charges passes from filament to plate, none being lost to the grid, if it is maintained at large or moderate negative potentials. It is important to study the changes which occur when one of the quantities is varied. It is the variations δI_a, δV_a, and δV_g which are important; these will be denoted by i_a, v_a, and v_g respectively.

In order to obtain a plane graphical record of the behaviour of the valve the relation between two variables is studied when the third is kept constant. Thus, with V_g constant a graph can be drawn showing the relation between I_a and V_a. Such a graph is described as a static characteristic of the valve. The term static implies that one of the variables is fixed and the record obtained is characteristic of the valve itself independent of the circuit into which it is introduced.

The fact that there is a relation between the variables implies a relation between certain important quantities which we measure. There is an analogous case in the study of the behaviour of gases. In this case the equation of state, the ideal gas equation, the equation of van der Waals, or whatever it may be, corresponds to the relation between I_a, V_a, and V_g.

Just as we study gases in isothermal, constant pressure, or constant volume conditions, so we study the behaviour of valves by their static characteristics.

It follows from the general equation that, if V_a and V_g are changed by amounts δV_a and δV_g, I_a changes by δI_a, where

$$\delta I_a = \frac{\partial I_a}{\partial V_a}\,\delta V_a + \frac{\partial I_a}{\partial V_g}\,\delta V_g,$$

or

$$i_a = \frac{\partial I_a}{\partial V_a}\,v_a + \frac{\partial I_a}{\partial V_g}\,v_g.$$

The differential coefficients in this relation are partial, and are thus

carried out with the third variable constant. Thus, for example, $\frac{\partial I_a}{\partial V_a}$ measures the change of I_a with V_a when V_g is constant.

Since there is a relation between the three variables, it follows that

$$\frac{\partial I_a}{\partial V_a}\frac{\partial V_a}{\partial V_g}\frac{\partial V_g}{\partial I_a} = -1.$$

The quantity $\frac{\partial I_a}{\partial V_a}$ or $\frac{i_a}{v_a}$ can be determined experimentally by keeping V_g constant. Since the application of an additional potential, v_a, on the anode produces an additional current, i_a, through the valve, it is appropriate to describe the ratio $\frac{v_a}{i_a}$ as the plate resistance, and to denote it by r_a. The application of an additional grid potential, δV_g or v_g, results in an additional anode potential, δV_a or v_a, and since the input potential difference is applied to the grid and the output arises from the plate, it is appropriate to describe $\frac{v_a}{v_g}$ as the voltage magnification. It is implied that this ratio is measured with the plate current constant. If the usual convention be applied to the operation $\frac{\partial V_a}{\partial V_g}$, the ratio turns out to be negative. Thus, to consider an actual example, it is found that for a constant plate current a change of grid potential from -25 volts to -10 volts, i.e. for an increase of potential of 15 volts, the change of plate potential is a drop of 45 volts. The interest is in the fact that the change is numerically increased, i.e. the change of 15 volts in input is associated with a change of 45 volts in output. The voltage amplification is three times. Thus, we write

$$\mu = -\frac{\partial V_a}{\partial V_g} = -\frac{v_a}{v_g}$$

in order to express this amplification, μ, by means of a positive number. The third ratio, $\frac{\partial I_a}{\partial V_g}$ is described as the mutual conductance, S_{ag}, and thus the following relation exists between these quantities:

$$\mu/S_{ag} = r_a.$$

The mutual conductance is some measure of the merit of the valve since the requirement is a low plate resistance and a high amplification factor. An indication of the attainment of this requirement is a high value of S_{ag}.

The Anode Current-Grid Voltage Characteristic

In this experiment let the anode potential be kept at a fixed value while variations of the grid potential are made. Grid potentials, V_g,

for ranges on both sides of zero should be used and should be measured by a voltmeter. The corresponding anode currents, I_a, should be measured by a milliammeter included in the plate circuit.

These observations are repeated for other anode potentials. In this way a series of curves (I_a against V_g) are obtained, and each should

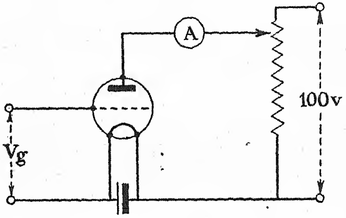

FIG. 446

be labelled with the appropriate anode potential, V_a, and all drawn on the same graph.

It is convenient to adjust the filament current so that an approach to saturation is obtained at the higher plate currents. This may not be

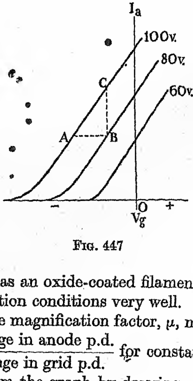

FIG. 447

possible if the valve has an oxide-coated filament. The older tungsten filament shows saturation conditions very well.

From the curves the magnification factor, μ, may be obtained. This means the ratio: $\dfrac{\text{change in anode p.d.}}{\text{change in grid p.d.}}$ for constant anode current.

This is obtained from the graph by drawing a line, AB, to cut two of the curves, e.g. the 80-volt and 100-volt curve from which it appears

that the length, AB, can be represented as a change in grid volts corresponding to a 20-volt change of anode volts. This factor depends chiefly on the structure of the grid and tends to be large if the grid shields the filament to a high degree from the anode. The contributions to the field at the filament arising from the grid and anode potentials are not, in general, relatively the same at all parts of the cathode. The result is that the magnification factor is not independent of the various voltages applied to the tube. It is only in an ideal case that the factor is constant. Thus, this quantity should strictly be measured by making the changes as small as possible.

By repeating the calculation for various parts of the curves the deviation from a mean value can be studied. The mutual conductance, S_{ag}, is by definition the slope of an $I_a - V_g$ characteristic for a constant value of the plate potential, and it can be obtained at various parts of the curves.

It should be recorded for various values of plate and grid potentials, and with the appropriate value of the magnification factor.

Finally, the plate resistance, $r_a = \left(\frac{\partial V_a}{\partial I_a}\right)$ can be determined for V_g constant by drawing the line BC parallel to the I_a axis. In the figure BC gives δI_a and $\delta V_a =$ 20 volts. From the values thus obtained examine the correctness of the formula $r_a = \mu S_{ag}$.

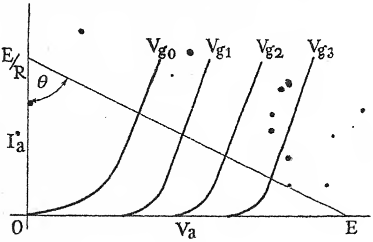

FIG. 448

The Relation between Anode Current and Anode Potential

This characteristic is perhaps the most important of the valve characteristics. It is obtained by maintaining the filament current constant throughout, and, for one set of readings, by keeping the grid potential constant. The latter is then altered and kept at the new value while a second set of observations of the anode current and anode potential is made.

The apparatus required is the same as in the previous experiment, but a voltmeter is required to measure the p.d. between the anode and filament. A variable p.d., which may be obtained from a small battery of cells with the poles joined by a resistance, is required for application to the grid. The resistance may be tapped at various points by a wire connected to the grid, and the applied voltage measured by a voltmeter joining the grid and filament. Beginning with this p.d. at zero, different series of readings may be obtained on both the positive and negative sides at intervals of about 2 volts. The current in the anode circuit is again measured by a milliammeter included in it.

A series of curves will be obtained like those illustrated in the figure. From these curves the slope $\left(\frac{\partial V_a}{\partial I_a}\right)_{V_g}$ gives the plate resistance and, if the work in this and the former experiment be carried out with the same valve, the results obtained may be compared. As in the previous case, r_a and S_{ag} can be obtained.

The Load Line

An important application of the characteristic just obtained is in the study of the conditions in the valve when a load is included in the anode circuit. Let this take the form of a resistance in series with the plate and the battery supplying the constant E.M.F. to the plate circuit.

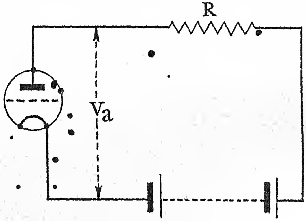

FIG. 449

Let this E.M.F. be denoted by E, so that the p.d., V_a, between the plate and filament is $(E - RI_a)$ where R includes the battery resistance. Thus, there is always the relation

$$V_a = E - RI_a$$

between V_a and I_a during the operation of the valve in these conditions. This relation may be represented on the graph by a straight line cutting the axis of V_a at the value, E, Fig 448, and the axis of I_a at the value, $\frac{E}{R}$.

The slope of this line may be given by the angle, θ, made with the current axis where $\tan\theta = R$. This line is known as the load line.

If the valve is to be worked at a particular potential, V_{g2}, the point, Q, where the load line cuts the V_{g2} characteristic is known as the operating point. The graph gives the values of V_a and I_a under these conditions, and it is interesting, after drawing these characteristics, to draw a load line for some resistance, noting the values of V_a and I_a which the curves give for particular grid potentials. The values may then be checked by applying the two potentials and reading the value of the plate current from the milliammeter.

The Dynamic Characteristic of the Triode

This is the name given to the curve relating the anode current to the grid volts when the potential on the anode does not remain constant. In contrast to the term static characteristic the dynamic characteristic could be described as the relation between two of the variables when the third is not maintained at a constant value. This relation is characteristic of the circuit in contrast to the static characteristic which is peculiar to the valve. The important case, however, is that in which the third variable is V_a and in which this variable varies with I_a on account of the presence of a load in the anode circuit. Thus V_a varies according to the relation

$$E = V_a + RI_a.$$

The static characteristics (V_a against I_a) with the load line will give a series of appropriate pairs of values of I_a and V_g. These can be obtained from the curves by passing along the load line and recording values of the two variables which have then to be plotted on a graph.

The Effects of Secondary Emission in Triodes

The chief source of electrons in these tubes is the heated cathode and the electrons produced in this way are called primary electrons. When they strike the molecules of a gas or solid objects, such as the plate of the triode, provided that they have sufficient energy, they produce electrons as the result of the impact. These are known as secondary electrons, and they have an important effect on the action of the tube.

Usually, the potential of the plate is much higher than that of the grid, so that any electrons knocked off the plate by the primary electrons are attracted back again and there is no effect on the anode current. If the potential of the grid approaches that of the plate, the grid succeeds in capturing some of the secondaries and the loss to the plate current is appreciable. The effect is still more marked when the grid potential is greater than that of the plate.

This effect can be studied by including milliammeters in the plate

and grid circuits and plotting on the same graph curves showing the relation between plate current and grid potential, and also between grid current and grid potential.

The circuit may be connected up as in the diagram. PP denotes a potentiometer across the high-tension terminals from which the

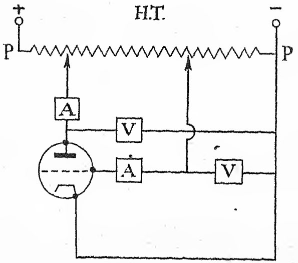

FIG. 450

potentials may be supplied to the plate and to the grid. The various meters measure these potentials and plate and grid currents.

A convenient plate potential is 50 volts, and the grid potential should be varied from —10 volts to about 150 volts.

The curves obtained on the graph will be of the character illustrated.

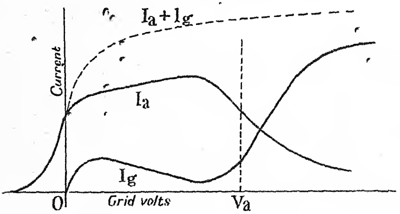

FIG. 451

It will be noticed that near the point where the grid potential approaches that of the plate the anode current diminishes and the grid current increases. This is due to the capture of the secondary electrons originating at the plate by the grid. Beyond this point the grid current increases at the expense of the plate current. It is to be

expected that the sum of the two currents will tend to a constant value since the grid collects charges which are lost by the plate. To test this point the sum of the currents should be plotted against the grid potential by adding the ordinates of the two curves. In general, this constancy is not attained, at any rate, not so soon as would be expected on this simple view of the case. The diagram shows the type of result that has been obtained in this experiment. The explanation is that with the change in the grid potential, there is a change in the state of the space charge which allows a greater supply of electrons to reach the grid and plate.

The Static Characteristics of a Tetrode

The tetrode is the simplest example of a screen grid tube and, as the name implies, it is a tube containing four electrodes. These are the cathode, control grid, screen grid, and the anode. The control grid acts in the same way as the grid of a triode, but the purpose of the screen grid is to act as an electrostatic shield to the anode, avoiding capacitative coupling between it and the control grid.

In operation, the screen grid is usually maintained at a fixed positive potential somewhat less than that of the plate.

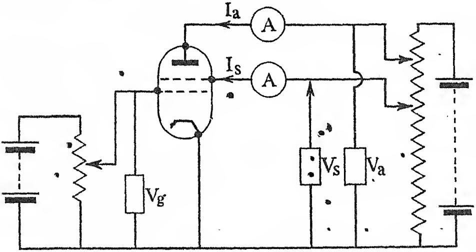

FIG. 452

Most of the electrons from the cathode pass through the meshes of the screen grid and are collected by the plate, so that while acting as a shield to the plate it does not diminish the current in the plate circuit to a great extent. This effect will be studied in the experiment. This is similar to the last, in which the effects of secondary emission at the plate were studied, but the currents to be measured are those in the plate and screen grid circuits.

The tetrode will have a value recommended for the screen grid potential. Set the screen grid potential somewhat below this in the

first part of the experiment and keep it fixed. Proceed to draw the curves of current against anode volts for at least three different control grid voltages. It is convenient to begin at zero and reduce to two negative values of about two or three volts.

The currents in the two cases, I_a and I_s, should be recorded at the same time as well as their sum. In each case three curves are to be drawn.

The experiment should then be repeated with the screen grid potential set at the value recommended.

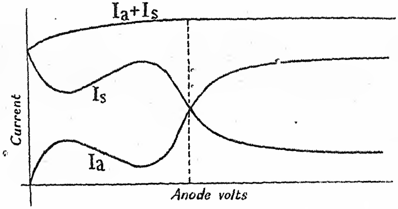

FIG. 453

The type of curves obtained are illustrated in fig. 453. It will be noted that when the anode potential rises beyond that of the screen grid the current lost to this grid is small. It will be noted that with the recommended potential the valve is more efficient in that the currents are large.

The character of the curves in the part before the anode and screen volts become equal shows that for a particular region the current may fall in the plate circuit as the plate voltage rises. This means that the plate resistance in this region is negative, or that the valve acts as a source of energy. In this condition the tube is described as a dynatron.

The Static Characteristics of the Pentode

In the pentode there are three electrodes, known as grids, besides the cathode and anode. The grid nearest the cathode is the control grid and it corresponds to the grid of a triode. The next grid is the screen grid, and the third, which lies nearest the anode, is the suppressor grid. In the ordinary use of the valve the control grid is maintained at a negative potential with respect to the cathode, the screen grid is kept at a positive potential, and the suppressor grid is connected to the cathode.

The two additional grids provide almost complete shielding of the plate from the control grid, so that capacity coupling between them

is almost completely avoided. The lead to the control grid is often at the top of the tube where a metal collar provides further electrostatic shielding.

This experiment is for the purpose of illustrating the effect of varying the potential on the suppressor grid from its usual value zero to a value comparable with that of the anode potential. This is to be followed by a study of the anode current/anode potential characteristic.

For this purpose a pentode is required in which there is an external terminal to the suppressor grid. The diagram illustrates the connexions and shows the various meters required.

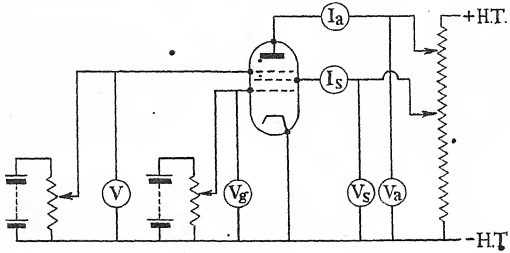

FIG. 454

A suitable range of anode potentials is from zero to 200 or 250 volts, and for the suppressor grid a maximum of 100 volts will be sufficient for the purpose of the experiment. The screen grid should be maintained at its normal working voltage. The control grid will require to be maintained at a series of potentials from zero to a few volts negative.

Let the suppressor grid be maintained at a constant potential of about 80 or 100 volts. The control grid should be maintained at a potential of zero or slightly negative.

Plot a graph showing the relation between plate current and plate voltage, and at the same time on the same graph plot screen grid current against plate voltage. The curves obtained are denoted by I_a and I_s in the diagram.

This is repeated, reducing the potential on the suppressor grid by one-half and finally reducing it to zero.

It is to be noted that as the potential on the suppressor grid is lowered the shape characteristic of the tetrode curves tends to disappear until it is suppressed completely at $V = 0$, V denoting the potential on the suppressor grid. Thus, in this condition the suppressor grid prevents secondary electrons from reaching the screen grid.

The experiment is now completed by drawing characteristics for the

anode current against the anode potential for various control grid voltages, e.g. from 0 to —2 volts in steps of 0·5 volt.

The potential on the suppressor grid is maintained at zero and the screen grid at the given normal potential.

From the curves deduce, as in the case of the experiment on the characteristics of the triode, the values of the amplification, the plate resistance, and mutual conductance of the valve. Compare the magnitudes of these quantities with those in the case of the triode, noting especially the high values of the amplification and resistance in the case of the pentode.

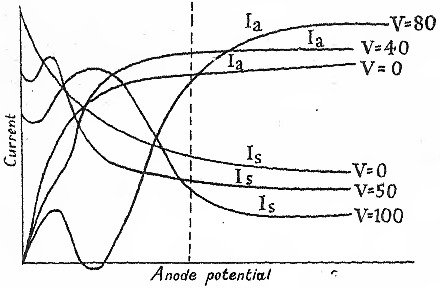

FIG. 455

BRIDGE METHODS FOR THE DETERMINATION OF VALVE CONSTANTS

It is possible to measure the values of the quantities necessary for a knowledge of the working conditions of valves by the use of bridge methods, the plotting of characteristics, as in the methods already described, being unnecessary.

The Measurement of the A.C. Resistance of a Triode

In this case the unknown arm of the bridge contains the valve and its circuit. The bridge is joined up, as illustrated in the diagram, as if to be supplied by current from the anode circuit.

The bridge is supplied with an alternating current, and since the valve circuit contains no alternating source the bridge will measure its alternating current impedance. This may be to some extent capacitative, and this reactive component must be balanced out by connecting a variable capacity across R or P. The need for this will appear when it is found difficult to obtain an exact balance in the bridge, as indicated

by the telephones. By varying P an exact balance can be obtained for a valve which acts as a pure resistor. The bridge arms must consist of non-reactive resistances.

An illustrative experiment can be made by keeping the voltage

In general this depends on the grid voltage and the ranges, but the quantity obtained in this way is useful, since it is a measure of the mutual conductance under working conditions of the valve.

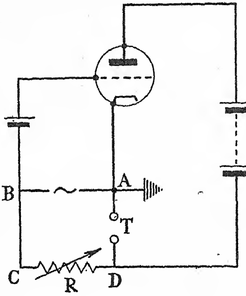

FIG. 457

Draw a graph of the ratio for various values of the grid potential, e.g. from −10 volts to zero.

The Measurement of the Amplification Factor of a Triode

Connect up the valve as in fig. 458, including a fixed resistance, P, in the grid circuit and a variable resistance, Q, in the anode circuit.

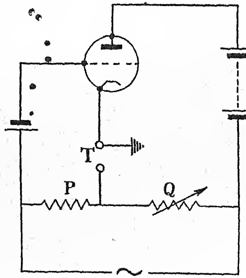

FIG. 458

Telephones are connected between the filament of the valve and a point between P and Q.

An alternating E.M.F. is introduced across PQ, and the part of this affecting the grid must not be so large that the grid p.d. becomes positive. The telephones, T, and the resistance, Q, must together have an impedance which is small compared with that of the valve, so that the working conditions are approximately those in which the plate potential is constant. By varying Q the telephones can be made silent. When this condition is attained the E.M.F. in the plate circuit which results from the change in potential in the grid circuit owing to the alternating current in P is balanced by the alternating current in Q. Thus $\frac{\delta V_a}{\delta V_g} = \frac{Q}{P}$, and the ratio measures on the average the amplification factor for this range of the variation under the conditions at which the valve is working.

Make a series of observations of the ratio for different values of the grid p.d. over a range of about -10 volts to zero for the normal working anode potential and plot a graph of amplification against grid volts.

Measurement of the Ionization Potential of the Gas in a Soft Valve

Method (1)

Find the relation between the anode current i_a and the anode potential v_a of a soft valve. (If a triode is used the grid and anode should be joined to form one electrode.) For values of the anode potential

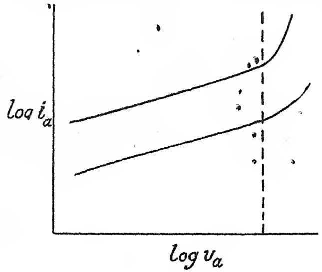

FIG. 459

below ionizing value, the relation between anode current and anode potential is represented by the Child-Langmuir formula for a space-charge controlled discharge of one type of ion, i.e.

$$i_a = Av_a^{\frac{3}{2}}.$$

If, however, the anode potential is increased to ionizing value, positive ions are formed, and because of their low mobility are very effective in neutralizing the electron space charge, so that the above

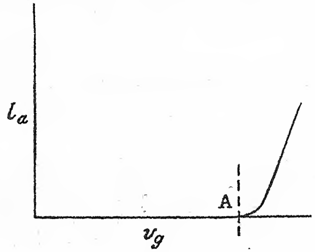

Fig. 460

relation no longer holds, the current increase with increase of potential being greater than that indicated by the equation. Thus, if the experimental values are plotted as log i_a against log v_a we get a straight line so long as the above relation holds (i.e. for values of v_a below ionizing value), but the curve departs from straightness at the ionizing value of v_a. Curves for different values of filament current should be obtained.

The break in the straight lines will occur at about the same value of log v_a as in the figure. From this, v_a corresponding to ionization is determined.

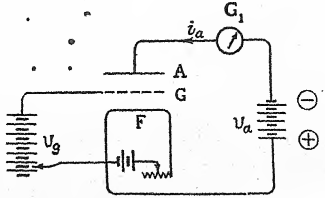

Fig. 461

Method (2)

Here a three-electrode valve must be used. The anode potential is made negative (e.g. -10 volts) with respect to the filament so that it will not collect electrons but will collect positive ions. Thus no current

will be registered by the anode current galvanometer, G_1, until positive ions are produced in the tube. The grid potential is made positive and gradually increased, the values of the anode current being noted. The relation between grid potential v_g and anode current i_a is then plotted (e.g. fig. 460). The ionization potential is that grid potential at which the positive ion anode current starts (e.g. at A in fig. 462).

The Determination of $\frac{e}{m}$ by Means of the Magnetron

A magnetron is a diode consisting of a filament situated along the axis of a cylindrical anode. When a magnetic field is applied along the axis of the cylinder the electrons traverse curved paths with greater curvatures the greater the field strength. Thus, for very strong fields the electrons emitted by the filament will fail to reach the anode.

In this experiment the anode current for a given anode potential is plotted against the magnetic field strength. The curve showing this

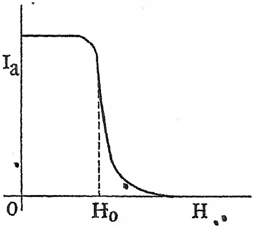

FIG. 462

relation is illustrated in fig. 462. The cut-off occurs at a certain field intensity, H_0.

With some simplifying assumptions, such as supposing that the electrons have no appreciable velocity on leaving the filament, it can be shown that

$$H_0 = \sqrt{\frac{8m}{e}} \cdot \frac{v_a^{\frac{1}{2}}}{a}.$$

In this formula a denotes the radius of the anode and v_a the anode potential.

It is thus clear that $\frac{e}{m}$ can be deduced from the value of H_0 obtained from the graph (fig. 462). Typical values that have occurred in practice are $v_a = 100$ volts, $a = 0{\cdot}5$ cm., $H_0 \simeq 135$ oersted.

In the formula H_0 is in oersteds and v_a in E.M. units.

It must be remembered that 1 volt $= 10^8$ E.M.U.

The magnetic field is produced conveniently by means of a long straight solenoid and the magnetron is placed well inside it to avoid end effects.

It will be found that a solenoid with fifty turns per cm. able to carry a current up to 2 amperes is convenient. The field $\mathrm{H} = \dfrac{4\pi n\mathrm{I}}{10}$, if I is in amperes. The rest of the apparatus is similar to that required in the experiment on the diode characteristics.

The equation $\mathrm{H}_0 = \sqrt{\dfrac{8m}{e}}\,\dfrac{v_a^{\frac{1}{2}}}{a}$ may be written:

$$\mathrm{H}_0 = \mathrm{A}v_a{}^n, \quad \text{where} \quad \mathrm{A} = \frac{1}{a}\sqrt{\frac{8m}{e}}$$

or
$$\log \mathrm{H}_0 = \log \mathrm{A} + n \log v_a.$$

The values of H_0 for several values of v_a (say 20 to 60 volts) are obtained experimentally and a graph is plotted of $\log \mathrm{H}_0$ against $\log v_a$.

The slope of the graph, n, should confirm that $n = \frac{1}{2}$. The intercept of the graph gives the value of A from which, using the known value of a, $\dfrac{e}{m}$ may be calculated.

CHAPTER XXVI

THE THERMIONIC VALVE AS A GENERATOR OF OSCILLATIONS

In order to illustrate the conditions which must be satisfied in a thermionic valve so that it may act as a generator of electromagnetic oscillations two experiments will be described. These are, firstly, the case in which a triode is used with a tuned grid circuit and, secondly, that in which it is used with a tuned anode. The condition to be satisfied is similar in the two cases.

The Study of the Condition to be Satisfied in a Tuned-grid Oscillator

Fig. 463 gives a diagrammatic representation of the apparatus. The circuit made up of L, R, and C is the seat of the oscillations. These

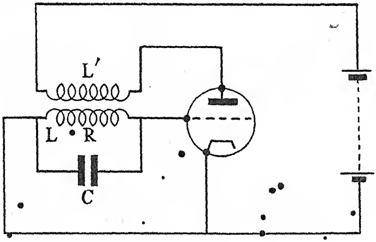

Fig. 463

are maintained by energy supplied by mutual induction from the coil, L′, of the anode circuit.

Let I denote the current in the circuit, LRC, and q the charge on C, so that $\mathrm{I} = \dfrac{dq}{dt}$.

Let I_a denote the current in the anode circuit, i.e. the current in L′, so that the E.M.F. in the oscillatory circuit is $\pm \mathrm{M}\dfrac{d\mathrm{I}_a}{dt}$. The sign depending upon the way the coils are introduced into their circuits.

The application of Ohm's law to the oscillatory circuit thus gives

$$\pm \mathrm{M}\frac{d\mathrm{I}_a}{dt} = \mathrm{L}\frac{d\mathrm{I}}{dt} + \mathrm{RI} + \frac{q}{\mathrm{C}}.$$

If s_{ag} denotes the dynamic anode-grid mutual conductance $\dfrac{dI_a}{dV_g} = s_{ag}$, where V_g denotes the grid potential which, in this case, is equal to $\dfrac{q}{C}$. In this case the dynamic and static values are approximately the same. Thus $\dfrac{dI_a}{dt} = \dfrac{s_{ag}}{C} I$ and, on substitution in the above equation, and making a further differentiation with respect to the time, we obtain

$$L\frac{d^2I}{dt^2} + \left(R \pm \frac{Ms_{ag}}{C}\right)\frac{dI}{dt} + \frac{I}{C} = 0.$$

This equation is familiar in physics in the form:

$$\frac{d^2I}{dt^2} + k\frac{dI}{dt} + n^2I = 0,$$

and the solution is

$$I = ae^{-\frac{k}{2}t}\sin\left(\sqrt{n^2 - \frac{k^2}{4}}\,t + \alpha\right),$$

where a and α are constants of integration. It is assumed that $n^2 > \dfrac{k^2}{4}$, so that oscillations occur. This condition is easily satisfied, for it implies simply that

$$\left(R \pm \frac{Ms_{ag}}{C}\right)^2 < \frac{4L}{C}.$$

In practice this is generally satisfied. The solution shows that the oscillations will decay exponentially if k is positive. If k is negative the solution suggests that the amplitude will increase with time. This requires that $\left(R \pm \dfrac{Ms_{ag}}{C}\right)$ should be negative. This will not be the case if the positive sign applies. It is thus important to have the coils connected in such a way that the negative sign applies in front of M and the condition for maintenance of oscillations is $Ms_{ag} > RC$.

If the condition, $Ms_{ag} = RC$, is satisfied, the suggestion is that simple harmonic oscillations are maintained, but it is not possible experimentally to keep to this exact equality. In practice $Ms_{ag} > RC$, and the solution now suggests that the amplitude of the oscillations builds up continuously with time. In this discussion it has been assumed tacitly that the mutual conductance, s_{ag}, is constant, which means that the characteristic relating I_a and V_g is linear in the region of operation of the valve. This is the case for small values of V_g, i.e. for values of a few volts, but with the increase of amplitude the grid potential acquires values for which these considerations do not apply.

The magnitude of s_{ag} diminishes as V_g increases, as may be seen from the I_a/V_g characteristic. Thus $\left(R - \frac{Ms_{ag}}{C}\right)$ will tend to increase and will finally become positive, the oscillations then ceasing. The result is that the amplitude cannot increase indefinitely and a steady state is reached.

In order to put these results to an experimental test it is convenient to arrange that the frequency generated lies in the range of audio-frequency. A coil connected to headphones placed near L′ can then be used to detect the presence of oscillations. The undamped frequency is $\frac{n}{2\pi}$ from the solutions we have obtained and this lies close to the frequency generated in the actual case. Thus, the frequency is $\frac{1}{2\pi\sqrt{LC}}$ and, if we require that this should be about 1500 cycles per second, we can use a coil of inductance $L = 20$ mH., and a capacity of 0·5 μF. for this gives a frequency of approximately 1600 cycles per second.

The valve must be operated over the linear part of the characteristic and the mutual inductance must be determined. The characteristic and s_{ag} should be obtained before using the valve. Let it be supposed that the value in a particular case is 5 mA. per volt. The magnitudes of R, C, and M should be chosen so that it is possible conveniently to obtain the condition $RC = Ms_{ag}$. In the present case this requires $RC = 5 \times 10^{-3}M$.

In order to obtain a variation of M a standard variable mutual inductance may be available, but if not, two coils should be taken and set up co-axially. The mutual inductance should be measured with the coils parallel and with various separations of their centres. A graph may then be drawn giving the value of M for various positions. Some idea of the values of R and C which are suitable for the experiment is then obtained.

Let L and L′ be placed so that the value of M is fairly large. The resistance, R, consists of that of the coil, L, and it may be varied by including a non-inductive resistance in series. The capacity, C, is a variable standard. It may happen that oscillations cannot be generated whatever the value of C. In this case reversal of the connexions of L′ will usually lead to the production of the necessary condition. If oscillations are produced immediately, the connexions to L′ should be reversed in order to show the importance of introducing the correct sign in the term containing M.

Adjust values of L, C, and M so that oscillations just appear and exhibit the result by plotting on a graph the values of RC against M in this limiting case. The graph should be linear and the slope should give the mutual conductance of the valve.

The Study of the Condition to be Satisfied in a Tuned-anode Oscillator

The equation for the variation of current in the inductance, L, can be obtained by a procedure similar to that of the last experiment.

Let I denote the current in the coil, L, and q the charge on the condenser, C. Then

$$I_a = I + \frac{dq}{dt}.$$

The E.M.F. induced in the grid-filament circuit is $M\frac{dI}{dt}$, so that an E.M.F. $\mu M\frac{dI}{dt}$ is generated in the anode filament circuit, μ denoting

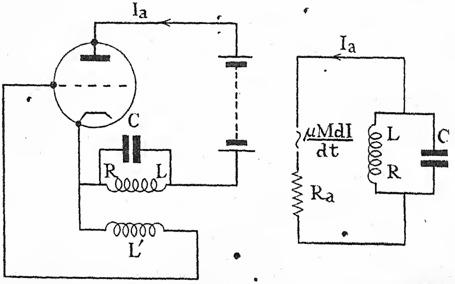

FIG. 464

the amplification factor. Thus, reference to the equivalent circuit (fig. 464) shows that for the anode-filament circuit

$$\pm \mu M\frac{dI}{dt} = R_a I_a + L\frac{dI}{dt} + RI,$$

where R_a denotes the plate resistance.

Equating the fall of potential through the capacity, C, to that through L and R:

$$\frac{q}{C} = L\frac{dI}{dt} + RI.$$

Thus $$I_a = I + LC\frac{d^2I}{dt^2} + RC\frac{dI}{dt},$$

and it then follows by substitution for I_a that

$$R_a CL\frac{d^2I}{dt^2} + (CRR_a + L \pm \mu M)\frac{dI}{dt} + (R_a + R)\,I = 0.$$

As in the previous example, the condition for the generation of oscillations requires the negative sign before μM, and

$$\mu M \nless CRR_a + L.$$

This relation suggests how the experiment may be carried out. Various values of M can be determined, as in the previous experiment, by separating the two coils. The graph drawn can be used for both cases.

The other quantity which is easily varied is the capacity. According to the equation the natural frequency when $\mu M = CRR_a + L$ is

$$\frac{1}{2\pi}\sqrt{\frac{\left(1+\frac{R}{R_a}\right)}{CL}}.$$

In practice this is approximately $\frac{1}{2\pi\sqrt{LC}}$. Thus, by varying C various frequencies of oscillation result.

If C is adjusted so that oscillations just fail to occur for various values of M, the relation between M and C should be linear and, if a graph is drawn of M against C, the intercept on the axis of M should be $\frac{L}{\mu}$ and the slope $\frac{RR_a}{\mu}$ or $\frac{R}{s_{ag}}$.

Thus for given values of μ and R_a, s_{ag}, L, and R, are known, so that the test can be made quantitatively. The oscillations may conveniently be obtained in the audio-frequency range and the condition of oscillation can be investigated by the use of earphones.

In each setting the frequency should be recorded.

When the frequency is in the audible range the value may be determined by means of the method described in Chapter XV, p. 426. A graph should be drawn of the square of the frequency to the resonating volume in order to obtain a variable frequency standard. By pouring water into the resonator it can be tuned to the note heard in the earphones. The simplest way to do this is to blow across the mouth of the resonator and compare the note given out with that generated in the valve circuit. By careful adjustment the notes may be brought into unison. The test of the frequency formula,

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(1+\frac{R}{R_a}\right)},$$

can then be made.

Suitable values of the various quantities can be estimated from the formulae. Thus, if the capacity be of the order of tenths of microfarads and the self-inductance about one-tenth henry, the frequency will be of the order of 1000 cycles per second. The value of R may be about 100 ohms and of R_a 10,000 ohms. Then, if $s_{ag} = 5$ mA. per volt, as in

the last experiment, the magnification is of the order 50, and this will correspond to a value of M of about 4 mH.

Verification of the Conditions of Oscillation in the case of the Dynatron

For the purpose of this experiment a tetrode may be used with a high-tension battery supplying the potentials to the anode and screen

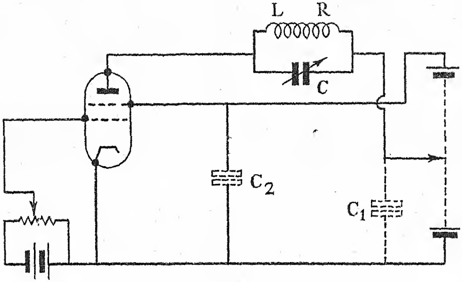

FIG. 465

grid. The latter may be connected so that the full potential of the battery is applied to it; somewhat lower potentials are required for the anode. The grid bias should also be variable and a potentiometer should be set up in order to make this variation continuous.

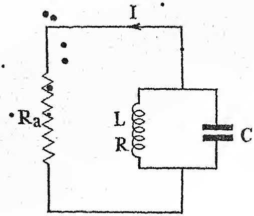

FIG. 466

The anode circuit contains a capacity in parallel with a coil of self-inductance, L, and of resistance, R. The condensers, C_1 and C_2, are inserted to provide by-passes to safeguard the battery.

The circuit of L, R, and C can be represented by fig. 466, R_a denoting the valve resistance.

For the circuit, by the application of Kirchhoff's laws:

$$\frac{q}{C} = R_a i_1 = L\frac{di}{dt} + Ri,$$

where q = the charge on the condenser and $\frac{dq}{dt} = i_2$.

In addition, $i + i_1 + i_2 = 0$.

From these relations the following equation can be deduced:

$$\frac{d^2i}{dt^2} + \left(\frac{R}{L} + \frac{1}{CR_a}\right)\frac{di}{dt} + \left(\frac{R_a + R}{LCR_a}\right)i = 0.$$

From the discussion in the case of the tuned grid and tuned anode circuits it follows that oscillations of frequency $\frac{1}{2\pi}\sqrt{\frac{R_a + R}{LCR_a}}$ will occur when $\left(\frac{R}{L} + \frac{1}{CR_a}\right)$ acquires a zero or negative value.

In practice R is small compared with R_a, so that the frequency may be taken to be

$$\frac{1}{2\pi}\sqrt{\frac{1}{LC}}.$$

The frequency may be measured as in the previous cases and the experimental value should be compared with that given by the theory.

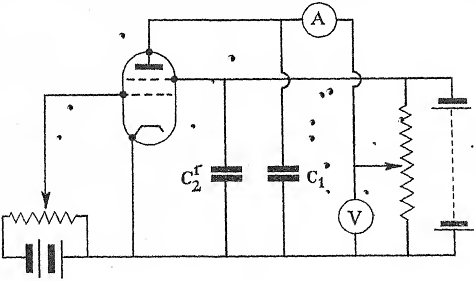

FIG. 467

In the experiment, L, C, and R are set at some convenient value, but the resistance, R_a, can be varied by means of the grid bias. This should be varied until oscillations occur. The critical value is

$$R_a = -\frac{L}{CR}.$$

This result should be checked by determining the value of R_a for the tetrode from the anode current/anode potential characteristic in the region of the critical point. It will be found that the characteristic has a negative slope at this point and the value $|R_a| = \frac{dV_a}{dI_a}$ should be compared with $\frac{L}{CR}$.

The characteristic may be obtained by removing the circuit, LRC, to prevent oscillations and introducing a direct current milliammeter, A, and a direct current voltmeter, V, as in fig. 467.

A sequence of values of L, R, and C, $\frac{L}{CR}$ and R_a should be tabulated together with the observed and calculated frequencies. It will be found that R_a can be varied over a wide range.

CHAPTER XXVII

RADIO-FREQUENCY MEASUREMENTS

WHEN measurements are made in which electrical oscillations occur at the high frequencies associated with radio communication, e.g. at a wave-length of 300 m. or 10^6 cycles per second, it becomes necessary to pay attention to certain facts that do not appear evident in connexion with direct or with low-frequency currents. Thus, owing to skin effect the resistance of a wire varies with the frequency, and a self-capacitance of part of the circuit may offer a path of low impedance at a high frequency. It is thus important when making measurements in this region to record the frequency of the electrical oscillations at which they are made.

In the case of experiments at high frequencies special apparatus is necessary, such as wavemeters and valve voltmeters. These have to be used with the same familiarity as batteries and galvanometers in experiments with direct currents. The student should know the principles on which the construction of these instruments are based, but should be able to use them without the necessity of going into every detail of their construction, which is sometimes intricate. In some of the experiments which follow, a calibrated wavemeter is essential. This is an oscillator working on principles that have been illustrated in the experiments studying the conditions of oscillation in various cases. It is usually enclosed in a box provided with suitable terminals and with a scale which indicates the frequency of the oscillations emitted with various settings. Such an oscillator is to be regarded as part of the laboratory equipment.

We begin this series of experiments with some which illustrate the action of valve voltmeters.

The Calibration of a Diode as a High-frequency Voltmeter

1. The relation between the potential applied to a diode at a low frequency and the rectified current through it is determined experimentally. It is assumed that the same relation holds for high frequencies so that a graph drawn from the results obtained for low frequencies can be used, on this assumption, for high-frequency measurements. In order to carry out the calibration, connect up the circuit as illustrated in fig. 468.

The resistance, R, is about two megohms and is inserted in order that the diode may not abstract any appreciable power from the source which is under investigation. It plays the same part as the high resistance of an ordinary voltmeter. A and B are connected to the source,

so that the voltmeter is in parallel with the part of the circuit under test. In order to provide a series of known potentials an auto-transformer may be used connected to the laboratory main supply, usually of a

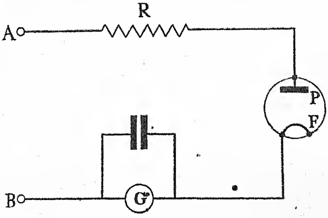

FIG. 468

frequency of 50 cycles. The potentials of the transformer can be determined by a low-frequency A.C. voltmeter and the currents by a sensitive D.C. galvanometer, illustrated by G. This galvanometer will present an impedance to the current on account of its windings, so that it must be shunted with a condenser to act as a by-pass.

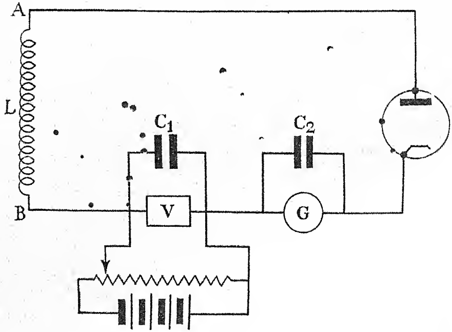

FIG. 469

A graph should be drawn showing the relation between the applied potential and the current in G. This will then serve as the calibration graph for the voltmeter. Plot the input p.d. against the square of the current, since this gives an approximately linear graph.

The inclusion of the high resistance, R, helps to bring about this approximately linear relation. The voltages measured are in this case average voltages of the supply.

2. Another interesting application of the diode as a voltmeter is by means of the so-called 'slide-back' method. In this case the voltmeter is an ideal one in the sense that it takes no current from the supply. Unlike the method just described, it measures the maximum or peak voltage of the supply, not the average voltage. This will be clear from the description of the action of the instrument.

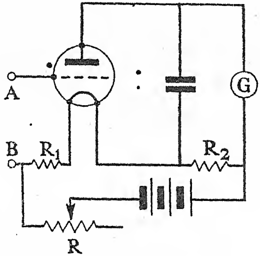

FIG. 470

The apparatus is connected up as shown in the diagram. V is a direct current voltmeter and G a sensitive galvanometer, each shorted by a high-frequency by-pass condenser.

The instrument measures the peak voltage developed in the coil, L. This is done by means of the potentiometer placed across the voltmeter which applies a potential difference between the anode and cathode. When this is adjusted so that the anode current is zero, as denoted by the galvanometer, the p.d. recorded by the voltmeter is equal to the amplitude developed across AB, but is opposite in sign. Thus the positive peak voltage between A and B is measured by the reading of V. The negative peak voltage can be measured by reversing the terminals of the coil. An important characteristic of this method is that no preliminary calibration by means of known alternating current voltages is necessary.

3. In this example of a voltmeter we take the case of the instrument introduced by Moullin in which a triode is used.

A feature of the apparatus is that only one battery is used for the supply to the filament and anode. This supply is provided by means of a potentiometer consisting of the resistances, R_1 and R_2, and a variable resistance, R, giving a maximum of about 6 volts. The anode

potential is due to the fall along R_2 and the grid potential to that along R_1. A series of known alternating potential differences is required for calibration, and these are applied across AB. The current in the galvanometer is recorded for various applied potential differences and the relation is represented graphically. The calibration can be made at low frequencies, and it is found to hold up to a frequency of 3×10^7 cycles per second. For high frequencies the grid bias resistance, R_1, should be shunted by a by-pass condenser.

An adjustment is necessary to reproduce the exact conditions of calibration. This is made by joining the terminals A and B by a copper wire and adjusting R so that the galvanometer causes a deflection to a marked point of the scale.

Calibration of a Variable Condenser. Method of Substitution

A standard condenser, C_0, is connected by short thick leads in parallel with the unknown condenser, C. If a fine adjustment is necessary this can be provided by a smaller variable standard condenser in parallel with C_0, so that these together provide the standard. One side of the

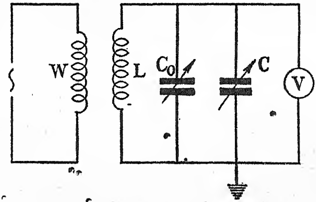

FIG. 471

condensers is earthed, as shown, in common with the filament of the valve voltmeter. In the diagram this instrument is represented by V, the current being read by means of a galvanometer or a micro-ammeter. A coil, L, is connected by stiff leads to the condensers so that no change in their positions can occur, and this is loosely coupled to a generator, which may conveniently be a wavemeter.

L is chosen so that with the condensers it provides a circuit which oscillates in tune with a frequency within the range of the wavemeter.

In operating the condensers it is important to avoid the addition of capacities through the medium of the operator, and the final condition of resonance must be obtained when the operator is well away from the capacities. It is convenient to make the final adjustment by means of a long insulated handle attached to the moving part.

There are two cases to consider: (i) when the unknown condenser

range runs over values within that of the standard, and (ii) when the range extends beyond that of the standard.

In the first case, L is suitably chosen and C_0 set near its maximum value, the generator being adjusted to a frequency in resonance with the natural frequency of the circuit of L and C_0. This resonance is detected by the valve voltmeter which gives a maximum reading under these conditions. The condition is that of voltage resonance, the voltage being measured across the condenser. So far the condenser, C, is not included. Next include C set at its minimum value and reduce C_0, making use of the fine adjustment if necessary, until resonance again occurs.

The capacitance introduced by C is equal to the diminution in C_0. Proceed in this way over the range and plot a calibration graph for C.

In the second case, the procedure is the same until C reaches its minimum value. C is then calibrated over the range of the standard, but there is a further range not yet investigated. Substitute a coil of smaller inductance so that resonance is obtained when the standard is raised to a point near its maximum value, the unknown condenser remaining at the point to which calibration has already been made. Then, by reducing the standard and raising the unknown capacitance, a further range equal to that of the standard can be covered. The process may be repeated if necessary, the final result being that the unknown condenser is calibrated for a particular frequency. This frequency should be recorded with the calibration graph.

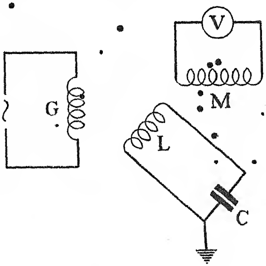

Fig. 472

Measurement of the Inductance and Self-capacitance of a Coil

The coil to be examined is L. It is connected with a variable standard capacity, C, which should measure capacities of the order of micro-micro-farads, since that is the order of magnitude of the self-capacitance

to be measured. To obtain a fine adjustment, if necessary, another condenser of the appropriate magnitude can be placed in parallel with C.

The wavemeter supplies the coil, G, with a sequence of known frequencies and the circuit of L and C is tuned to these by varying C.

Let c denote the self-capacitance of L, regarded as in parallel with C. When resonance occurs the following relation holds:

$$(C + c)\,L\omega^2 = 1,$$

where $\omega = 2\pi f$, f denoting the frequency.

The condition of resonance is detected by means of the valve voltmeter loosely coupled to L. Coupling between the generator and valve voltmeter circuit must be avoided, and to obtain this condition the

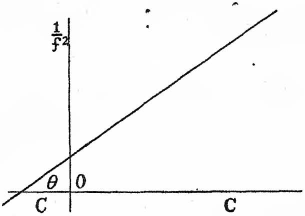

FIG. 473

coils G and M are placed at right angles to each other and suitably separated. The absence of effect between G and M should be tested by observing whether any change in the reading occurs when, in the absence of L, G is excited. The coil, L, should be placed at 45° to each of the other coils and, if it is required, L may be placed within G to obtain a high degree of coupling.

All that is now required is to plot the reciprocal of the square of the frequency against the known capacity which produces resonance. A linear graph is obtained giving the value of c by the intercept on the axis of C.

The relation between C and f is

$$\frac{1}{f^2} = 4\pi^2 L\,(C + c).$$

The value of L can be found from the slope of the graph

$$\tan\theta = 4\pi^2 L.$$

A useful exercise is to wind a solenoid on a former to give a self-inductance of about 1·5 mH. This will usually have a self-capacity of

the right order and the value of the self-inductance can be calculated approximately from the formula for the solenoid,

$$L = 4\pi^2 n^2 a^2 l,$$

where a = radius, n = number of turns per cm., and l = length.

The calculated and experimental value can be compared.

The value of L depends upon the frequency, so that the graph is not exactly linear, but L remains approximately constant over a wide range below $0 \cdot 5 \times 10^6$ cycles per second.

The method is inaccurate if the intercept is small. In this case the value of L may be obtained from the slope of the graph and the approximate value of the self-capacity from the intercept.

The coil will have a natural frequency of oscillation of approximately $\frac{1}{2\pi\sqrt{Lc}}$, and it will behave like a coil of inductance, L, in series with a capacitance, c.

A wavemeter is placed to excite oscillations in the coil, first setting it at the frequency calculated from L and the approximate value of c, and then bringing the coil into resonance by adjustment of the wavemeter. This will ensure that the coil is resounding to the fundamental and not to a harmonic. The state of resonance is shown by means of a valve voltmeter coupled with the coil, L, and to the coil of the wavemeter in the usual way. c is then calculated from the formula

$$\frac{1}{f} = 2\pi\sqrt{Lc}.$$

The Measurement of the High-frequency Resistance of a Circuit

A coil, L, which provides the principal unknown resistance of the circuit is placed in series with a condenser of variable capacity, C. Another variable condenser to act as a fine adjustment may be placed in parallel with C, if necessary. A standard variable pure resistance, R, is included in the circuit in series. A wavemeter is used as a generator of oscillations and the coil, L, is loosely coupled to it. There must be no direct coupling between the wavemeter and valve voltmeter.

Tune the circuit to resonance for a particular frequency generated by the wavemeter, WM. A deflection, θ, will be recorded by the voltmeter which is approximately proportional to the square of the current in L. If the amplitude of this current be denoted by i,

$$i = \frac{E}{R + r},$$

where E denotes the amplitude of the E.M.F. induced in L.

Since the circuit is in the condition of resonance it acts as a pure resistance, r denoting the resistance of the circuit additional to R.

Thus $R + r = k\theta^{-\frac{1}{2}}$, assuming that the voltmeter operates according to a square law, where k is constant and different values of θ result from varying R.

If a graph be plotted with $\theta^{-\frac{1}{2}}$ as ordinate and R as abscissa, the correctness of the assumption $i^2 \propto \theta$ can be checked. Should the

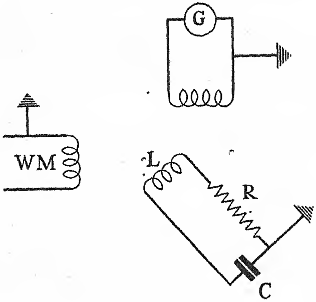

FIG. 474

condition of linearity not hold, the voltmeter is unsuitable for the present purpose. It would be necessary in this case to calibrate the instrument in order to investigate the law relating the deflection to the applied voltage. For the present purpose an instrument should be chosen which conforms to the simple condition.

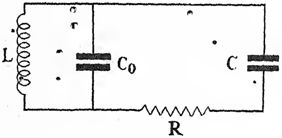

FIG. 475 (*a*)

The value of r is obtained from the intercept on the axis of R. This is the value for the particular frequency at which the generator is set.

The experiment should be repeated at a series of frequencies and r plotted against the frequency.

In this discussion of the experiment it has been assumed that the coil, L, has no appreciable self-capacity. If this is not the case closer consideration is necessary. This self-capacity can be regarded as

existing in parallel with the self-inductance, L. It was treated in this way in the experiment described above for its determination.

Thus the circuit can be represented as in the diagram (fig. 475 (*a*)). The theory applies to the circuit of fig. 475 (*b*), so that we require the values of R′ and C′ in series which are together equal to C and R in parallel with C_0.

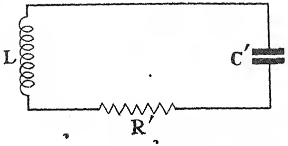

FIG. 475·(*b*)

This is represented by the equation:

$$R' - \frac{j}{C'\omega} = \frac{-\left(R - \frac{j}{C\omega}\right)\frac{j}{C_0\omega}}{R - \frac{j}{C\omega} - \frac{j}{C_0\omega}}.$$

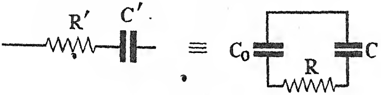

FIG. 476

By equating real and imaginary parts it follows that

$$R' = R\left(\frac{C}{C + C_0}\right)^2.$$

The relation between θ and R′ is

$$R' + r = k\theta^{-\frac{1}{2}}.$$

Thus the relation between R and θ is

$$R\left(\frac{C}{C + C_0}\right)^2 + r = k\theta^{-\frac{1}{2}}.$$

If R_0 denotes the numerical value of the intercept the value of r is

$$r = R_0\left(\frac{C}{C + C_0}\right)^2.$$

Thus the self-capacitance of L is required, and it should be obtained by the method described.

The Determination of the High-frequency Resistance of a Circuit by a Resonance Curve

The circuit is set up as in the last experiment, excluding the standard resistance, R.

Set the wavemeter at a frequency in tune with the circuit when the condenser is at about the middle of its range. With this frequency constant, vary the capacity and draw a graph showing the relation between the voltmeter deflection, θ, and the capacity.

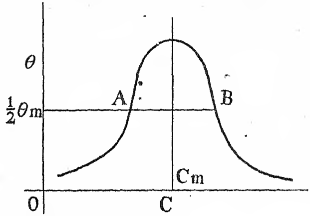

Fig. 477

If E_0 is the amplitude of the sinusoidal E.M.F. induced in the circuit, the amplitude of the current is

$$i = \frac{E_0}{\left[r^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]^{\frac{1}{2}}}.$$

Thus, assuming the square law for the voltmeter,

$$\theta = \frac{A}{r^2 + \left(L\omega - \frac{1}{C\omega}\right)^2},$$

where A is a constant.

At resonance the maximum deflection, θ_m, is

$$\theta_m = \frac{A}{r^2}.$$

Thus
$$\frac{\theta_m}{\theta} = \frac{\left\{r^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right\}}{r^2}.$$

Let the capacity at resonance be C_m, so that $LC_m\omega^2 = 1$, and let C_m be increased by c, the deflection becoming θ_1. If it be assumed

that this change occurs for a value of c small in comparison with C_m, it follows that

$$\left(\frac{\theta_m}{\theta_1} - 1\right)^{\frac{1}{2}} r = \frac{c}{\omega C_m{}^2}.$$

The curve is not quite symmetric about the maximum, but with the same approximation a diminution of the capacity by the same amount will give the same value of the deflection.

If the resonance curve is fairly sharp the approximation will be a close one, and the appearance of the curve will indicate the validity of the result.

A convenient method of carrying out the experiment is to obtain the curve and draw across it the line $\theta = \frac{1}{2}\theta_m$. The distance, AB, on the graph is $2c$. This can be measured directly and denoted by c'. Then, since $\theta_1 = \frac{1}{2}\theta_m$ in this case, it follows that

$$r = \frac{c'}{4\pi f C_m{}^2},$$

where f denotes the frequency of the oscillations.

The values required for the calculation should also be obtained by an actual setting of the apparatus. The maximum should be carefully traversed and C_m recorded from the condenser. The deflection should be halved by two settings of the condenser and the difference taken. The result calculated in this way from the formula should be compared with the result obtained from the observations on the graph.

CHAPTER XXVIII

THE CATHODE-RAY OSCILLOGRAPH

The Cathode-ray Oscillograph

THE cathode-ray oscillograph is an apparatus which has only recently become generally available for the teaching of practical physics. It has a wide range of applications and students of physics can now regard it as a piece of laboratory apparatus in the same way as they look upon galvanometers, stop-watches, or spectrometers.

The instrument will be described briefly and some experiments suggested in order that familiarity with its use may be gained. Fuller details in which the student may be interested must be sought in catalogues and books which deal with the structure of the various types in use.

This instrument, like many others used in experimental work in physics, is provided with a pointer and sometimes also with a scale, but the particular characteristic is that the pointer is without inertia, and thus responds at once to impulses imposed upon it. It can thus follow very rapid changes and portray them on a suitable screen.

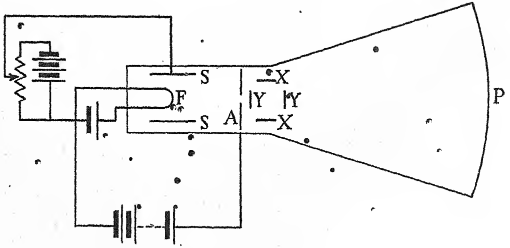

FIG. 478

The Construction of a Simple Cathode-ray Oscillograph

The diagram represents a soft cathode-ray oscillograph which contains an inert gas such as argon. Electrons are produced from the filament, F, which is heated by a low-tension current represented in the diagram as originating from a battery. These electrons enter into the field between the filament and anode, A, the difference of potential being, in this case, 500 to 1000 volts. They are thus accelerated and pass through the hole in A with considerable energy, and on striking

the fluorescent screen, P, they make it luminous at the point of impact. The screen marked S is usually kept at a potential a little lower than that of the filament, so that electrons which tend to diverge are made to concentrate along the centre of the beam. This is an example of electrostatic focusing, and its effect is to direct more electrons through the hole in the anode.

The focusing in this case is not enough to produce a sharply defined small spot on the screen, and in the soft tube additional definition results from the presence of the inert gas. The gas is ionized by the stream of energetic electrons and the electrons produced in the process add themselves to the main stream. The resulting positively charged ions produce a field which tends to keep the electrons clustered round them and so prevent spreading.

The plates, XX and YY, are arranged in pairs with the plane of one pair perpendicular to that of the other, both parallel to the direction of the electron stream, as illustrated diagrammatically in figure 478. One pair of plates is horizontal and the other vertical, and one pair is at a short distance from the other. By the application of fields to these plates the electron beam is changed, and since the beam has negligible

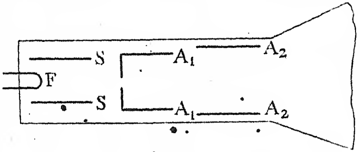

FIG. 479

inertia it follows the variation of the field instantaneously. The character of the variation is thus portrayed on the screen by a luminescent line.

In the hard tube the focusing is of a different character. The tube is exhausted to the state of a high vacuum and the focusing is obtained either by electrostatic or magnetic fields.

In the case of electrostatic focusing, the filament is surrounded by the control shield in the way already described, but there is a succession of anodes. Fig. 479 illustrates a two-anode focusing device.

The second anode is the main acclerating device in this case, and is at a potential of 1000 to 2000 volts above that of the filament. The first anode, which lies between the filament and second anode, is maintained at about a quarter of the potential difference.

The arrangement can be described as an electron gun. For still better focusing, an additional anode is required and the best potentials to employ are indicated with the apparatus.

In the case of magnetic focusing, the anode serves as an accelerator, as described in the case of the soft tube, and a screen surrounds the filament as before. The focusing is now by means of a coil of wire round the neck of the tube, which carries an electric current and thus generates a magnetic field along the axis. Any electrons not moving in this

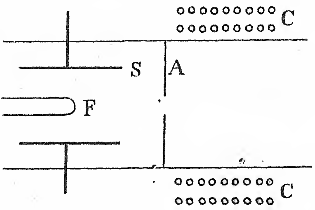

FIG. 480

direction spiral round the lines of force, and as their velocity components at right angles to the axis of the tube are not large, they spiral in paths of small radius and tend to come to the axis along which they move with the velocity acquired by the fall between the filament and anode.

It will be observed in any but the very simple forms of soft tubes that the X- and Y-plates are not so simple as has been illustrated so

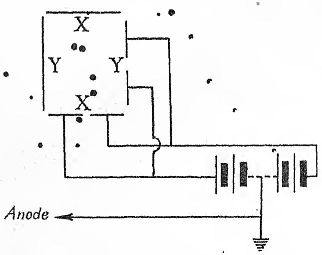

FIG. 481

far. The form often taken in practice is illustrated in fig. 481. The reason for this is to eliminate a form of distortion known as origin distortion. One of each pair of plates is divided and the parts maintained at a difference of potential. In the diagram is shown a

battery supply, but in practice the required difference is obtained from the tube power pack.

Thus the average p.d. through which the electrons fall is kept constant and the sensitivity is unaltered.

The Power Supply for a Cathode-ray Tube

The diagram, (fig. 482), illustrates a circuit suitable for the supply to a hard tube. The mains supply is transformed to give both the low and high tension. The former is of the order of 1 volt and heats the

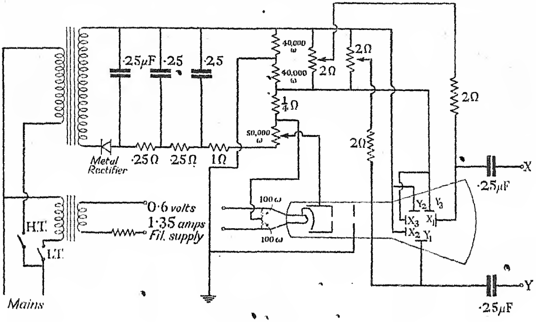

FIG. 482

filament, providing the recommended number of amperes. This may require an increase as the tube ages.

The high tension is rectified and smoothed. In the diagram rectification is shown by means of a metal rectifier and the network of resistances and capacities smooth the rectified current.

Time Bases

In using the cathode-ray tube the potential difference to be investigated is connected across one pair of plates known as the Y-plates. If an alternating p.d. is connected to the plates and the X-plates are at equal potentials, the spot on the fluorescent screen will move to and fro in time with the supply, the extent of the swing being a measure of the amplitude of the p.d. This examination gives no indication of how the p.d. varies with time. Information on this point can be obtained if the spot is made to cross the field in a direction at right angles to the oscillations. If the motion in this direction is made at

a uniform speed the fluorescent track is a time-displacement graph of the motion and is thus a graph of the p.d. against the time. If the spot can be caused to fly back the trace can be repeated. By synchronization of the motion due to the X-plates with that due to the Y-plates, a stationary picture is obtained for the motion is so rapid that the fluorescence persists. In this way the screen is made to present a graph of the p.d. dependence on time which can be examined under static conditions.

The application of the required p.d. to the X-plates in order to produce the constant sweep provides a time scale of measurements known as a time base. In order to understand the principle of time bases we begin with a simple form based on the property of the neon lamp. The apparatus necessary is described as a neon time base.

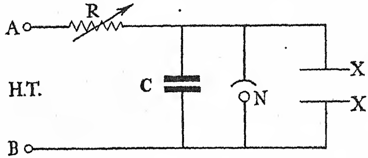

FIG. 483

The potential is applied at points A, B, which are connected through a variable resistance, R, and a condenser, C. The condenser plates are connected to the X-plates (fig. 483) and to a neon lamp.

Suppose a potential, V_0, is applied at AB. The condenser is charged through the resistance, R, and takes a certain time to acquire a potential, V. Theoretically, it takes an infinitely long time to attain the full voltage, V_0. The potential, V, at any time, t, after the beginning of charging is

$$V = V_0\left(1 - e^{-\frac{t}{CR}}\right).$$

The neon lamp flashes when a certain potential, V_1, is applied to it. This causes the condenser to discharge, and it continues to do so until a certain potential, V_2, is reached, when the neon lamp ceases to glow and the condenser begins to charge again. When the potential, V_1, is reached the process is repeated.

Thus the deflector plates gradually attain the potential difference, V_1, and then suddenly fall to V_2, the process repeating itself as long as the tension is applied.

Thus the spot moves across the plate a distance corresponding to a p.d. of V_1, then flies back to that corresponding to V_2.

In the case of the neon lamp, these potentials are about 170 volts and 130 volts, so that a range of 40 volts is provided.

There are two disadvantages associated with this time base. The first is that the range is small, and the second that the sweep is not linear. This means that the p.d. between the plates is not proportional to the time taken to acquire it.

The variation of potential applied to the plates, given by the formula, can be represented graphically. At first the p.d. rises from

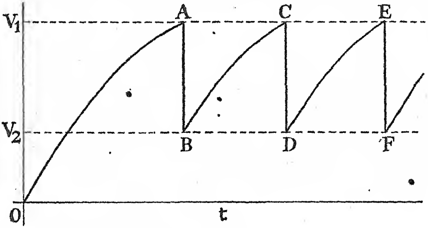

FIG. 484

zero to V_1 along OA. It then falls almost instantaneously to V_2 along AB, and the charging then continues according to the graph along BC, DE, etc.

The time taken for the sweep is denoted by BD, DF, etc. This depends upon the product, CR, according to the formula, and can be varied conveniently by altering the resistance. In this way the period of the sweep can be made equal to that of the p.d. applied to the Y-plates. It will be noted that the dependence of V on t is not linear. This is seen from the graph, where the curves BC, DE, etc., are not straight. They can be made nearly straight by making use of only short parts of them. This requires the use of a short sweep and is a disadvantage in general use. It is for this reason that the use of this simple method has been superseded.

Improvements are required in two main directions. In the first place, the range of the sweep should be greater and should be capable of easy adjustment. One way of obtaining improvement in this respect is to replace the neon lamp by a more flexible discharging device. Mercury vapour or gas-filled relays are used for this purpose. These instruments are triode valves and are generally described as thyratrons, the name used originally for the mercury vapour relay.

The discharge results from the ionization of the gas, the ionization potential of mercury vapour being 15 to 20 volts. It is the function of the grid to control the anode potential corresponding to which the

ionization takes place. When a negative potential is placed on the grid the anode potential has to be raised sufficiently to overcome the negative field at the cathode before ionization occurs. The anode is connected to the condenser, so that the anode potential is the charging potential of the condenser. The discharge is caused by the ionization of the gas in the thyratron, so that this occurs at different anode potentials, according to the potential on the grid.

A means is provided in this way for varying the potential reached by the condenser before it discharges. The circuit of the thyratron is illustrated in fig. 485.

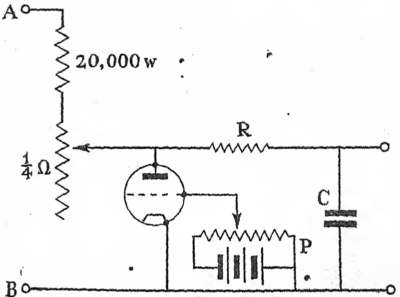

FIG. 485

The H.T. supply is placed across AB and the anode of the thyratron is charged through a resistance, part of which is fixed, the values in a particular example being shown in the figure, in which the quarter megohm is divisible.

The grid potential is variable by means of the potentiometer, P, various negative potentials can be applied to the grid in this way. The condenser discharges through the resistance, R, when the anode, and consequently the condenser, attains the potential corresponding to the thyratron discharge. The p.d. between the condenser plates is applied to the X-plates of the oscillograph.

In the second place, it is necessary to arrange for a linear relation between the potential of the condenser and the time of charging. To obtain this condition a charging device has to be introduced which provides a current independent of the voltage. The diode in the state of saturation is a possible device for this purpose but the use of a pentode is preferred, and is more usual on account of the ease with which its impedance can be controlled.

The Investigation of the Electrostatic Sensitivity of a Cathode-ray Tube

The sensitivity of a cathode-ray tube is determined by the deflection of the spot of light on the screen per unit potential difference between the plates. This is not an absolute measure of the sensitivity, since the deflection depends on the velocity of the electrons in the beam. This depends on the potential difference, V, through which the electrons have fallen in their flight from the filament to the anode.

The length of the tube beyond the deflecting plates is also a factor in determining the magnitude of the deflection, as also is the distance

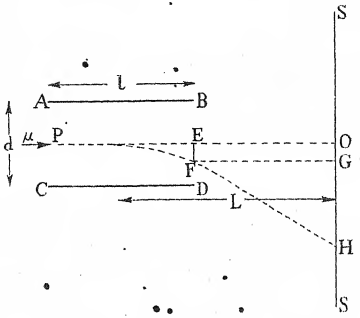

Fig. 486

between the plates. The latter determines the strength of the field between the plates, and consequently the magnitude of the displacement of the beam is influenced by the separation.

When the tube is made up the operator has only one of these variables under control, viz. the potential difference between the filament and anode.

The way the various factors influence the sensitivity quantitatively may be appreciated by a simple consideration in the following way.

Let the plates be denoted by AB and CD (fig. 486), and let their length in the direction of the beam be l. Let the distance between their mid-points and the screen, SS, be L, and their separation be d.

If the electrons have fallen through a potential difference, V, they will have a velocity, u, given by

$$\tfrac{1}{2}mu^2 = eV,$$

where m denotes their mass and e their charge.

Thus the electrons arrive at the point, P, where they enter the field, E, between the plates with the velocity, u. From this point until they leave this field at F, they move horizontally with constant velocity, u, and vertically with acceleration, $\frac{Ee}{m}$.

Thus
$$EF = \frac{1}{2}\frac{Ee}{m} \cdot \left(\frac{l}{u}\right)^2,$$
$\frac{l}{u}$ denoting the time taken to traverse the field.

The vertical velocity on reaching F is $\frac{Ee}{m}\frac{l}{u}$, and they travel from this point to the plate with this constant velocity vertically and with u horizontally.

Thus
$$\frac{HG}{FG} = \frac{Eel}{mu^2}$$
and
$$OH = EF + HG = \frac{EelL}{mu^2}.$$
since
$$FG = L - \tfrac{1}{2}l.$$

Thus, substituting for mu^2 in terms of V,
$$OH = \frac{ElL}{2V} = \frac{UlL}{2Vd},$$
if U denotes the p.d. between the plates.

The sensitivity is thus measured by $\frac{OH}{U}$ and this quantity is evidently inversely proportional to the p.d. between filament and anode.

The experiment to be described will determine the value of the deflection OH for various values of U when a fixed accelerating voltage, V, is applied.

The observations will be repeated for different values of V, and graphs will be drawn for each value of V. The mean value of $\frac{OH}{U}$ will be found for each case, and by plotting the logarithm of this ratio against log V it can be verified whether the slope of the graph is negative and equal to unity as the formula suggests.

If the tube is gas-filled and of a simple type the influence of origin distortion to which a reference has been made (p. 728) will be observed.

The diagram (fig. 487) illustrates the arrangement of the apparatus which consists chiefly of the provision of variable values of V and U by means of potentiometers denoted by Q and R.

The potentiometer, P, is for the purpose of providing a potential to the shield, S, which for many tubes will be negative. The sign of

this potential and its value will depend on the make of the tube. Variations may be used to control the intensity of the spot on the screen. When a suitable value has been chosen it should remain fixed during the experiment.

Voltmeters V and U are required to measure the p.d.s already referred to by these letters.

The potentials may be obtained from batteries the order of value in the case of V being about 100 to 750 volts for a gas focus tube.

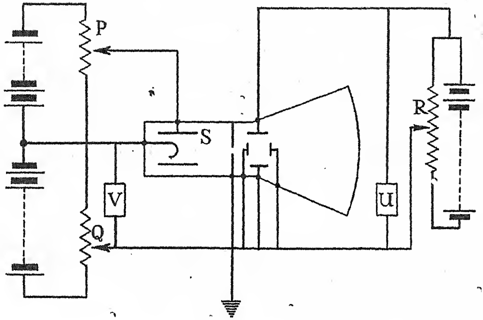

FIG. 487

The Investigation of the Magnetic Sensitivity of a Cathode-ray Tube

A magnetic field is produced across the neck of the tube by a pair of coils connected in series one at each side. These are provided with a current which is variable and they are designed to produce a magnetic field as uniform as possible for a length, l, of the tube across which the electrons pass.

In considering the question of sensitivity in this case, it will be assumed that the field is of strength, H, in this region of length, l, and that it falls abruptly to zero outside the region. This is treating the problem in an ideal way, but the result gives a good approximation to that obtained experimentally, and illustrates the principles concerned.

Let ABO denote the direction of the undisturbed beam and let AB denote the extent of the magnetic field, supposed to be directed into the paper at right angles to it. Each electron of e electrostatic units moving with velocity, u, is equivalent to a current of $\frac{eu}{c}$ electromagnetic units flowing in the direction BA. Thus the field, H, produces a force

$\frac{Heu}{c}$ lying in the plane of the paper at right angles to the direction of motion of the electron. Its motion is thus along the circular arc, AC, so long as it remains in the field. On leaving it at C it flies off at a tangent to strike the screen at X, the deflection being OX. The tangent, XC, cuts the line, AB, at its middle point, D, since both DA and DC are tangents, and it may be supposed that the deviation, BC, is small enough to make BD and DC approximately equal.

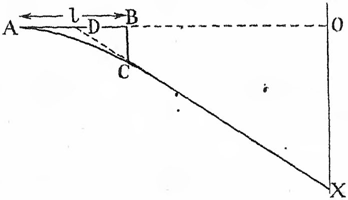

Fig. 488

If OD is denoted by L, and BC by x,

$$OX = \frac{2Lx}{l}.$$

If R denote the radius of the circle of which AC is an arc, we can write

$$2Rx \doteq l^2,$$

again assuming that x is small.

The equation of motion is

$$\frac{Heu}{c} = \frac{mu^2}{R},$$

thus

$$\frac{1}{R} = \frac{He}{cmu}$$

and

$$x = \frac{Hel^2}{2cmu}.$$

If the p.d. between anode and filament is V,

$$Ve = \tfrac{1}{2}mu^2,$$

from which it follows that

$$x = \frac{Hl^2}{2c}\sqrt{\frac{e}{2Vm}}.$$

Thus the deflection on the screen

$$X = OX = \frac{HlL}{c}\sqrt{\frac{e}{2Vm}}.$$

In carrying out the experiment the potential, V, and the focusing potential to the screen may be applied to the tube as before.

The magnetic coils are joined to a battery, a variable resistance and an ammeter in series.

Since the field, H, is proportional to the current, it will be sufficient to plot the ammeter readings against the deflection for a series of values of V. The graphs which result should be linear.

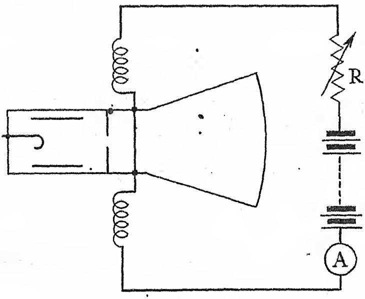

FIG. 489

In order to test the dependence upon V, values of V and X corresponding to a constant deflecting current should be obtained. These may be read from the graphs already plotted.

If log X be plotted as ordinate and log V as abscissa the graph should be linear with a negative slope of value $\frac{1}{2}$.

A further interesting calculation can be made if the field between the coils can be obtained from a knowledge of the current in them. This depends on the dimensions and the winding of the coils, and may be given by means of a graph or table.

If this is not the case and the coils can be removed, the field midway between them can be found by means of a fluxmeter. When H is known, together with the lengths l and L and the p.d. V, the value of $\frac{e}{m}$ can be determined.

The Examination of a Power Pack

A power pack is, in general, required to convert an oscillatory supply of electricity, such as the mains supply of about 250 volts at 50 cycles per second into a steady direct potential.

The object of this experiment is to study the structure of a convenient power pack circuit and to examine the contribution of various parts of it in attaining the desired result.

The figure shows a typical power pack circuit. The mains supply is shown on the left connected to a transformer, T_1T_2. In the figure the transformer is represented as supplying two low-tension voltages by means of L_1 and L_2, for heating filaments, and a high tension by means of H. R denotes a full wave diode rectifier. When A_1 is positive the current flows from A_1 to F, and when A_2 is positive it flows from A_2

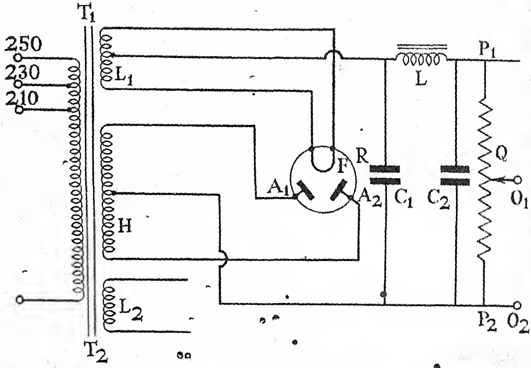

Fig. 490

to F. Thus the current passes through R in the same direction in each half of the cycle.

The E.M.F. supplied is represented in fig. 491.

This curve of E.M.F. against time can be expressed by means of a Fourier series consisting of a constant term and an infinite number of others which are harmonic in the time. The expression may be written in the form:

$$E = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + a_n \cos n\omega t + \ldots$$

If it is possible to suppress the variable terms a constant value, a_0, of the E.M.F. remains.

We can regard the E.M.F. resulting from the rectification as consisting of a direct E.M.F. and a series of waves of alternating E.M.F. of frequencies $\frac{\omega}{2\pi}$, $\frac{2\omega}{2\pi}$, etc.

The capacities, C_1 and C_2, and the inductance, L, form a wave filter which ideally refuses passage to all these waves and thus a constant E.M.F. of magnitude, a_0, results.

In the experiment we shall examine the difference in the final results which follows from cutting out various elements of the filter.

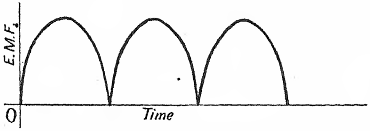

FIG. 491

It will be observed that if the lead to A_1 or A_2 is broken the current in half a cycle fails to get through the rectifier and half-wave rectification results (fig. 492). The effect on smoothing which results from half-wave rectification will be examined.

The potentiometer, P_1P_2, allows the output voltage taken between O_1 and O_2 to be varied. These points are connected to the plates of a cathode-ray oscillograph.

Fig. 493 illustrates a method for easy examination of the effects produced by various elements of the power pack.

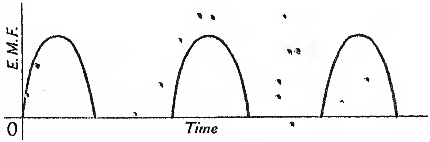

FIG. 492

It will be noted that fig. 493 is the same as fig. 490, with the addition of five switches.

If S_1 is brought into contact with A and the others closed with S_4 in contact with D, the circuits of the two figures are identical. The small coil of the transformer shown disconnected serves as a supply of additional low tension for heating purposes should it be required.

1. To examine the character of the unaltered wave-form as it is supplied by the transformer the switch, S_1, should be closed with contact at B. The switch, S_3, should be closed to cut out the inductance. L, and S_5 left open to cut out the capacities. S_2 can be in either position,

The switch, S_1, must be such that it is impossible to supply a filament current and make contact at B at the same time.

With the arrangements described, the supply is admitted to the cathode-ray oscillograph through O_1 and O_2 without smoothing or rectification.

A trace illustrating the wave-form of the supply should be made. After this examination S_1 should be closed with contact at A and the following tests carried out.

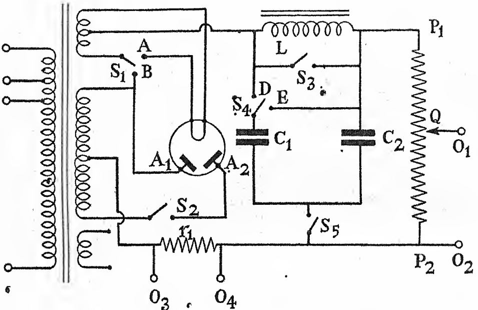

FIG. 493

2. To examine half-wave rectification without smoothing, let the switch, S_2, be open. Close S_3 to cut out the inductance and open S_4 to cut out the capacities.

3. To examine full-wave rectification, close S_2 and leave the other switches unaltered.

Draw diagrams of the traces observed.

The following experiments should be made for half-wave and full-wave rectification:

1. Examine the effect of introducing the capacities in parallel across the leads by closing S_3 to cut out the inductance, by placing switch, S_4, to make contact with D, and by closing S_5. Note the traces and examine the degree of smoothing, contrasting the cases of half- and full-wave rectification.

2. Insert the inductance by opening switch, S_3. This is the usual working position of the filter. Again examine the two cases and make a record of the result by drawing the traces observed.

3. Place switch, S_4, to make contact with E so that the capacity, C_2, is increased by C_1. Contrast this with the previous case.

These observations record the output voltage wave-forms of the power pack.

The current wave-forms can be observed by inserting a resistance, r_1, and connecting the ends, O_3, O_4, to the cathode-ray oscillograph.

Examine the current wave-forms in all the cases described above.

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